Mathematics 2000: Assignment #8, Winter 2006

1. Use the Chain Rule to find $\frac{dw}{dt}$, given $w = \cos(x - y)$ with $x = t^2$ and $y = 1$.

$$\frac{dw}{dt} = -\sin(x-y)(2t) + \sin(x-y)(0) = -2t\sin(x-y) = -2t\sin(t^2-1)$$

2. Use the Chain Rule to find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$, given $w = x\cos(yz)$, with $x = s^2$,$y = t^2$ and $z = s - 2t$.

$$\frac{\partial w}{\partial s} = \cos(yz)(2s) - xz\sin(yz)(0) - xy\sin(yz)(1)$$

$$= \cos(st^2 - 2t^3)2s - s^2t^2\sin(st^2 - 2t^3)$$

$$\frac{\partial w}{\partial t} = \cos(yz)(0) - xz\sin(yz)(2t) - xy\sin(yz)(-2) =$$

$$= -2s^2t(s-2t)\sin(st^2 - 2t^3) + 2s^2t^2\sin(st^2 - 2t^3)$$

$$= (6s^2t^2 - 2s^3t)\sin(t^2s - 2t^3)$$

3. Let $w = \frac{yz}{x}$ with $x = \theta^2, y = r + \theta$ and $z = r - \theta$. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$, by:

   (a) using the Chain Rule

   (b) converting w to a function of r and θ before differentiating.

(a) $$\frac{\partial w}{\partial r} = \frac{-yz}{x^2}(0) + \frac{z}{x}(1) + \frac{y}{x}(1) = \frac{z+y}{x} = \frac{2r}{\theta^2}$$

$$\frac{\partial w}{\partial \theta} = \frac{-yz}{x^2}(2\theta) + \frac{z}{x}(1) + \frac{y}{x}(1) = \frac{-(r+\theta)(r-\theta)}{\theta^4}(2\theta)$$

$$+\frac{(r-\theta)-(r+\theta)}{\theta^2} = \frac{-2r^2}{\theta^3}$$

(b) $$w = \frac{yz}{x} = \frac{(r+\theta)(r-\theta)}{\theta^2} = \frac{r^2-\theta^2}{\theta^2} = \left(\frac{r}{\theta}\right)^2 - 1$$

$$\frac{\partial w}{\partial r} = \frac{2r}{\theta^2}; \qquad \frac{\partial w}{\partial \theta} = \frac{-2r^2}{\theta^2}$$

4. Show that $w = (x - y)\sin(y - x)$ satisfies the equation $\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$, (where $w = f(x, y), x = u - v, y = v - u$).

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{dx}{dv} + \frac{\partial w}{\partial y}\frac{dy}{dv} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$$

5. Differentiate implicitly to find $\frac{dy}{dx}$, given $\cos x + \tan xy + 5 = 0$.

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)} = -\frac{-\sin x + y\sec^2(xy)}{x\sec^2(xy)}$$

6. Differentiate implicitly to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

(a) $z = e^x \sin(z+y)$

$$\frac{\partial z}{\partial x} = \frac{e^x \sin(z+y)}{1 - e^x \cos(z+y)}$$

$$\frac{\partial z}{\partial y} = \frac{e^x \cos(z+y)}{1 - e^x \cos(z+y)}$$

(b) $x \ln y + y^2 z + z^2 = 8$.

$$\frac{\partial z}{\partial x} = \frac{-\ln y}{y^2 + 2z}, \qquad \frac{\partial z}{\partial y} = -\frac{x + 2y^2 z}{y^3 + 2yz}$$

7. Examine each function for relative extrema.

(a) $f(x,y) = |x+y| - 2$

Since $f(x,y) \geq -2$ for all (x,y), the relative minima consists of all points (x,y) satisfying $x + y = 0$.

(b) $z = -3x^2 - 2y^2 + 3x - 4y + 5$

$f_x = -6x + 3 = 0$ when $x = \frac{1}{2}$ $f_{xx} = -6$

$f_y = -4y - 4 = 0$ when $y = -1$ $f_{yy} = -4$ $f_{xy} = 0$

At the critical point $(\frac{1}{2}, -1)$; $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

Therefore $(\frac{1}{2}, -1, \frac{31}{4})$ is a relative maximum.

8. Determine whether there is a relative maximum, a relative minimum, a saddle point, or insufficient information to determine the nature of the function $f(x, y)$ at the critical point $f_{xx}(x_0, y_0) = -3, f_{yy}(x_0, y_0) = -8, f_{xy}(x_0, y_0) = 2$.

   $f_{xx} < 0$, and $f_{xx}f_{yy} - (f_{xy})^2 = (-3)(-8) - 2^2 > 0 \Rightarrow f$ has a relative maximum at (x_0, y_0)

9. Find the absolute extrema for the function f over the region R. (In each case R contains the boundaries).

   (a) $f(x, y) = 2x - 2xy + y^2$,
   R : the region in the xy-plane bounded by the graphs of $y = x^2$ and $y = 1$.

   $f_x = 2 - 2y = 0 \Rightarrow y = 1$

   $f_y = 2y - 2x = 0 \Rightarrow y = x \Rightarrow x = 1 \Rightarrow f(1, 1) = 1$

   On the line $y = 1; -1 \leq x \leq 1, f(x, y = 1) = 2x - 2x + 1 = 1$

   On the curve $y = x^2; -1 \leq x \leq 1;$

   $g(x) = f(x, y = x^2) = 2x - 2x(x^2) + (x^2)^2 = x^4 - 2x^3 + 2x$. Set $g'(x) = 0$ to find critical points. They are at $x = 1$ and $x = -0.5$

   Evaluate $g(-1) = 1, g(1) = 1, g(-0.5) = -11/16$.

   Thus the maximum is 1, the minimum is $\dfrac{-11}{16}$

   Absolute maximum: 1 at on $y = 1, -1 \leq x \leq 1$

   Absolute minimum: $\dfrac{-11}{16}$ at $(-\dfrac{1}{2}, \dfrac{1}{4})$

   (b) $f(x, y) = x^2 - 4xy + 5$,
   $R = \{(x, y) : 0 \leq x \leq 4, 0 \leq y \leq \sqrt{x}\}$.

   $f_x = 2x - 4y = 0 \qquad f_y = -4x = 0$

   $\Rightarrow x = y = 0$; is critical point; $f(0, 0) = 5$

   Along $y = 0; x \in [0, 4]; g(x) = f(x, 0) = x^2 + 5$, critical point is at $x = 0$ and $g(0) = f(0, 0) = 5$, $g(4) = f(4, 0) = 21$

Along $x = 4, y \in [0, 2], g(y) = f(4, y) = 16 - 16y + 5 = 21 - 16y$;
$g'(y) = -16 \neq 0$; No critical points; $g(0) = f(4, 0) = 21$,
$g(2) = f(4, 2) = -11$

Along $y = \sqrt{x}$; $x \in [0, 4]$; $g(x) = f(x, \sqrt{x}) = x^2 - 4x^{\frac{3}{2}} + 5$;
$g' = 2x - 6x^{\frac{1}{2}} = 0$, critical point $x = 0$ on $[0, 4]$
$g(0) = f(0, 0) = 5; g(4) = f(4, 2) = -11$

Hence, the maximum is $f(4, 0) = 21$ and
the minimum is $f(4, 2) = -11$