Math 2000, Winter 2011—Assignment 10—Solution

Polar coordinates and triple integrals

1. Calculate the following double integrals using polar coordinates.

(a) $\int\int_D xy\,dA$ where D is the disk with center the origin and radius 3;
Solution:

$$\int\int_D xy\,dA = \int_0^{2\pi}\int_0^3 (r\cos\theta)(r\sin\theta)r dr d\theta = \Big(\int_0^{2\pi}\sin\theta\,\cos\theta\,d\theta\Big)\int_0^3 r^3 dr = 0.$$

(b) $\int\int_R (x+y)dA$ where R is the region that lies to the left of the y-axis between the circles $x^2+y^2=1$ and $x^2+y^2=4$;
Solution:

$$\int\int_R (x+y)dA = \int_{\pi/2}^{3\pi/2}\int_1^2 (r\cos\theta + r\sin\theta) r dr d\theta = \Big(\int_{\pi/2}^{3\pi/2}(\sin\theta+\cos\theta)\,d\theta\Big)\int_1^2 r^2 dr = -\frac{14}{3}.$$

(c) $\int\int_R \cos(x^2+y^2)\,dA$ where R is the region that lies above the x-axis within the circle $x^2+y^2=9$;
Solution:

$$\int\int_R \cos(x^2+y^2)\,dA = \int_0^{\pi}\int_0^3 \cos(r^2)\,r dr d\theta = (\int_0^{\pi} d\theta)(\int_0^3 r\cos(r^2)\,dr) = \frac{\pi}{2}\sin 9.$$

(d) $\int\int_D e^{-x^2-y^2}\,dA$ where D is the region bounded by the semicircle $x=\sqrt{4-y^2}$;
Solution:

$$\int\int_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2} r dr d\theta = \Big(\int_{-\pi/2}^{\pi/2} d\theta\Big)\int_0^2 e^{-r^2} r dr = \frac{\pi}{2}(1-e^{-4}).$$

(e) $\int\int_D ye^x\,dA$ where D is the region in the first quadrant enclosed by the circle $x^2+y^2=25$;
Solution:

$$\int\int_D ye^x\,dA = \int_0^{\pi/2}\int_0^5 (r\sin\theta)e^{r\cos\theta} r dr d\theta.$$

We first integrate $\int_0^{\pi/2} r^2(\sin\theta)e^{r\cos\theta}d\theta$ through letting $u=r\cos\theta$, and get that it equals

$$\int_{u=r}^{u=0}(-re^u)\,du = re^r - r.$$

Hence, by integration-by-parts, we get

$$\int\int_D ye^x\,dA = \int_0^5 (re^r - r)dr = (re^r - e^r - r^2/2)|_0^5 = 4e^5 - \frac{23}{2}.$$

2. Evaluate the following triple or iterated integrals.

(f) $\int\int\int_E (xz - y^3)\, dV$ where $E = \{(x,y,z):\ -1 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 1\}$;
Solution:

$$\begin{aligned}\int\int\int_E (xz - y^3)\, dV &= \int_{-1}^{2}\int_0^2\int_0^1 (xz - y^3) dz dy dx \\ &= \int_{-1}^{1}\int_0^2 (\frac{xz^2}{2} - y^3 z)|_{z=0}^{z=1}\, dy dz \\ &= \int_{-1}^{1}\int_0^2 (\frac{x}{2} - y^3) dy dx = \int_{-1}^{1} (x-4) dx = -8.\end{aligned}$$

(g) $\int\int\int_E 2x\, dV$ where $E = \{(x,y,z):\ 0 \le y \le 2,\ 0 \le x \le \sqrt{4-y^2},\ 0 \le z \le y\}$;
Solution:

$$\int\int\int_E 2x\, dV = \int_0^2\int_0^{\sqrt{4-y^2}}\int_0^y 2x dz dx dy = \int_0^2\int_0^{\sqrt{4-y^2}} 2xy dx dy = \int_0^2 (4-y^2) y dy = 4.$$

(h) $\int_0^1\int_0^z\int_0^{z+z} 6zx\, dy dx dz$;
Solution:

$$\int_0^1\int_0^z\int_0^{z+z} 6zx\, dy dx dz = \int_0^1\int_0^z 6sz(x+a)\, dx dz = \int_0^1 (2z^4 + 3z^4) dz = 1.$$

(i) $\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}} ze^y\, dz dz dy$;
Solution:

$$\int_0^3\int_0^1\int_0^{\sqrt{1-z^2}} ze^y\, dz dz dy = \int_0^3\int_0^1 ze^y\sqrt{1-z^2} dz dy = \int_0^3 \frac{d^y}{3} dy = \frac{e^3-1}{3}.$$

(j) $\int_0^{\pi/2}\int_0^y\int_0^x \cos(x+y+z)\, dz dx dy$.
Solution:

$$\begin{aligned}\int_0^{\pi/2}\int_0^y\int_0^x \cos(x+y+z)\, dz dx dy &= \int_0^{\pi/2}\int_0^y (\sin(2x+y) - \sin(x+y))\, dx dy \\ &= \int_0^{\pi/2} (-\frac{\cos 3y}{2} + \cos 2y + \frac{\cos y}{2} - \cos y) dy = -\frac{1}{3}.\end{aligned}$$