Assignment 2 answer key

1. Determine which of the following series is divergent and explain why.

   Solution: We will use the statement that if $\lim_{n\to\infty} a_n \neq 0$ then series $\sum_{n=1}^{\infty} a_n$ is divergent.

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$; $\displaystyle\lim_{n\to\infty} \frac{n+1}{2n+1} = \frac{1}{2} \neq 0$. The series diverges.

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3}$; By L'Hospital's Rule
   $$\lim_{n\to\infty} \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{(\ln 3)3^n}{3n^2} = \lim_{n\to\infty} \frac{(\ln 3)^2 3^n}{6n} = \lim_{n\to\infty} \frac{(\ln 3)^3 3^n}{6} = \infty$$
   The series diverges.

   c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{100}$, geometric series with r=2. The series diverges.

   d) $\displaystyle\sum_{n=1}^{\infty} (1+\frac{k}{n})^n$,

   For all k the limit $\displaystyle\lim_{n\to\infty} \left(1+\frac{k}{n}\right)^n = e^k \neq 0$. The series diverges.

   e) $\displaystyle\sum_{n=1}^{\infty} (\frac{n}{n+3})^{-2n}$. The limit $\displaystyle\lim_{n\to\infty} (\frac{n}{n+3})^{-2n} \lim_{n\to\infty} (\frac{n+3}{n})^{2n} = e^6 \neq 0$.
   The series diverges.

2. For each of the following series, find the sum of the convergent series.
   Solution All of the following series are geometric related. Here we use $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

   (a) $$\sum_{n=1}^{\infty} [(0.7)^n + (0.9)^n] = \sum_{n=0}^{\infty} \left(\frac{7}{10}\right)^n + \sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n - 2$$
   $$= \frac{1}{1-\frac{7}{10}} + \frac{1}{1-\frac{9}{10}} - 2 = \frac{10}{3} + 10 - 2 = \frac{34}{3}$$

   (b) $$4 - 2 + 1 - \frac{1}{2} + ... = \sum_{n=0}^{\infty} 4\left(\frac{-1}{2}\right)^n = \frac{4}{1-(-\frac{1}{2})} = \frac{4}{\frac{3}{2}} = \frac{8}{3}$$

   (c) $$\sum_{n=0}^{\infty} (-2)(\frac{-2}{3})^n = \frac{-2}{1-(-\frac{2}{3})} = -\frac{6}{5}.$$

3. Find the values of x for which the series converges, and find the sum of the series for those values of x.

   (a) $\sum_{n=0}^{\infty} \frac{(4x-1)^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{4x-1}{5}\right)^n = \sum_{n=0}^{\infty} r^n$; where $r = \frac{4x-1}{5}$
   $\Rightarrow$ The series converges iff $|r| < 1 \Leftrightarrow |\frac{4x-1}{5}| < 1 \Leftrightarrow$
   $|4x-1| < 5 \Leftrightarrow -5 < 4x - 1 < 5 \Leftrightarrow -1 < x < 3/2$
   So, the series converges iff x belongs to the interval $(-1, \frac{3}{2})$ and the sum is
   S$=\frac{1}{1-r} = \frac{1}{1-\frac{4x-1}{5}} = \frac{5}{5-(4-1)} = \frac{5}{6-4x}$.

   (b) $\sum_{n=0}^{\infty} (\cos x)^n = \sum_{n=0}^{\infty} r^n$, where $r = \cos x \Rightarrow$ the series converges iff $|\cos x| < 1$ $\Leftrightarrow -1 < \cos x < 1 \Leftrightarrow x$ is any number from the interval $[0, 2\pi]$ except $0, \pi$ and 2π. The sum of the series is $\frac{1}{1-\cos x}$
   P.S. Function $\cos x$ is $2\pi-$periodic function thus it is enough to consider its restriction on its period.

4. Express the repeating decimal as a geometric series and write its sum as the ratio of two integers.

   (a) $0.21515515515... = 0.21 + .000\overline{515} = \frac{21}{100} + \frac{1}{100} \cdot 0.\overline{515} = \frac{21}{100} + \frac{1}{100} \sum_{n=1}^{\infty} \frac{515}{(1000)^n} = \frac{21}{100} + \frac{1}{100} \cdot \frac{515}{1000} \cdot \sum_{n=0}^{\infty} \frac{1}{(1000)^n} = \frac{21}{100} + \frac{515}{10^5} \cdot \frac{1}{1-\frac{1}{10^3}} = \frac{21}{100} + \frac{515}{10^5} \cdot \frac{10^3}{999} = \frac{21494}{99900}$

   (b) $1.818181818181818181818... = 1.8\overline{181} = 1 + 0.8\overline{181} = 1 + \sum_{n=0}^{\infty} \frac{81}{100}\left(\frac{1}{100}\right)^n$
   $\Leftrightarrow$ geom. series. $\sum_{n=0}^{\infty} \frac{81}{100}\left(\frac{1}{100}\right)^n$; $r = \frac{1}{100}$ $a = \frac{81}{100}$
   S$= \frac{a}{1-r} = \frac{\frac{81}{100}}{1-\frac{1}{100}} = \frac{9}{11}$ so, $1.8\overline{181} = 1 + \frac{9}{11} = \frac{20}{11}$

5. Use the Integral test to determine the convergence or divergence of the series.

   Solution: The integral test is applicable for series with $a_n = f(n)$, where $f(x)$ is a positive and continuous (for $x \geq 1$), and decreasing (for $x \geq a \geq 1$) function. Then convergence of the improper integral $\int_1^{\infty} f(x)\,dx$ implies convergence of the series $\sum_{n=1}^{\infty} a_n$.

   (a) $\sum_{n=1}^{\infty} (n)^k (e)^{-n}$, for all integer $k \geq 1$;

Case $k = 1$ we did in class. Now consider case $k = 2$. We have $\sum_{n=1}^{\infty} n^2 e^{-n}$; Then $f(x) = x^2 e^{-x}$ is positive and continuous. To check that the function is decreasing we find its derivative $f'(x) = e^{-x}(-x^2 + 2x)$ and show that $f'(x) < 0$ for all $x > 2$.
The integral can be evaluated by parts
$\int_1^\infty x^2 e^{-x} dx = -e^{-x}(x^2+2x+2)|_1^\infty = \frac{5}{e}$. Thus the series $\sum_{n=1}^\infty n^2 e^{-n}$ converges.

Similarly, for $k = 3$. We have $\sum_{n=1}^{\infty} n^3 e^{-n}$; Then $f(x) = x^3 e^{-x}$ is positive (for $x > 0$)and continuous. To check that the function is decreasing we find its derivative $f'(x) = e^{-x}(-x^3 + 3x)$ and show that $f'(x) < 0$ for all $x > 3$.
The integral can be evaluated by parts
$\int_1^\infty x^3 e^{-x} dx = -e^{-x}(x^3 + 3x^2 + 6x + 6)|_1^\infty = \frac{16}{e}$. Thus the series $\sum_{n=1}^\infty n^3 e^{-n}$ converges.
For any integer $k \geq 1$ we have: $f(x) = x^k e^{-x}$ is positive (for $x > 0$)and continuous. To check that the function is decreasing we find its derivative $f'(x) = e^{-x}(-x^k + kx^{k-1})$ and show that $f'(x) < 0$ for all $x > k$.
Using integration by parts and mathematical induction we show that $\displaystyle\int_1^\infty x^k e^{-x} dx =$
$$= \left[-x^k e^{-x}\right]_1^\infty + k\int_1^\infty x^{k-1} e^{-x} dx = \frac{1}{e} + \frac{k}{e} + \frac{k(k-1)}{e} + \cdots + \frac{k!}{e}.$$
Thus the series converges for any $k \geq 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$.

Function $f(x) = \dfrac{1}{x^{1/3}}$ is positive, continuous and decreasing for $x > 0$.
$\displaystyle\int_1^\infty \frac{1}{x^{1/3}} dx = \lim_{A\to\infty} \left[\frac{3}{2} x^{\frac{2}{3}}\right]_1^A = \infty$. Thus the series diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\pi}$ Convergent p-series with $p = \pi > 1$.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$. Function $f(x) = \frac{\ln x}{x^3}$ is continous and positive for $x \geq 2$.
$f'(x) = \frac{1-3\ln x}{x^4} < 0$ for $x > e^{1/3}$. Thus the function is decreasing.
$$\int_2^\infty f(x)dx = \lim_{t\to\infty} \int_2^t \frac{\ln x}{x^3} dx = \lim_{t\to\infty} \left[-\frac{1}{2}\frac{\ln x}{x^2} - \frac{1}{4x^2}\right]_2^t$$
$$= \lim_{t\to\infty} \left(\frac{-\ln t}{2t^2} - \frac{1}{4t^2} + \frac{\ln 2}{8} + \frac{1}{16}\right) = \frac{1 + 2\ln 2}{16} - \lim_{t\to\infty} \frac{1}{4t^2} - \lim_{t\to\infty} \frac{\ln t}{2t^2} =$$
$$\frac{1 + 2\ln 2}{16} - 0 - 0 = \frac{1 + \ln 4}{16}$$ $\Rightarrow$ the series is convergent.