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A Universality Property of Gaussian Analytic Functions

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Abstract We consider random analytic functions defined on the unit disk of the complex plane $f(z)=\sum_{n=0}^{\infty} a_n X_n z^n$, where the X_n's are i.i.d., complex-valued random variables with mean zero and unit variance. The coefficients a_n are chosen so that $f(z)$ is defined on a domain of $\mathbb{C}$ carrying a planar or hyperbolic geometry, and $\mathbf{E} f(z)\overline{f(w)}$ is covariant with respect to the isometry group. The corresponding Gaussian analytic functions have been much studied, and their zero sets have been considered in detail in a monograph by Hough, Krishnapur, Peres, and Virág. We show that for non-Gaussian coefficients, the zero set converges in distribution to that of the Gaussian analytic functions as one transports isometrically to the boundary of the domain. The proof is elementary and general.

Keywords Random analytic functions · Gaussian analytic functions

Mathematics Subject Classification (2000) 30B20 · 60B12 · 60G15

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1 Main Result

Random analytic functions are a topic of classical interest [2, 11], which gained renewed interest as a toy model for quantum chaos following work of Bogomolny, Bohigas and Leboeuf [6, 7]. A recent short review of Gaussian analytic functions is entitled, "What is ... a Gaussian entire function," [13]. A monograph devoted to the study of their zeros has appeared several years ago by Hough, Krishanpur, Peres and Virág [9].

Given a parameter $\kappa \le 0$, and a sequence of coefficients $\boldsymbol{x} = (x_0, x_1, x_2, \dots)$, one may define the power series

$$f_\kappa(\boldsymbol{x}, z) = \sum_{n=0}^{\infty} a_{\kappa,n} x_n z^n,$$

where

$$a_{\kappa,n} = \prod_{j=1}^{n} \left[\frac{1-(j-1)\kappa}{j}\right]^{1/2}.$$

We consider random analytic functions (RAF's) defined by choosing a coefficient sequence $\boldsymbol{X} = (X_0, X_1, \dots)$ where $X_0, X_1, \dots$ are i.i.d., complex-valued random variables, with mean zero and unit variance, such that

$$\mathbf{E}\big[(\mathrm{Re}[X_i])^2\big] = \mathbf{E}\big[(\mathrm{Im}[X_i])^2\big] = 1, \qquad \mathbf{E}\mathrm{Re}[X_i]\mathrm{Im}[X_i] = 0. \tag{1.1}$$

A number of important properties hold for these models, which we describe now. An excellent reference, with complete proofs is [9].

We write $\mathbb{U}(z, r)$ for the open disk $\{w \in \mathbb{C} : |w - z| < r\}$. The natural domain of convergence for $f_\kappa(\boldsymbol{X}, z)$ is $\mathbb{U}(0, \rho_\kappa)$, a.s., where $\rho_\kappa = |\kappa|^{-1/2}$. By (1.1), $\mathbf{E}[f_\kappa(\boldsymbol{X}, z) f_\kappa(\boldsymbol{X}, w)] = 0$, and

$$\mathbf{E}\big[f_\kappa(\boldsymbol{X}, z)\overline{f_\kappa(\boldsymbol{X}, w)}\big] = Q_\kappa(z, w) := \begin{cases} (1+\kappa z\overline{w})^{1/\kappa} & \text{for } \kappa \neq 0, \\ e^{z\overline{w}} & \text{for } \kappa = 0. \end{cases} \tag{1.2}$$

The function Q_κ possesses important symmetries. For $|u| < \rho_\kappa$, consider the Möbius transformation

$$\Phi_\kappa^u(z) = \frac{z-u}{1+\kappa\overline{u}z},$$

which is a univalent mapping of $\mathbb{U}(0, \rho_\kappa)$ to itself. This is an isometry relative to a metric with Gauss curvature 4κ. Moreover,

$$Q_\kappa\big(\Phi_\kappa^u(z), \Phi_\kappa^u(w)\big) = \Delta_\kappa^u(z)\overline{\Delta_\kappa^u(w)}Q_\kappa(z, w),$$

where

$$\Delta_\kappa^u(z) = \begin{cases} (1+\kappa|u|^2)^{1/(2\kappa)}(1+\kappa\overline{u}z)^{-1/\kappa} & \text{for } \kappa \neq 0, \\ \exp(\frac{1}{2}|u|^2 - \overline{u}z) & \text{for } \kappa = 0. \end{cases}$$

While Q_κ is not *invariant* with respect to the isometries Φ_κ^u, one says it is *covariant* because of this property. Also note that for any $z \in \mathbb{U}(0, \rho_\kappa)$,

$$|\Phi_\kappa^u(z)| \to \rho_\kappa \quad \text{as } |u| \to \rho_\kappa.$$

Taking u to the boundary of the domain $\mathbb{U}(0, \rho_\kappa)$, conformally maps neighborhoods of 0 to domains approaching the boundary.

Gaussian analytic functions (GAF's) are important special cases of RAF's. Their zero sets have been studied in [9], and many interesting questions about these zero sets continue to be studied. Some of the ensembles had been introduced before in [6, 7, 12, 16]. An indication of these point processes is given by the case $\kappa = -1$. A seminal observation of Peres and Virág completely classified the distribution [14]:

Theorem 1.1 (Peres and Virág 2005) *Suppose that $Z_0, Z_1, \dots$ are i.i.d., and each Z_i has density on $\mathbb{C}$ given by $\pi^{-1}\exp(-|z|^2)$. Then the zero set of $f_{-1}(\boldsymbol{Z}, z)$ is a determinantal point process on $\mathbb{U}(0, 1)$ with kernel given by the Bergman kernel $K(z, w) = \pi(1 - z\overline{w})^{-2}$.*

Because of results such as these, it is useful to compare the zero sets for non-Gaussian RAF's to those of Gaussian RAF's. The expected number of zeros for non-Gaussian RAF's has been studied extensively in the literature. See for example [8, 10, 15]. But the correlations of the zeros of non-Gaussian RAF's has been considered less frequently. That is what we consider. Our main result proves convergence in distribution of the zero sets of the RAF's, for a sequence of neighborhoods converging to the boundary.

Theorem 1.2 (Main result) *Suppose that $X_0, X_1, \dots$ are i.i.d., complex-valued random variables with mean zero and satisfying (1.1). Let $\boldsymbol{Z}$ be i.i.d., complex Gaussians as in Theorem 1.1. For each $\kappa \leq 0$, and any continuous function φ whose support is a compact subset of $\mathbb{U}(0, \rho_\kappa)$, the random variables*

$$\sum_{\xi : f_\kappa(\boldsymbol{X}, \xi) = 0} \varphi\big(\Phi_\kappa^u(\xi)\big)$$

converge in distribution, in the limit $|u| \to \rho_\kappa$, to the random variable

$$\sum_{\xi : f_\kappa(\boldsymbol{Z}, \xi) = 0} \varphi(\xi).$$

Since Q_κ is covariant with respect to the mappings Φ_κ^u, and since the distribution of a Gaussian process is determined by its covariance, the zeros of the GAF, $\{\xi : f_\kappa(\boldsymbol{Z}, \xi) = 0\}$ is a stationary point process with respect to these mappings.

Remark 1.3 The mapping Φ_κ^u was defined so that $\Phi_\kappa^u(u) = 0$. Therefore, mapping the zeros by Φ_κ^u maps the zeros in a neighborhood of u to a neighborhood of 0. We have an illustration in Fig. 1 to indicate this. The test function φ is nonzero only in a window around 0.

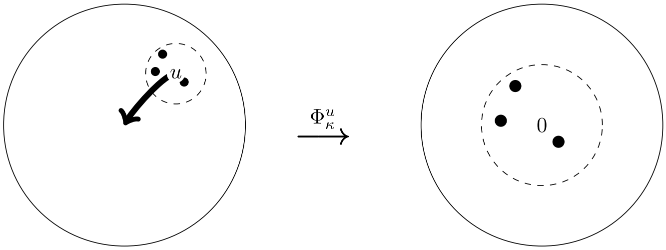

Fig. 1 To study the zeros in a neighborhood of a point u near the boundary of $\mathbb{U}(0, \rho_\kappa)$, take the image under the map Φ_κ^u, which maps u to 0, and consider the positions of the zeros under this mapping. This is described in Remark 1.3

As mentioned before, there have been many papers proving convergence of the first intensity measure of zeros, which is related to the expected number of zeros. There are very precise and general results for that. Our main result, Theorem 1.2, addresses a complementary issue because it allows one to deduce convergence for the correlations between zeros. It shows that for a general class of non-Gaussian RAF's, if you take the correct limit, the zero set converges back to that of the corresponding GAF. In this sense, Theorem 1.2 is one result of the type showing that GAF's have a universal property, much as the central limit theorem may be interpreted as a universality result for Gaussian random variables.

There is another important group of papers proving universality for GAF's. These are by Bleher, Di, Shiffman and Zelditch [3–5]. They prove convergence of the entire set of correlations for a class of GAF's. Their results are more precise and detailed than ours. But they are in a different context.

In Sect. 2 we give the simple proof of Theorem 1.2. In Sect. 3, we consider an elementary application for a discrete family of RAF's which have been considered recently.

2 Proof of the Main Result

The main result is a simple application of the CLT, especially using the Lindeberg–Feller condition. A main step in proving Theorem 1.2 is the following elementary observation.

Lemma 2.1 *Let* $\boldsymbol{Z}$ *be i.i.d., complex Gaussians as in Theorem* 1.1. *Then for any* $N \in \mathbb{N}$, *any* $z_1, \ldots, z_N \in \mathbb{U}(0, \rho_\kappa)$ *and any* $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, *the random variables*

$$\sum_{k=1}^{N} \lambda_k \frac{f_\kappa(\boldsymbol{X}, \Phi_\kappa^u(z_k))}{\Delta_\kappa^u(z_k)}$$

converge in distribution, in the limit $|u| \to \rho_\kappa$, *to the random variable* $\sum_{k=1}^{N} \lambda_k f_\kappa(\boldsymbol{Z}, z_k)$.

Proof We may write

$$\sum_{k=1}^{N} \lambda_k \frac{f_\kappa(\boldsymbol{X}, \Phi_\kappa^u(z_k))}{\Delta_\kappa^u(z_k)} = \sum_{k=1}^{N} \frac{\lambda_k}{\Delta_\kappa^u(z_k)} \sum_{n=0}^{\infty} a_{n,\kappa} X_n \left(\Phi_\kappa^u(z_k)\right)^n = \sum_{n=0}^{\infty} \alpha_{n,\kappa}(u) X_n,$$

where we have left the dependence of $\alpha_{n,\kappa}(u)$ on $\lambda_1, \dots, \lambda_n$ and $z_1, \dots, z_n$ implicit for the coefficients

$$\alpha_{n,\kappa}(u) = a_{n,\kappa} \sum_{k=1}^{N} \frac{\lambda_k (\Phi_\kappa^u(z_k))^n}{\Delta_\kappa^u(z_k)}.$$

Since Q_κ is *covariant* with respect to the transformations Φ_κ^u, this implies that

$$\sum_{n=0}^{\infty} |\alpha_{n,\kappa}(u)|^2 = \sum_{j,k=1}^{N} \lambda_j Q_\kappa(z_j, z_k) \overline{\lambda}_k.$$

For the same reason the variance of the random variables in question is a constant function of u.

To apply the Lindeberg–Feller conditions, we need to check that the sequence $\alpha_{n,\kappa}(u)$ converges to 0 in ℓ^∞. It suffices to check this in ℓ^p for any $p > 2$. We note that for $p = 4$

$$\sum_{n=0}^{\infty} |\alpha_n(u)|^4 \le N^4 \max_{k=1,\dots,N} |\lambda_k|^4 \sum_{n=0}^{\infty} a_{n,\kappa}^4 \frac{|\Phi_\kappa^u(z_k)|^{4n}}{|\Delta_\kappa^u(z_k)|^4}.$$

Cauchy's integral formula implies

$$\sum_{n=0}^{\infty} a_{n,\kappa}^4 \frac{|\Phi_\kappa^u(z)|^{4n}}{|\Delta_\kappa^u(z)|^4} = \frac{1}{2\pi i} \oint_{C(0,1)} \left| \sum_{n=0}^{\infty} a_{n,\kappa}^2 \frac{|\Phi_\kappa^u(z)|^{2n}}{|\Delta_\kappa^u(z)|^2} \zeta^n \right|^2 d\zeta,$$

for each fixed $z \in \mathbb{U}(0, \rho_\kappa)$. For $\kappa < 0$ the series sums to

$$\sum_{n=0}^{\infty} a_{n,\kappa}^2 \frac{|\Phi_\kappa^u(z)|^{2n}}{|\Delta_\kappa^u(z)|^2} \zeta^n = \left(1 + \kappa |z|^2\right)^{1/\kappa} \left(\frac{1 + \kappa \left|\Phi_\kappa^u(z)\right|^2 \zeta}{1 + \kappa |\Phi_\kappa^u(z)|^2} \right)^{1/\kappa}.$$

The second factor on the right hand side has norm bounded by 1 for all $\zeta \in C(0, 1)$. Moreover since $|\Phi_\kappa^u(z)|$ converges to $\rho_\kappa = |\kappa|^{-1/2}$ in the limit $|u| \to \rho_\kappa$, the second factor converges pointwise to 0 in that limit, for every $\zeta \in C(0, 1) \setminus \{1\}$. For $\kappa = 0$ the series sums to

$$e^{|z|^2} \exp\left((\zeta - 1)|\Phi_\kappa^u(z)|^2\right).$$

But for $\kappa = 0$, we know $\rho_0 = \infty$ and $|\Phi_0^u(z)| \to \infty$ in the limit $|u| \to \infty$. Since the real part of $(\zeta - 1)$ is non-positive, the same conclusion follows. In either case, the dominated convergence gives the desired result. □

Lemma 2.1 implies that the random analytic functions $[\Delta_\kappa^u(z)]^{-1} f_\kappa(\boldsymbol{X}, \Phi_\kappa^u(z))$ converge in distribution to the random analytic function $f_\kappa(\boldsymbol{Z}, z)$, in the limit $|u| \to$

ρ_κ, in the sense that the finite dimensional marginals of the function values converge. This also implies convergence in distribution of the zero sets. A clear and elegant proof of this fact has been provided by Valko and Virág in a recent paper they wrote on random Schrödinger operators [17].

Lemma 2.2 (Valko and Virág 2010) *Let* $f_n(\omega, z)$ *be a sequence of random analytic functions on a domain* D (*which is open, connected and simply connected*) *such that* $\mathbf{E}h(|f_n(z)|) < g(z)$ *for some increasing unbounded function* h *and a locally bounded function* g. *Assume that* $f_n(z) \Rightarrow f(z)$ *in the sense of finite dimensional distributions. Then* f *has a unique analytic version and* $f_n \Rightarrow f$ *in distribution with respect to local-uniform convergence.*

Because of this result we see that $[\Delta_\kappa^u(z)]^{-1} f_\kappa(\boldsymbol{X}, \Phi_\kappa^u(z))$ converges in distribution to $f_\kappa(\boldsymbol{Z}, z)$, with respect to the local-uniform convergence. But by standard results in complex analysis, such as Hurwitz's theorem and Rouché's theorem, this implies that the zero sets also converge in distribution, relative to the local, weak topology. For a sequence of locally finite measures μ_n on a locally compact, Hausdorff space, μ_n converges to μ locally, weakly if and only if, for each continuous function φ with compact support, the integrals $\int \varphi \, d\mu_n$ converge to $\int \varphi \, d\mu$. This explains the statement of our main result. Note that $\Delta_\kappa^u(z)$ is finite and non-vanishing for $z \in \mathbb{U}(0, \rho_\kappa)$. Therefore the zero set is just the zero set of $f_\kappa(\boldsymbol{X}, \Phi_\kappa^u(z))$.

3 An Application

The simplest non-Gaussian RAF is given by

$$f_{-1}(\boldsymbol{X}, z) = \sum_{n=0}^{\infty} X_n z^n,$$

where $X_0, X_1, \ldots$ are i.i.d., random variable chosen from the set $\{1, -1, i, -i\}$ with equal probabilities. Our theorem implies that the zero set of such a RAF near the unit circle is has a distribution which is close to that of the corresponding GAF. Due to Peres and Virág's result, Theorem 1.1, this means that the zero set is asymptotically determinantal.

On his blog, John Baez has reproduced some numerical plots by Sam Derbyshire of the zeros of the polynomial

$$p_n((x_0, \ldots, x_n), z) = \sum_{k=0}^{n} x_k z^k,$$

but where the coefficients $x_0, \ldots, x_n \in \{1, -1\}$ [1]. Derbyshire plotted the union of the zero sets

$$\bigcup_{y_0, \ldots, y_n \in \{1, -1\}} \{\xi : p_n((y_0, \ldots, y_n), \xi) = 0\}.$$

One can consider this from another perspective. For a fixed value of z, we may define the set of function values for all possible coefficients

$$C_n(z) = \{p_n((x_0, \ldots, x_n), z) : x_0, \ldots, x_n \in \{1, -1\}\}.$$

This satisfies the recurrence relation

$$C_n(z) = \{1 + zw : w \in C_{n-1}(z)\} \cup \{-1 + zw : w \in C_{n-1}(z)\}.$$

Defining the set

$$C(z) = \{f_{-1}(\boldsymbol{y}, z) : y_0, y_1, \ldots \in \{1, -1\}\},$$

one can see that $C(z) = \{1 + zw : w \in C(z)\} \cup \{-1 + zw : w \in C(z)\}$. This implies the Hausdorff dimension satisfies the bound

$$\dim C(z) \le \frac{\log(2)}{\log(1/|z|)}.$$

For $r \in \mathbb{R}$, we know that $C(r) \subseteq \mathbb{R}$. Therefore, it seems reasonable to conjecture that

$$\dim C(z) = \begin{cases} \min\{2, \log(2)/\log(1/|z|)\} & \text{for } z \in \mathbb{C} \setminus \mathbb{R}, \\ \min\{1, \log(2)/\log(1/|z|)\} & \text{for } z \in \mathbb{R}. \end{cases}$$

It is easy to see that $C(1/3)$ is the middle-thirds Cantor set; whereas, $C(i/\sqrt{2})$ is the rectangle $\{x + iy : x \in [-2, 2], y \in [-\sqrt{2}, \sqrt{2}]\}$. In Fig. 2(a) we have displayed $C_{15}(z)$ for $z = e^{i\pi/4}/\sqrt{2}$.

The fractal dimension of $C(z)$ may pertain to Baez's conjectures that the zero sets $\{z : 0 \in C_n(z)\}$ have a fractal structure associated to z. In analogy to the dimension of varieties of smooth curves, one might guess that "typically" if z satisfies $C(z) \ni 0$ then the set of w's in a neighborhood of z with $C(w) \ni 0$ has dimension "approximately" equal to $\dim C(z)$.

Similarly, defining

$$B_n(z) = \{p_n((x_1, \ldots, x_n), z) : x_0, x_1, \ldots \in \{1, -1, i, -i\}\},$$

we have $B_n(z) = \bigcup_{u \in \{1,-1,i,-i\}} \{u + zw : w \in B_{n-1}(z)\}$. Defining $B(z) = \{f_{-1}(\boldsymbol{x}, z) : x_0, x_1, \ldots \in \{1, -1, i, -i\}\}$, this implies that $\dim B(z) \le 2\log(2)/\log(1/|z|)$. It seems reasonable to guess that $\dim B(z) = \min\{2, 2\log(2)/\log(1/|z|)\}$. An easy calculation shows $B(1/2) = \{x + iy : x, y \in [-2, 2]\}$. In Fig. 2(b) we have plotted $B_8(z)$ for $z = e^{i\pi/8}/2$.

Similar to Derbyshire's plots, one can consider $\{z : B(z) \ni 0\}$. In addition to the question about what this set equals, one can also ask, for z such that $B(z) \ni 0$, about the structure of the coefficients $\{(x_0, x_1, \ldots) : x_0, x_1, \ldots \in \{1, -1, i, -i\}\}$. One guess is that as $|z| \to 1$, the distribution converges in some sense to "uniform" with the density appropriate for the intensity measure of the zeros at that point. An interesting possibility is to try to study this question by considering instead the correlations of zeros of $f_{-1}(\boldsymbol{X}, z)$. For instance, repulsion of zeros of $f(\boldsymbol{X}, z)$ holds because the zero set is asymptotically determinantal. It is tempting to believe that repulsion of the

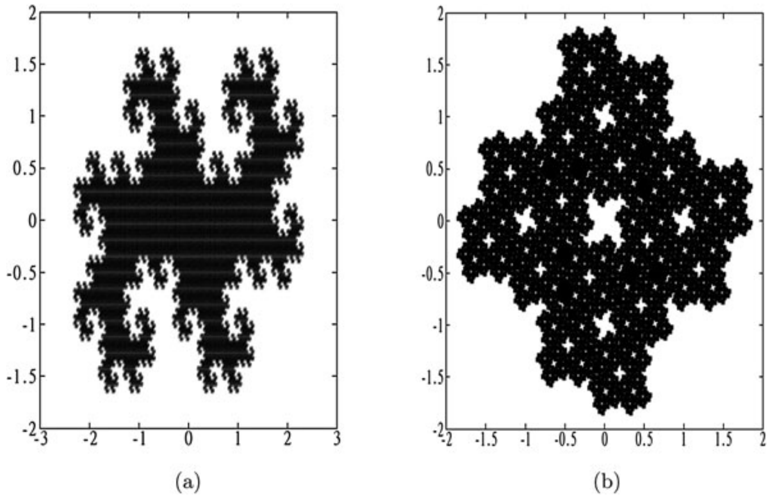

Fig. 2 (**a**) The set $C_{15}(z)$ for $z = e^{i\pi/4}/\sqrt{2}$. (**b**) The set $B_8(z)$ for $z = e^{i\pi/8}\sqrt{2}$

zeros might be used to prove that the set $\{\boldsymbol{x} : f(\boldsymbol{x}, z) = 0\}$ is "spread out" in some sense. However, this remains a challenge for the future.

We can answer a simpler question. Baez notes several "holes" in the sets $C_n(z)$ centered at points on the unit circle, such as 1. In plots, one can also see holes in the sets $B_n(z)$, as well. But the holes for $B_n(z)$ must all close as $n \to \infty$. The intensity measure of the zeros of the GAF $f_{-1}(\boldsymbol{Z}, z)$ has full support on $\mathbb{U}(0, 1)$. The intensity measure of $f_{-1}(\boldsymbol{X}, z)$ is asymptotically close to that of $f_{-1}(\boldsymbol{Z}, z)$, in the weak topology, near the unit circle. This is the simplest consequence of Theorem 1.2. There cannot be any holes in $B(z)$ centered at the unit circle.

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