

$$1. \text{ (a) } \int \frac{x^3}{x^2-1} dx \stackrel{\text{divide out}}{=} \int \left(x + \frac{x}{x^2-1} \right) dx = \frac{x^2}{2} + \frac{1}{2} \ln|x^2-1| + C$$

$$\text{(b) } \frac{10x+9}{2x^2+5x-3} = \frac{10x+9}{(2x-1)(x+3)} = \frac{A}{2x-1} + \frac{B}{x+3}$$

$$\begin{aligned} 10x+9 &= A(x+3) + B(2x-1) \\ &= (A+2B)x + (3A-B) \end{aligned}$$

$$\begin{aligned} A+2B &= 10 \\ 3A-B &= 9 \end{aligned} \Rightarrow \begin{aligned} A &= 4 \\ B &= 3 \end{aligned}$$

$$\int \frac{10x+9}{2x^2+5x-3} dx = \int \left(\frac{4}{2x-1} + \frac{3}{x+3} \right) dx = 2 \ln|2x-1| + 3 \ln|x+3| + C$$

$$\text{(c) } \frac{1}{x(x+1)(2x+3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{2x+3}$$

$$\begin{aligned} 1 &= A(x+1)(2x+3) + Bx(2x+3) + Cx(x+1) \\ &= (2A+2B+C)x^2 + (5A+3B+C)x + 3A \end{aligned}$$

$$\begin{aligned} 2A+2B+C &= 0 & A &= 1/3 \\ 5A+3B+C &= 0 & \Rightarrow B &= -1 \\ 3A &= 1 & C &= 4/3 \end{aligned}$$

$$\int \frac{1}{x(x+1)(2x+3)} dx = \int \left(\frac{1/3}{x} - \frac{1}{x+1} + \frac{4/3}{2x+3} \right) dx = \frac{1}{3} \ln|x| - \ln|x+1| + \frac{2}{3} \ln|2x+3| + C$$

$$\text{(d) } \frac{3x^2-2}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

$$\begin{aligned} 3x^2-2 &= A(x+2)^2 + B(x+2) + C \\ &= Ax^2 + (4A+B)x + (4A+2B+C) \end{aligned}$$

$$\begin{aligned} A &= 3 & A &= 3 \\ 4A+B &= 0 & \Rightarrow B &= -12 \\ 4A+2B+C &= -2 & C &= 10 \end{aligned}$$

$$\int \frac{3x^2-2}{(x+2)^3} dx = \int \left[\frac{3}{x+2} - \frac{12}{(x+2)^2} + \frac{10}{(x+2)^3} \right] dx = 3 \ln|x+2| + \frac{12}{x+2} - \frac{5}{(x+2)^2} + C$$

$$(e) \frac{1}{(x-1)^2(x+4)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+4}$$

$$\begin{aligned} 1 &= A(x-1)(x+4) + B(x+4) + C(x-1)^2 \\ &= (A+C)x^2 + (3A+B-2C)x + (-4A+4B+C) \end{aligned}$$

$$\begin{aligned} A + C &= 0 & A &= -1/25 \\ 3A + B - 2C &= 0 & \Rightarrow B &= 1/5 \\ -4A + 4B + C &= 1 & C &= 1/25 \end{aligned}$$

$$\begin{aligned} \int \frac{1}{(x-1)^2(x+4)} dx &= \int \left[\frac{-\frac{1}{25}}{x-1} + \frac{\frac{1}{5}}{(x-1)^2} + \frac{\frac{1}{25}}{x+4} \right] dx \\ &= -\frac{1}{25} \ln|x-1| - \frac{1}{5(x-1)} + \frac{1}{25} \ln|x+4| + C \end{aligned}$$

$$(f) \frac{2x^2+3}{x^4-2x^2+1} = \frac{2x^2+3}{(x^2-1)^2} = \frac{2x^2+3}{(x-1)^2(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$$\begin{aligned} 2x^2+3 &= A(x-1)(x+1)^2 + B(x+1)^2 + C(x+1)(x-1)^2 + D(x-1)^2 \\ &= (A+C)x^3 + (A+B-C+D)x^2 + (-A+2B-C-2D)x + (-A+B+C+D) \end{aligned}$$

$$\begin{aligned} A + C &= 0 & A &= -1/4 \\ A + B - C + D &= 2 & \Rightarrow B &= 5/4 \\ -A + 2B - C - 2D &= 0 & C &= 1/4 \\ -A + B + C + D &= 3 & D &= 5/4 \end{aligned}$$

$$\begin{aligned} \int \frac{2x^2+3}{x^4-2x^2+1} dx &= \int \left[\frac{-\frac{1}{4}}{x-1} + \frac{\frac{5}{4}}{(x-1)^2} + \frac{\frac{1}{4}}{x+1} + \frac{\frac{5}{4}}{(x+1)^2} \right] dx \\ &= -\frac{1}{4} \ln|x-1| - \frac{5}{4(x-1)} + \frac{1}{4} \ln|x+1| - \frac{5}{4(x+1)} + C \end{aligned}$$

$$(g) \frac{2x+3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$\begin{aligned} 2x+3 &= A(x^2+3) + (Bx+C)x & \Rightarrow \begin{aligned} A+B &= 0 \\ C &= 2 \end{aligned} & \Rightarrow \begin{aligned} A &= 1 \\ B &= -1 \\ C &= 2 \end{aligned} \\ &= (A+B)x^2 + Cx + 3A & \Rightarrow \begin{aligned} 3A &= 3 \end{aligned} \end{aligned}$$

$$\begin{aligned} \int \frac{2x+3}{x(x^2+3)} dx &= \int \left(\frac{1}{x} + \frac{-x+2}{x^2+3} \right) dx = \int \left(\frac{1}{x} - \frac{x}{x^2+3} + \frac{2}{x^2+3} \right) dx \\ &= \ln|x| - \frac{1}{2} \ln(x^2+3) + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} + C \end{aligned}$$

$$(h) \quad \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$\begin{aligned} x^2 - 2x - 1 &= A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2 \\ &= (A+C)x^3 + (-A+B-2C+D)x^2 + (A+C-2D)x + (-A+B+D) \end{aligned}$$

$$\begin{aligned} A + C &= 0 & A &= 1 \\ -A + B - 2C + D &= 1 & B &= -1 \\ A + C - 2D &= -2 & C &= -1 \\ -A + B + D &= -1 & D &= 1 \end{aligned} \Rightarrow$$

$$\begin{aligned} \int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)} dx &= \int \left[\frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{-x+1}{x^2+1} \right] dx = \int \left[\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x}{x^2+1} + \frac{1}{x^2+1} \right] dx \\ &= \ln|x-1| + \frac{1}{x-1} - \frac{1}{2} \ln(x^2+1) + \tan^{-1} x + C \end{aligned}$$

$$(i) \quad \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} = \frac{x^3 - 2x^2 + x + 1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

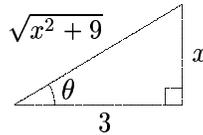
$$\begin{aligned} x^3 - 2x^2 + x + 1 &= (Ax+B)(x^2+4) + (Cx+D)(x^2+1) \\ &= (A+C)x^3 + (B+D)x^2 + (4A+C)x + (4B+D) \end{aligned}$$

$$\begin{aligned} A + C &= 1 & A &= 0 \\ B + D &= -2 & B &= 1 \\ 4A + C &= 1 & C &= 1 \\ 4B + D &= 1 & D &= -3 \end{aligned} \Rightarrow$$

$$\begin{aligned} \int \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} dx &= \int \left(\frac{1}{x^2+1} + \frac{x-3}{x^2+4} \right) dx = \int \left(\frac{1}{x^2+1} + \frac{x}{x^2+4} - \frac{3}{x^2+4} \right) dx \\ &= \tan^{-1} x + \frac{1}{2} \ln(x^2+4) - \frac{3}{2} \tan^{-1} \frac{x}{2} + C \end{aligned}$$

$$(j) \quad \int \frac{x^2}{(x^2+9)^2} dx$$

$$\begin{aligned} x &= 3 \tan \theta \\ dx &= 3 \sec^2 \theta d\theta \\ \sqrt{x^2+9} &= 3 \sec \theta \end{aligned}$$



$$= \int \frac{(3 \tan \theta)^2}{(3 \sec \theta)^4} \cdot 3 \sec^2 \theta d\theta = \frac{1}{3} \int \frac{\tan^2 \theta}{\sec^2 \theta} d\theta = \frac{1}{3} \int \sin^2 \theta d\theta = \frac{1}{6} \int (1 - \cos \theta) d\theta = \frac{1}{6} \left(\theta - \frac{1}{2} \sin 2\theta \right) + C$$

$$= \frac{1}{6} (\theta - \sin \theta \cos \theta) + C = \frac{1}{6} \left(\tan^{-1} \frac{x}{3} - \frac{x}{\sqrt{x^2+9}} \cdot \frac{3}{\sqrt{x^2+9}} \right) + C = \frac{1}{6} \tan^{-1} \frac{x}{3} - \frac{x}{2(x^2+9)} + C$$

$$(k) \quad \frac{12}{x^3 + 8} = \frac{12}{(x+2)(x^2 - 2x + 4)} = \frac{A}{x+2} + \frac{Bx + C}{x^2 - 2x + 4}$$

$$\begin{aligned} 12 &= A(x^2 - 2x + 4) + (Bx + C)(x + 2) \\ &= (A + B)x^2 + (-2A + 2B + C)x + (4A + 2C) \end{aligned}$$

$$\begin{aligned} A + B &= 0 & A &= 1 \\ -2A + 2B + C &= 0 & \Rightarrow B &= -1 \\ 4A &+ 2C = 12 & C &= 4 \end{aligned}$$

$$\begin{aligned} \int \frac{12}{x^3 + 8} dx &= \int \left(\frac{1}{x+2} + \frac{-x+4}{x^2 - 2x + 4} \right) dx = \ln|x+2| + \int \frac{-x+4}{(x-1)^2 + 3} dx & \begin{array}{l} u = x - 1 \\ du = dx \end{array} \\ &= \ln|x+2| + \int \frac{-(u+1)+4}{u^2 + 3} du = \ln|x+2| + \int \frac{-u+3}{u^2 + 3} du \\ &= \ln|x+2| + \int \left(\frac{-u}{u^2 + 3} + \frac{3}{u^2 + 3} \right) du = \ln|x+2| - \frac{1}{2} \ln(u^2 + 3) + \frac{3}{\sqrt{3}} \tan^{-1} \frac{u}{\sqrt{3}} + C \\ &= \ln|x+2| - \frac{1}{2} \ln(x^2 - 2x + 4) + \frac{3}{\sqrt{3}} \tan^{-1} \frac{x-1}{\sqrt{3}} + C \end{aligned}$$

$$2. \quad \frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$\begin{aligned} 1 &= A(x+a) + B(x-a) \\ &= (A+B)x + (aA - aB) \end{aligned}$$

$$\begin{aligned} A + B &= 0 & A &= \frac{1}{2a} \\ aA - aB &= 1 & \Rightarrow B &= -\frac{1}{2a} \end{aligned}$$

$$\begin{aligned} \int \frac{1}{x^2 - a^2} dx &= \int \left(\frac{\frac{1}{2a}}{x-a} - \frac{\frac{1}{2a}}{x+a} \right) dx = \frac{1}{2a} \ln|x-a| - \frac{1}{2a} \ln|x+a| + C \\ &= \frac{1}{2a} (\ln|x-a| - \ln|x+a|) + C = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C \end{aligned}$$

$$3. \quad \int \frac{1}{e^x + 4} dx$$

$$\begin{aligned} u &= e^x + 4 \Rightarrow e^x = u - 4 \\ du &= e^x dx \Rightarrow dx = \frac{1}{e^x} dx = \frac{1}{u-4} du \end{aligned}$$

$$= \int \frac{1}{u(u-4)} du = \int \left[\frac{A}{u} + \frac{B}{u-4} \right] du = \int \left(\frac{-\frac{1}{4}}{u} + \frac{\frac{1}{4}}{u-4} \right) du = -\frac{1}{4} \ln|u| + \frac{1}{4} \ln|u-4| + C$$

$$= -\frac{1}{4} \ln|e^x + 4| + \frac{1}{4} \ln|(e^x + 4) - 4| + C = -\frac{1}{4} \ln|e^x + 4| + \frac{1}{4} \ln|e^x| + C = \frac{1}{4} \ln \left(\frac{e^x}{e^x + 4} \right) + C$$