

M6310  
Functional Analysis Graduate Course  
Problem List  
Problems 25-28, due on December 1st (in class)

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25. Let  $\mathcal{H}$  be a Hilbert space.
- (a) Let  $M$  be a subspace of  $\mathcal{H}$ . Show that  $(M^\perp)^\perp$  is the closure of  $M$ .
  - (b) Let  $A$  be a subset of  $\mathcal{H}$ . Show that  $(A^\perp)^\perp$  is the smallest closed subspace of  $\mathcal{H}$  which contains  $A$ .
26. Let  $\mathcal{A}_\alpha$ ,  $\alpha \in I$  (index set) be a collection of  $\sigma$ -algebras on a set  $\Omega$ . Show that  $\mathcal{A} := \bigcap_{\alpha \in I} \mathcal{A}_\alpha$  is a  $\sigma$ -algebra on  $\Omega$ .
27. **Bounded Extension Theorem.** Let  $X$  be a normed vector space, and  $\mathcal{B}$  a complete normed vector space. Let  $\mathcal{D}$  be a dense subset of  $X$ . Suppose  $\ell : \mathcal{D} \rightarrow \mathcal{B}$  is a bounded linear functional on  $\mathcal{D}$ , i.e., satisfying

$$C := \sup \{ \|\ell(x)\|_{\mathcal{B}} : x \in \mathcal{D}, \|x\|_X = 1 \} < \infty.$$

Show that there exists a unique bounded extension  $\bar{\ell}$  of  $\ell$ , defined on all of  $X$ , meaning that there is a unique bounded  $\bar{\ell} : X \rightarrow \mathcal{B}$  s.t.  $\bar{\ell}(x) = \ell(x)$  for all  $x \in \mathcal{D}$ . Show that  $\|\bar{\ell}\| = C$ .

[Hint: If define  $\bar{\ell}(x) = \lim_n \ell(x_n)$ , where  $x_n$  is a sequence in  $\mathcal{D}$  converging to  $x$ .]

28. **Adjoint of densely defined operator.** Let  $\mathcal{H}$  be a Hilbert space and  $\mathcal{D}$  a dense set. Suppose  $A : \mathcal{D} \rightarrow \mathcal{H}$  is a linear map (defined on  $\mathcal{D}$ ).
- (a) Define the subset  $\mathcal{D}^*$  of  $\mathcal{H}$  by

$$\mathcal{D}^* = \{y \in \mathcal{H} : x \mapsto \langle y, Ax \rangle \text{ is bounded on } \mathcal{D}\}$$

(bounded meaning  $\sup_{0 \neq x \in \mathcal{D}} |\langle y, Ax \rangle| / \|x\| < \infty$ ).

Show that for each  $y \in \mathcal{D}^*$ ,  $x \mapsto \langle y, Ax \rangle$  has a unique bounded extension  $\ell_y \in \mathcal{H}^*$ .

- (b) Show that  $y \mapsto \ell_y$  is an anti-linear map from  $\mathcal{D}^*$  to  $\mathcal{H}$ . This shows that we can define a linear operator  $A^* : \mathcal{D}^* \rightarrow \mathcal{H}$  satisfying  $\langle y, Ax \rangle = \langle A^*y, x \rangle$  for all  $x \in \mathcal{D}$  and all  $y \in \mathcal{D}^*$ .

25 (1) follows from (2). Denote by  $[A]$  the smallest closed subspace containing  $A$ .

Take  $a \in A^\perp$  ( $A^\perp$  contains at least 0). We have  $A \subset \{a\}^\perp$  (since  $X \subset Y^\perp \Leftrightarrow X^\perp \supseteq Y$  which is easily seen)

Now  $\{a\}^\perp$  is a closed subspace (easy to see) which contains  $A$ , so  $[A] \subset \{a\}^\perp$ . Hence again (as above)

$$[A]^\perp \supseteq \{a\}.$$

Since  $a \in A^\perp$  was arbitrary, we get

$$A^\perp \subset [A]^\perp.$$

On the other hand,  $A \subset [A]$ , so  $A^\perp \supseteq [A]^\perp$  ( $X \subset Y \Rightarrow X^\perp \supseteq Y^\perp$  as is easily seen). Thus

$$[A]^\perp = A^\perp.$$

i.e., the orthogonal complement of  $A$  and  $[A]$  is the same. We get  $(A^\perp)^\perp = ([A]^\perp)^\perp$  and it remains to show that  $([A]^\perp)^\perp = [A]$ . Let  $M = [A]$ .  $M$  is a closed subspace, so

$$\mathbb{H} = M \oplus M^\perp.$$

Take  $x \in (M^\perp)^\perp$ , we show  $x \in M$ . (The inclusion  $(M^\perp)^\perp \supseteq M$  is trivial). There are unique  $y \in M, z \in M^\perp$  s.t.  $x = y + z$ . Since  $x \in (M^\perp)^\perp$  and  $y \in M \subset (M^\perp)^\perp$  we have  $z = x - y \in (M^\perp)^\perp$ . But also,  $z \in M^\perp$ , so  $z \in (M^\perp)^\perp \cap M^\perp = \{0\}$ . Thus  $x = y$  and hence  $x \in M$ . □

26 Clearly,  $\mathcal{A} \subset \mathcal{P}(\Omega)$ . Let  $A \in \mathcal{A}$ , then  $A \in \mathcal{A}_\alpha \forall \alpha \in I$ , so  $A^c = \Omega \setminus A \in \mathcal{A}_\alpha \forall \alpha \in I$  (since each  $\mathcal{A}_\alpha$  is closed under complements)  $\Rightarrow A \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$ .

Next, let  $\{A_j\}_{j=1}^\infty$  be a sequence in  $\mathcal{A}$ . Then

$A_j \in \mathcal{A}_\alpha \forall j, \forall \alpha$ , so  $\bigcup_{j=1}^\infty A_j \in \mathcal{A}_\alpha \forall \alpha$

Hence  $\bigcup_{j=1}^\infty A_j \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$  □

27 Let  $x \in X$ . Since  $\mathcal{D}$  is dense, there is a sequence  $x_n \in \mathcal{D}$ , s.t.  $x_n \rightarrow x$  (in norm topology of  $X$ ).

The sequence  $l(x_n)$  in  $\mathcal{B}$  is Cauchy, since

$$\|l(x_m) - l(x_n)\|_{\mathcal{B}} = \|l(x_m - x_n)\|_{\mathcal{B}} \leq C \|x_m - x_n\|_X \xrightarrow{m, n \rightarrow \infty} 0$$

Thus  $l(x_n)$  converges in  $\mathcal{B}$ , as  $n \rightarrow \infty$ . Next suppose that  $x_n \in \mathcal{D}$ ,  $y_n \in \mathcal{D}$  both converge to  $x \in X$ . Then

$$\|l(x_n) - l(y_n)\|_{\mathcal{B}} \leq C \|x_n - y_n\|_X \leq C (\|x_n - x\|_X + \|y_n - x\|_X) \xrightarrow{m, n \rightarrow \infty} 0$$

Hence  $\bar{l}(x) := \lim_{n \rightarrow \infty} l(x_n)$  is indep. of approximating sequence. It's easy to see that  $x \mapsto \bar{l}(x)$  is linear. Moreover, if  $x \in \mathcal{D}$ , then the constant sequence  $x_n \equiv x$  converges to  $x$  and so  $\bar{l}(x) = l(x)$ . Finally,

$$\|\bar{l}(x)\|_{\mathcal{B}} = \lim_{n \rightarrow \infty} \|l(x_n)\|_{\mathcal{B}} \leq C \lim_{n \rightarrow \infty} \|x_n\|_X = C \|x\|_X$$

so  $\|\bar{l}\| \leq C$ . That  $\|\bar{l}\| \geq C$  is clear since  $\bar{l}$  is an extension of  $l$  and from the definition of  $C$ .  $\square$

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(a) Fix  $y \in \mathcal{D}^*$ . Then  $x \mapsto \langle y, Ax \rangle$  has a unique bounded extension, accordingly to previous exercise, and this extension  $l_y: \mathcal{H} \rightarrow \mathbb{C}$  satisfies

$$\|l_y\| = \sup_{x \in \mathcal{D}, \|x\|=1} \frac{|\langle y, Ax \rangle|}{\|x\|}$$

(b) Note that  $\mathcal{D}^*$  is a subspace. Suppose  $y_1, y_2 \in \mathcal{D}^*$ . Then  $\alpha_1 y_1 + \alpha_2 y_2 \in \mathcal{D}^*$ , and

$$l_{\alpha_1 y_1 + \alpha_2 y_2}(x) = \lim_{n \rightarrow \infty} \langle \alpha_1 y_1 + \alpha_2 y_2, Ax_n \rangle,$$

where  $x_n \in \mathcal{D}$ ,  $x_n \rightarrow x$  in  $\mathcal{H}$ .

$$= \alpha_1 l_{y_1}(x) + \alpha_2 l_{y_2}(x)$$

Hence  $\langle A^*(\alpha_1 y_1 + \alpha_2 y_2), x \rangle$

$$= \langle \alpha_1 A^* y_1 + \alpha_2 A^* y_2, x \rangle \quad \forall x \in \mathcal{D}$$

Since  $\mathcal{D}$  is dense  $\Rightarrow A^*(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 A^* y_1 + \alpha_2 A^* y_2$

□