Math 2000: Solutions for Assignment #1, W 2006

1a) First five terms of the sequence $a_n = \frac{n+1}{3n+1}$, where $n = 1, 2, 3, 4, 5$ are

$$\left\{\frac{1}{2}, \frac{3}{7}, \frac{2}{5}, \frac{5}{13}, \frac{3}{8}\right\}.$$

1b) First five terms of the sequence given by the reccurence relation $a_{n+1} = \frac{a_n}{a_n - 1}$, where $a_1 = 4$ are

$$\left\{4, \frac{4}{3}, 4, \frac{4}{3}, 4\right\}$$

2. Formulas for general terms are

a) $a_n = \dfrac{3^{n-1}}{2^n}$, $n = 1, 2, ...$ b) $a_n = (-1)^{n+1}(\dfrac{2n-1}{2n})$, $n = 1, 2, ...$

3.

a) $\displaystyle\lim_{n\to\infty} \frac{n+1}{3n+1} = \lim_{n\to\infty} \frac{1+1/n}{3+1/n} = \frac{1}{3}$

b) $\displaystyle\lim_{n\to\infty} \frac{3^n}{n!} = \lim_{n\to\infty} \left(\frac{3}{n} \cdot \frac{3}{n-1} \cdot \frac{3}{n-2} \cdot \frac{3}{n-3} \cdots \frac{3}{3} \cdot \frac{3}{2} \cdot \frac{3}{1}\right)$

$\Rightarrow$ each of terms $\frac{3}{k} \leq 1$, for $k \geq 4 \Rightarrow$ the total product $\frac{3}{4}\frac{3}{5} \cdots \frac{3}{n-1}$ is also less then 1

So,

$\displaystyle\leq \lim_{n\to\infty} \frac{3}{n}\left(\frac{3}{2} \cdot \frac{3}{1}\right) = \lim_{n\to\infty} \frac{27}{2n} = 0$

$\Rightarrow 0 \leq \dfrac{3^n}{n!} \leq \dfrac{27}{2n}$ So by the squeeze therom, $\displaystyle\lim_{n\to\infty} \frac{3^n}{n!} = 0$

c) $c_n = (-1)^n \dfrac{n}{n+1}$ doesn't converge. $\dfrac{n}{n+1} \to 1$ as $n \to \infty$

so the $(-1)^n$ causes late terms in the series to alternate between $+1$ and -1, approaching bot

d) $\displaystyle b_n = \frac{n!}{(n+1)!} = \frac{1}{n+1} \Rightarrow \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

e) $c_n = \cos\left(\dfrac{2}{n}\right)$ As $n \to \infty, \dfrac{2}{n} \to 0 \Rightarrow \displaystyle\lim_{n\to\infty} \cos(\frac{2}{n}) = \cos(0) = 1$

f) $\displaystyle a_n = \frac{\sqrt{n}}{\sqrt{n+1}} \to \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}} = 1$

g) $c_n = \dfrac{(-3)^n}{(n+1)!}$ Note that $\lim\limits_{n\to\infty} \dfrac{3^n}{(n+1)!} = \lim\limits_{n\to\infty} \dfrac{1}{n+1} \lim\limits_{n\to\infty} \dfrac{3^n}{n!}$

From problem 3. b) We know that $\lim\limits_{n\to\infty} \dfrac{3^n}{n!} = 0$

Hence, by the absolute value theorem, $\lim\limits_{n\to\infty} c_n = 0$

h) $a_n = \arctan(2n) \to \lim\limits_{n\to\infty} a_n = \arctan(\infty) = \dfrac{\pi}{2}$

i) $b_n = \dfrac{\ln(n)}{\ln(2n)} = \dfrac{\ln(n)}{\ln(2) + \ln(n)}$ dividing everything by $\ln(n)$ yeilds $\dfrac{1}{\frac{\ln(2)}{\ln(n)} + 1}$

As $n \to \infty$, $\dfrac{\ln(2)}{\ln(n)} = 0, so \lim\limits_{n\to\infty} b_n = 1$

4.

a) $a_n = \dfrac{1}{3^n} \to$ this is decreasing since $a_{n+1} = \dfrac{1}{3^{n+1}} = \dfrac{1}{3}a_n$

Therefore, it is bounded from above by its first term $\to \frac{1}{3}$ and is bounded from below by 0

b) $b_n = \dfrac{2n-3}{3n+4} \to$ this is increaseing since $\dfrac{d}{dx}\left(\dfrac{2x-3}{3x+4}\right) = \dfrac{17}{3x+4} > 0$ for $x \geq 1$.

Therefore it is bounded from below by its first term: $b_1 = \dfrac{1}{7}$, and is bounded from above by $\frac{2}{3}$

because $\lim\limits_{n\to\infty} \dfrac{2n-3}{3n+4} = \dfrac{2}{3}$

c) This sequence is not monotonic since $\sin(\dfrac{\pi n}{4})$ alternates between 1, $\dfrac{\sqrt{2}}{2}$, 0, $\dfrac{-\sqrt{2}}{2}$,and 1

However it is bound from above by 1 and below by -1(or some tighter bound)

5. This is a harder one. We proceed by induction. If

$$\begin{aligned}
&a_{n+1} < a_n \\
&-a_{n+1} > -a_n \\
&3 - a_{n+1} > 3 - a_n \\
&\frac{1}{3 - a_{n+1}} < \frac{1}{3 - a_n}
\end{aligned}$$

$a_{n+2} < a_{n+1}$

So if $a_1 \geq a_2$, then $a_2 \geq a_3 \Rightarrow a_3 \geq a_4 \Rightarrow a_4 \geq a_5$, etc

$a_1 = 2, a_2 = \dfrac{1}{3-a_n} = \dfrac{1}{3-2} = 1$ so, $a_1 > a_2$ and in general $a_{n+1} < a_n \Rightarrow$ sequence is decreasing and monotonic. Therefore it is bound from above by its first term. Is it bound from below as well?

Yes, $a_n \leq 2$ for all n. Thus $a_{n+1} = \frac{1}{3-a_n}$ is positive. So, the sequence is bounded $0 \leq a_n \leq 2$ and monotonic, thus it converges.

Next, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{1}{3-a_n} \Rightarrow L = \dfrac{1}{3-L} \Rightarrow L(3-L) = 1$

$\Rightarrow L^2 - 3L + 1 = 0 \Rightarrow L = \dfrac{3+\sqrt{5}}{2}$ or $\frac{3-\sqrt{5}}{2}$. Now $\frac{3+\sqrt{5}}{2} > 2$ so that isn't the limit.

Therfore $\lim\limits_{n\to\infty} a_n = \dfrac{3-\sqrt{5}}{2}$.

6.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^2+1} \to S_n =$

$\left(\frac{1}{4}, \frac{41}{52}, \frac{127}{91}, \frac{1292}{637}, \frac{129405}{48412}, 3.324..., 3.979..., 4.637..., 5.297..., 5.958...\right)$

This series is divergent because $\displaystyle\lim_{n\to\infty} \frac{2n^2-1}{3n^2+1} = \frac{2}{3} \neq 0$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}} =\!\to S_n = \left(2, \frac{10}{3}, \frac{38}{9}, \frac{130}{27}, ...\right)$

$$\sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}} = 2\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = 2\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n} = \frac{2}{1-2/3} = 6$$

$\to$ This is a geometric series, $r = \frac{2}{3} < 1 \Rightarrow$ The series is convergent.