Exam topics

Spectral representation of $A \in \mathcal{L}(X)$

1. State result (including uniqueness) $(A = S + D)$

2. What is spec rep of $A = \begin{bmatrix} i & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

(From before, $S = i P_i - P_{-1} =$

$$= \begin{bmatrix} i & \frac{i+1}{i+1} & \frac{i+1}{i+1} + \frac{i+1}{(i+1)^2} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} i & 1 & 1 + \frac{1}{i+1} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$D_i = (A - i) P_i = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1-i & 1 \\ 0 & 0 & -1-i \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{i+1} & \frac{1}{i+1} + \frac{1}{(i+1)^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

(no surprise, since $m_i = n_i$)

$$D_{-1} \underset{(A+1)P_{-1}}{=} \begin{bmatrix} i+1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{-1}{i+1} & \frac{-1}{1+i} - \frac{1}{(1+i)^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{1+i} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$(D_{-1}^2 = 0)$

Expansion of resolvent (Laurent, partial fractions)

1. analyticity properties of $z \mapsto (z-A)^{-1}$

2. eigenvalues = poles $\Rightarrow$ $(z-A)^{-1} = \sum_{n=-M}^{\infty} A_n (z-\lambda_j)^n$

(around λ_j) $A_n = \frac{1}{2\pi i} \int_{C_{\lambda_j}} (z-\lambda_j)^{-n-1} (z-A)^{-1} dz$

3. Find formula for A_{-1} $(=P_j)$ & A_{-k}, $k \geq 2$

$$A_{-k} = \frac{1}{2\pi i} \int_C (z-\lambda_j)^{k-1} (z-A)^{-1} dz \underset{\substack{\uparrow \\ \text{funct. calc.}}}{=} (z-A) P_j = D_j$$

$\Rightarrow$ order of pole is ν_j

Singular part (at λ_j) $\frac{P_j}{z-\lambda_j} + \frac{D_j}{(z-\lambda_j)^2} + \cdots + \frac{D_j^{\nu_j - 1}}{(z-\lambda_j)^{\nu_j}}$

4. Ex: $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ what is order of pole $z=1$ of $(z-A)^{-1}$ $\left(A = \underset{S}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} + \underset{D}{\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}} ;\ D^2 = 0 \Rightarrow \underline{\underline{\nu = 2}} \right)$

5. Regular part of Laurent series at λ_j: $\enclose{circle}{n \geq 0}$: $A_n = (-1)^n S_0^{n+1}$

where $S_0 = A_0 = \frac{1}{2\pi i} \int_C (z-\lambda_0)^{-1} R(z) dz$ ("reduced resolvent")

What's S_0 for $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$?

$\rightarrow$ $(z-A)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ $\Rightarrow$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} z-1 & 0 \\ -2 & z-1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

so $f_1 = \frac{1}{z-1}$, $f_2 = \frac{2f_1}{z-1} = \frac{2}{(z-1)^2}$

$\rightarrow$ $A_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2\pi i} \int (z-1)^{-1} \frac{1}{z-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$$\Rightarrow \quad (-1)^n S_0^{n+1} = (-1)^n \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & n = 0 \\ 0 & n \geqslant 1 \end{cases}$$

$$(z-A)^{-1} = (z-1)^0 S_0 + \frac{P}{z-1} + \frac{D}{(z-1)^2}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{z-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{(z-1)^2} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

↑ see 4. ↗

$$= \begin{bmatrix} \frac{1}{z-1} & 0 \\ \frac{2}{(z-1)^2} & \frac{1}{z-1} \end{bmatrix}$$

Test: multiply by $(z-A) = \begin{bmatrix} z-1 & 0 \\ -2 & z-1 \end{bmatrix}$ to get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Inner product spaces, adjoint op's

1. Give an example of an inner product space (& inner product induces norm $\|x\|^2 = \langle x,x\rangle$)

2. Define the adjoint of $A \in \mathcal{L}(X)$, X an IPS

3. What's the spectral rep. of A in terms of that of A^*?

$$\rightarrow \quad A^* = \sum_{j=1}^{r} \bar{\lambda}_j P_j^* + \sum_{j=1}^{r} D_j^*$$

4. Show that spectrum of self-adjoint op. is real

$$\left(Ax = \lambda x \Rightarrow \langle x, Ax\rangle = \lambda \|x\| = \langle Ax, x\rangle = \bar{\lambda} \|x\| \right)$$

5. Show that eigenvectors of $A = A^*$ assoc. to different eigenvalues are orthogonal. ($Ax = \lambda x$, $Ax' = \lambda' x'$:

$$(\lambda - \lambda')\langle x, x'\rangle \underset{\lambda, \lambda' \in \mathbb{R}}{=} \langle Ax, x'\rangle - \langle x, Ax'\rangle = 0 \quad \text{since } A = A^*)$$

6. What's special about spec. rep. of a normal op.?

(A normal op is diagonalizable & its eigenproj. are orthogonal proj.)

$$A = \sum \lambda_j P_j \quad , A^* = \sum \bar{\lambda}_j P_j$$

Let f be analytic on spec(A), what is $f(A)$ for A normal?

$$f(A) = \frac{1}{2\pi i} \int f(z) (z - A)^{-1} dz \quad \leftarrow \quad (z-A)^{-1} = \sum (z - \lambda_j)^{-1} P_j$$

$$= \frac{1}{2\pi i} \sum_j \int f(z) (z - \lambda_j)^{-1} P_j \, dz = \sum_{j=1}^{r} f(\lambda_j) P_j$$

Minimax principle & Its consequences

1. State minimax principle

2. Consequence : if $S \le T$, then $\lambda_k(S) \le \lambda_k(T)$, $\forall k$, where $\lambda_1(S) \le \dots \le \lambda_N(S)$, $\lambda_1(T) \le \dots \le \lambda_N(T)$.

Another Consequence S n-a. on X ($\dim X = N$), M subsp. P ON proj. onto M, $T = PSP \restriction \operatorname{Ran} P$, Then

$$\lambda_k(S) \le \lambda_k(T) \le \lambda_{k+\operatorname{codim} M}(S)$$

3. Example $X = \mathbb{C}^4$, $S = \left[\begin{array}{cc|cc} 0 & i & 0 & \\ -i & 1 & & \\ \hline & & 2 & 1 \\ 0 & & 1 & 3 \end{array}\right]$, $M = \operatorname{span}\{e_2, e_3\}$

(a) find $T = PSP \restriction \operatorname{Ran} P$; basis of M : $\{e_2, e_3\}$

$$Se_2 = \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad Se_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad PSe_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$PSe_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow T = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\lambda_1(T) = 1 < \lambda_2(T) = 2$$

(b) find spec S : $-\lambda(1-\lambda) = 1$, $\lambda^2 - \lambda - 1 = 0$ $\lambda = \dfrac{1 \pm \sqrt{5}}{2}$

or $(2-\lambda)(3-\lambda) = 1$, $\lambda^2 - 5\lambda + 5 = 0$, $\lambda = \dfrac{5 \pm \sqrt{25-20}}{2}$

$\lambda_1(S) = \dfrac{1-\sqrt{5}}{2}$, $\lambda_2(S) = \dfrac{5-\sqrt{5}}{2}$, $\lambda_3(S) = \dfrac{1+\sqrt{5}}{\sqrt{2}}$, $\lambda_4 = \dfrac{5+\sqrt{5}}{2}$

$1+\sqrt{5} \ge 5-\sqrt{5} \Leftrightarrow 4 \le 2\sqrt{5} \Leftrightarrow 2 \le \sqrt{5} \Leftrightarrow 4 \le 5$ ✓

(c) verify result of thm : $\begin{cases} k=1: & \dfrac{1-\sqrt{5}}{2} \le 1 \le \dfrac{2+\sqrt{5}}{2} \checkmark \\ k=2: & \dfrac{5-\sqrt{5}}{2} \le 2 \le \dfrac{5+\sqrt{5}}{2} \checkmark \end{cases}$

Perron – Frobenius theorem

1. State (weak form of) P-F thm: $A \in M_n(\mathbb{R})$ non-negative $(a_{ij} \geq 0) \Rightarrow$ spr(A) is eigenvalue of A with nonnegative eigenvector

2. Suppose A has non-negative off-diagonals, Then e^A has eigenvector $x \geq 0$ w/ eigenvalue spr(e^A).

(Sol^{n}: $\delta = \max |a_{ii}|$: $e^{A+\delta 1} = e^{\delta} e^{A}$ and since $A + \delta 1 \geq 0$ (entrywise)
$\Rightarrow$ $e^{A+\delta 1} \geq 0$ (series expansion) also spec $(e^{A+\delta 1}) = e^{\delta}$ spec $e^A \Rightarrow$ spr $e^{A+\delta 1}$
$= e^{\delta}$ spr(e^A). By P-F $\exists x \geq 0$ s.th. $e^{A+\delta 1}x =$ spr$(e^{A+\delta 1})x$
$\Leftrightarrow$ $e^A x = ($spr $e^A)x$)

3. Spec of $A \geq 0$ (in above sense) always real; No; spr(A) in gen. not eigenvalue, but spr(A) = $\|A\|$ for A normal
(b/c spr(A) = $\lim_{n\to\infty}\sup \|A^n\|^{1/n} = \|A\|$; since $\|A^n\| = \|A\|^n$ for A normal)

4. Example
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$
fits into framework 2.

spec: -1 and $(-4-\lambda)(2-\lambda) - 3 = 0$, $\lambda^2 + 2\lambda - 11 = 0$
$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{48}}{2} = -1 \pm 2\sqrt{3}; \quad \text{spec}(e^A) = \{e^{-1}, e^{-1 \pm 2\sqrt{3}}\}$$
$\Rightarrow$ spr$(e^A) = e^{-1+2\sqrt{3}}$ can we find associated eigenvector $x \geq 0$?

Check eigenvector of A w/ eval $-1 + 2\sqrt{3}$. must come from 2x2 block
$$x = \begin{bmatrix} 1 \\ a \end{bmatrix} \qquad \begin{bmatrix} -4 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = (-1 + 2\sqrt{3})\begin{bmatrix} 1 \\ a \end{bmatrix}$$

$-4 + a = -1 + 2\sqrt{3}$ & $3 + 2a = (-1 + 2\sqrt{3})a$

$a = 3 + 2\sqrt{3}$ & $a = 3$ $a \geq 0$

U-GRADE

Linear forms

1. Def of dual space X^*? give a concrete example!
2. $\dim X = \dim X^*$
3. identification $X = X^{**}$ (exhibit <u>isometric</u> isom.)
4. convergence of vectors $x_n \to x$ in X

   Ex. Does $\begin{bmatrix} \frac{1}{n} & 0 \\ 1 & \frac{\ln(n)}{n} \end{bmatrix}$ converge in $M_2(\mathbb{R})$?

   (test on basis $\underset{= e_{ij}}{f_{ij}}$ of X^*, dual p. = Trace)

5. f analytic, $A \in M_n$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$

   $\Rightarrow f(A) = \sum_{n=0}^{\infty} a_n A^n$ conv. of series?

   If f analytic at 0 with R r. of. cge for Taylor series, then $f(A) = \sum_{n=0}^{\infty} a_n A^n$ well defined for A s.t. $\|A\| < R$.

Functions of operators, Riesz funct. calc.

1. Def of $f(A)$ in Riesz functional calculus setting
$\left(f \text{ piecewise analytic on spec}(A) : f(A) = \frac{1}{2\pi i} \int_{\mathcal{C}} \underset{\uparrow f(z)}{} (z-A)^{-1} dz \right)$

2. What are Riesz projection $(P_j = \chi_j(A))$

3. Let $f(z) = z^2$. Show $f(A) = A^2$
$\left(\frac{1}{2\pi i} \underset{\nwarrow \text{take } \mathcal{C} \text{ large s.t. spec}(A) \subset \text{int}(\mathcal{C})}{\int_{\mathcal{C}}} z^2 (z-A)^{-1} dz \overset{\text{unif conv}}{=} \sum_{n \geq 0} \frac{1}{2\pi i} \int_{\mathcal{C}} z^2 \left(\frac{A}{z}\right)^n \frac{1}{z} dz \right.$

$\left. = \sum_n A^n \frac{1}{2\pi i} \int_{\mathcal{C}} \underbrace{z^{1-n}}_{\text{nonzero only if } 1-n=-1} dz = A^2 \right)$

4. State spectral mapping theorem. Let $A = \begin{bmatrix} i & 1 & 1 \\ 0 & -1 & 1 \\ & & -1 \end{bmatrix}$
Let $f(z) = \begin{cases} 1, & |z+1| < 1/2 \\ 0, & \text{else} \end{cases}$
What is spec$(f(A))$? Answ: $= f(\text{spec}(A)) = \{1\}$

5. What is $f(A)$ in above $q^{\underline{n}}$ (it is P_{-1})
calc. $f(A)$: $\lambda = -1$ geom. mult 1, evect $\begin{bmatrix} \frac{-1}{i+1} \\ 1 \\ 0 \end{bmatrix}$ $n_{-1} = 1$, $m_{-1} = 2$
(also $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $n_i = 1 = m_i$)

$P_{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2\pi i} \int_{\mathcal{C}_i} (z-i)^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} dz = 0$

$(z-A)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} z-i & -1 & -1 \\ 0 & z+1 & -1 \\ 0 & 0 & z+1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$

$$(z-A)^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix} \Rightarrow \quad f_3 = \frac{1}{z+1}, \qquad f_2 = \frac{1}{(z+1)^2}$$

$$0=(z-i)f_1 - f_2 - f_3 \Rightarrow f_1 = \frac{1}{z-i}\,\frac{1}{z+1}\left[1+\frac{1}{z+1}\right]$$

$$\Rightarrow \quad P_{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{-1-i} + (-1)(z-i)^{-2}\big|_{z=-1}\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{1}{1+i} - \frac{1}{(1+i)^2}\\0\\1\end{bmatrix} \Rightarrow P_{-1} = \begin{bmatrix}0 & \frac{-1}{i+1} & -\frac{1}{1+i} - \frac{1}{(1+i)^2}\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$$P_i\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad P_i\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{i+1}\\0\\0\end{bmatrix}, \quad P_i\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{1+i}\left[1+\frac{1}{i+1}\right]\\0\\0\end{bmatrix}$$

$$P_i = \begin{bmatrix}1 & \frac{1}{i+1} & \frac{1}{i+1}+\frac{1}{(i+1)^2}\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

(check: $P_i + P_{-1} = 1$ ✓
$P_i \cdot P_{-1} = 0$ ✓)