A little excursion into perturbation theory of operators. The Kato-Rellich theorem (in finite dimensions).

$A \in \mathcal{L}(X)$, $\sigma(A) = \{\lambda_1, \dots, \lambda_r\}$ (distinct)

Let $\mathcal{C}$ be a simple closed contour in $\mathbb{C}$, $\mathcal{C} \cap \sigma(A) = \phi$.

Set $P := \frac{1}{2\pi i} \int_{\mathcal{C}} (z-A)^{-1} dz$. By the partial fraction decomposition of the resolvent,

$$P = \sum_{j=1}^{r} \frac{1}{2\pi i} \int_{\mathcal{C}} \left\{ \frac{P_j}{z-\lambda_j} + \sum_{k=1}^{\nu_j - 1} \frac{D_j^k}{(z-\lambda_j)^{k+1}} \right\} dz$$

We evaluate the integrals explicitly and obtain

$$P = \sum_{\{j : \lambda_j \text{ inside } \mathcal{C}\}} P_j . \qquad (1)$$

Of course, P_j, D_j are nilpotents assoc. to λ_j.

(1) $\Rightarrow$ $\dim P = \sum_{\{j : \lambda_j \text{ inside } \mathcal{C}\}} \dim P_j$ (since the P_j are disjoint projections). Since $\dim P_j = m_j$ is the algebraic multiplicity of λ_j, we have that $\dim P = \sum_{\{j : \lambda_j \text{ inside } \mathcal{C}\}} m_j$.

In particular, $\dim P = 1 \Rightarrow A$ has a unique eigenvalue inside $\mathcal{C}$, and this eigenvalue is simple.

Let $A, B \in \mathcal{L}(X)$. Formally, we have

$$(A+B-z)^{-1} = \left[(\mathbb{1} + B(A-z)^{-1})(A-z) \right]^{-1}$$
$$= (A-z)^{-1} \sum_{k=0}^{\infty} (-1)^k \left[B(A-z)^{-1} \right]^k .$$

Suppose $z \in \rho(A)$. If $\| B(A-z)^{-1} \| < 1$, then the series converges (absolutely) and it is easily verified that it equals $(A+B-z)^{-1}$. Hence

$$\{ z \in \rho(A) : \| B(A-z)^{-1} \| < 1 \} \subset \rho(A+B) \qquad (2)$$

Suppose $\varkappa \mapsto A(\varkappa)$ is analytic at $\varkappa_0 \in \mathbb{C}$, where $A(\varkappa) \in \mathcal{L}(X)$. Then $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $|\varkappa - \varkappa_0| < \delta$, then

$$\left\| \frac{A(\varkappa) - A(\varkappa_0)}{\varkappa - \varkappa_0} - A'(\varkappa_0) \right\| < \varepsilon,$$

where $A'(\varkappa) = \frac{d}{d\varkappa} A(\varkappa)$. Thus

$$A(\varkappa) = A(\varkappa_0) + (\varkappa - \varkappa_0) A'(\varkappa_0) + T,$$

where $\|T\| < \varepsilon$. Take $z \in \rho(A(\varkappa_0))$. By (2), $z \in \rho(A(\varkappa))$ if $\| \{ (\varkappa - \varkappa_0) A'(\varkappa_0) + T \} (A(\varkappa_0) - z)^{-1} \| < 1$.

Let $\mathcal{C}$ be a simple closed contour in $\mathbb{C}$, $\mathcal{C} \cap \sigma(A(x_0)) = \phi$. Then, since $z \mapsto (A(x_0)-z)^{-1}$ is continuous (it's even analytic!) and $\mathcal{C}$ is compact, we have

$$\sup_{z \in \mathcal{C}} \| (A(x_0)-z)^{-1} \| = q < \infty .$$

It follows that $\forall z \in \mathcal{C}$,

$$\| \{(x-x_0) A'(x_0) + T\} (A(x_0)-z)^{-1} \|$$

$$\leq \| (x-x_0) A'(x_0) + T \| \, q$$

$$\leq \left(|x-x_0| \, \|A'(x_0)\| + \varepsilon \right) q$$

$$< 1,$$

provided $|x-x_0| < \eta$, for some $\eta > 0$. This shows that if $|x-x_0| < \eta$, then $\mathcal{C} \subset \rho(A(x))$. So we can define

$$P(x) := \frac{1}{2\pi i} \int_{\mathcal{C}} (z - A(x))^{-1} \, dz . \qquad (3)$$

The map $x \mapsto P(x)$ is analytic at x_0. To show this, note that (resolvent identity)

$$\frac{P(x)-P(x_0)}{x-x_0} = \frac{1}{2\pi i}\int_{\mathcal{C}} (z-A(x))^{-1}\,\frac{A(x)-A(x_0)}{x-x_0}\,(z-A(x_0))^{-1}\,dz$$

and that $\lim_{x\to x_0} (z-A(x))^{-1}\,\dfrac{A(x)-A(x_0)}{x-x_0}\,(z-A(x_0))^{-1}$

$$= (z-A(x_0))^{-1}\,A'(x_0)\,(z-A(x_0))^{-1}$$

the limit converging uniformly in $z \in \mathcal{C}$.

For $|x-x_0|$ small, we have $\dim P(x) = \dim P(x_0)$. Indeed, if P is a projection on a finite-dimensional space, then $\dim P = \operatorname{Tr} P$ (trace), because P can be represented by a matrix having zeroes everywhere except for an identity block of size $\dim P$. Since $x \mapsto \operatorname{Tr} P(x)$ is continuous at x_0 (even analytic!) this means that $x \mapsto \dim P(x)$ is continuous at x_0. However, $\dim P(x)$ can only take integer values, so for it to be continuous it must be constant.

Combining this with the results on page 1, we see that the number of eigenvalues of $A(x_0)$ and $A(x)$, counting (algebraic) multiplicity, inside $\mathcal{C}$, is the same, provided $|\varkappa - \varkappa_0|$ is small enough.

Suppose that $A(\varkappa_0)$ has a simple eigenvalue a_0. Then $P(\varkappa_0)$, (3), with $\mathcal{C}$ a small circle around a_0, has dimension one. Consequently, $\dim P(\varkappa) = 1$ for $|\varkappa - \varkappa_0| < \varepsilon$, some $\varepsilon > 0$. Thus $A(\varkappa)$ has exactly one eigenvalue, $a(\varkappa)$, inside $\mathcal{C}$, for $|\varkappa - \varkappa_0| < \varepsilon$, and $a(\varkappa_0) = a_0$.

The eigenvalue $a(\varkappa)$ is analytic at $\varkappa_0$. Indeed, take $x \in X$ s.t. $P(\varkappa_0) x \neq 0$. Then for small $\varkappa$ we have $y(\varkappa) := P(\varkappa) x \neq 0$, and $y(\varkappa) \in \operatorname{Ran} P(\varkappa)$. Since $a(\varkappa)$ is a simple eigenvalue of $A(\varkappa)$, we have $A(\varkappa) P(\varkappa) = a(\varkappa) P(\varkappa)$ (the associated nilpotent vanishes). Thus

$$A(\varkappa)\, y(\varkappa) = a(\varkappa)\, y(\varkappa). \tag{4}$$

Take an $f \in X^*$ s.t. $(f, P(\varkappa_0) x) \neq 0$. Then, by continuity, $(f, y(\varkappa)) \neq 0$ for $\varkappa$ close to $\varkappa_0$. So (4) implies

$$a(\varkappa) = \frac{(f, A(\varkappa)\, y(\varkappa))}{(f, y(\varkappa))},$$

which is analytic at x_0 (since $A(x)$, $y(x)$ are).

This shows the following result:

<u>Theorem</u> (Kato-Rellich)
Suppose $A(x) \in \mathcal{L}(X)$ is analytic at x_0 ($\dim X < \infty$). Let a_0 be a simple eigenvalue of $A(x_0)$. Then, for x close to x_0, $A(x)$ has exactly one simple eigenvalue $a(x)$ close to a_0, with an analytic eigenvector $y(x)$.

This is a very simple first result in <u>perturbation theory</u> of linear operators, which deals with the behaviour of the spectrum of an operator when the latter is modified slightly.

<u>Example</u> $A(x) = \begin{bmatrix} 1 & x \\ x^2 & 2 \end{bmatrix}$ $a_0 = 1$, $b_0 = 2$ are the eigenvalues of $A(0)$. What are $a(x)$, $b(x)$?

$$(1-\lambda)(2-\lambda) - x^3 = 0 \qquad \lambda^2 - 3\lambda + 2 - x^3 = 0$$

$$\lambda = \frac{3 + \left[9 - 4(2-x^3)\right]^{1/2}}{2} = \frac{3}{2} + \frac{1}{2}\left[1 + 4x^3\right]^{1/2}$$

For x close to zero $\left(|x|^3 < \frac{1}{4} \right)$ the square root $[1 + 4x^3]^{1/2}$ is analytic and takes the values $\pm\sqrt{1+4x^3}$ (where $\sqrt{}$ is the principal branch with branch cut on the negative real axis). Hence

$$\begin{cases} a(x) = \frac{3}{2} - \frac{1}{2}\sqrt{1+4x^3} \\ b(x) = \frac{3}{2} + \frac{1}{2}\sqrt{1+4x^3} \end{cases}, \quad \text{for } |x| < 2^{-2/3}.$$

<u>Task</u>: calculate eigenvectors and check that they are analytic (i.e., that you can choose them to be analytic).