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Homogeneous endpoint Besov space embeddings by Hausdorff capacity and heat equation ☆

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Abstract

Two embeddings of a homogeneous endpoint Besov space are established via the Hausdorff capacity and the heat equation. Meanwhile, a co-capacity formula and a trace inequality are derived from the Besov space.

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1. Statement of theorems

As a prelude to our main results of this paper, we state two motives that originate from geometric measure theory and its applications in partial differential equations.

The first is Adams' inequality in [2, Theorem B] over the Euclidean space $\mathbb{R}^n$, $n > 1$: for any natural number $k \in (0, n)$,

$$\int_0^\infty H_{n-k}\big(\big\{x \in \mathbb{R}^n\colon\ \big|f(x)\big| > t\big\}\big)\,dt \lesssim \big\|\nabla^k f\big\|_{L^1}, \quad f \in C_0^\infty. \tag{1.1}$$

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Here $H_\beta(\cdot)$ is the Hausdorff capacity of dimension $\beta \in (0,n)$; $U \lesssim V$ means that there is a constant $\kappa > 0$ such that $U \leqslant \kappa V$—moreover, if $U \lesssim V$ and $V \lesssim U$ then we say $U \approx V$, i.e., U is comparable to V; $\nabla^k f$ denotes the vector of all kth order derivatives of f; $\|\cdot\|_{L^p}$ stands for the p-Lebesgue norm on $\mathbb{R}^n$; and $C_0^\infty = C_0^\infty(\mathbb{R}^n)$ represents the class of all infinitely differentiable functions with compact support in $\mathbb{R}^n$.

The second is an a priori estimate for the solution to the homogeneous heat equation. More precisely, suppose $W_t(z) = (4\pi t)^{-n/2}\exp(-|z|^2/(4t))$ is the heat kernel. Then

$$w(t,x) = W_t * f(x) = \int_{\mathbb{R}^n} W_t(x-y) f(y)\,dy$$

is the solution of the Dirichlet problem for the heat equation:

$$\begin{cases} (\partial_t - \Delta_x) w(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n = \mathbb{R}^{1+n}_+, \\ w(0,x) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

where $\Delta_x = \sum_{j=1}^n \partial^2/\partial x_j^2$, $\partial_t^2 = \partial^2/\partial t^2$ and $\partial_t = \partial/\partial t$. Looking over [20, Theorem 1.3], we can read off that if μ is a nonnegative Radon measure on $\mathbb{R}^{1+n}_+$ and $p \geqslant 1$, then

$$\left(\int_{\mathbb{R}^{1+n}_+} \left| w\left(t^2, x\right) \right|^p d\mu(t,x) \right)^{1/p} \lesssim \|\nabla f\|_{L^1}, \quad f \in C_0^\infty$$

$$\Leftrightarrow \quad \sup_{r>0,\, x\in\mathbb{R}^n} \frac{(\mu((0,r) \times B(x,r)))^{1/p}}{H_{n-1}(B(x,r))} < \infty. \tag{1.2}$$

As usual, $B(x,r)$ expresses the open ball of radius $r > 0$ about $x \in \mathbb{R}^n$.

Due to the usefulness of (1.1) and (1.2), as well as the embedding (cf. [18, p. 47]): for odd dimension n,

$$\left\| \nabla^{(n-1)/2} f \right\|_{L^1} \lesssim \|f\|_{\dot{\Lambda}^{1,1}_{(n-1)/2}}, \quad f \in C_0^\infty,$$

where $\dot{\Lambda}^{1,1}_\alpha = \dot{\Lambda}^{1,1}_\alpha(\mathbb{R}^n)$, $\alpha \in (0,n)$, is the homogeneous endpoint Besov space on $\mathbb{R}^n$, a natural and compelling question is whether (1.1) and (1.2) admit extensions to fractional derivatives. Accordingly, solving this question by using homogeneous endpoint Besov space becomes the main objective of this paper.

To begin with, we have the following fractional-order extension of (1.1).

Theorem 1.1. *Let $\alpha \in (0,n)$. Then*

$$\int_0^\infty H_{n-\alpha}\left(\left\{x \in \mathbb{R}^n \colon \left|f(x)\right| > t\right\}\right) dt \lesssim \|f\|_{\dot{\Lambda}^{1,1}_\alpha}, \quad f \in C_0^\infty. \tag{1.3}$$

Note that $H_{n-\alpha}(\cdot)$ is comparable to its dyadic counterpart $\tilde{H}_{n-\alpha}(\cdot)$ which is strongly subadditive; see also [2]. So, from the viewpoint of space embedding, (1.3) means that $\dot{\Lambda}_\alpha^{1,1}$ embeds the Choquet space $L^1(\tilde{H}_{n-\alpha})$ which consists of all $\tilde{H}_{n-\alpha}$-quasi-continuous functions f on $\mathbb{R}^n$ with

$$\int_{\mathbb{R}^n} |f|\,d\tilde{H}_{n-\alpha} = \int_0^\infty \tilde{H}_{n-\alpha}\big(\big\{x \in \mathbb{R}^n:\ \big|f(x)\big| > t\big\}\big)\,dt < \infty.$$

Importantly, this leads to a co-capacity formula of dimension $n-\alpha$, that is,

Theorem 1.2. *Let $\alpha \in (0,n)$ and $B_0 = B_0(\mathbb{R}^n)$ be the class of bounded functions with compact support in $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} |f|\,d\tilde{H}_{n-\alpha} \approx \inf\big\{\|\phi\|_{\dot{\Lambda}_\alpha^{1,1}}:\ \phi \in C_0^\infty,\ \phi \geqslant |f|\big\} \tag{1.4}$$

for every lower semi-continuous function $f \in B_0$.

In other words, (1.4) provides a principle dominating the norm $\int_{\mathbb{R}^n} |f|\,d\tilde{H}_{n-\alpha}$. Actually, this illustrates that Adams' variational capacity idea (cf. [4, p. 28]) of determining a domination principle is realizable at least for $q=1$ and $\alpha = k$; see [2] for the (H^1, BMO) method, and [13] for the $(n-k)$-dimensional co-area approach.

Next, having the fact—$\|f\|_{\dot{\Lambda}_\alpha^{1,1}}$ generalizes $\|\nabla^k f\|_{L^1}$—in mind, we get an analogue of (1.2) for the fractional derivatives.

Theorem 1.3. *Let $\alpha \in (0,n)$, $p \in [1,\infty)$ and μ be a nonnegative Radon measure on $\mathbb{R}_+^{1+n}$. Then*

$$\begin{aligned} &\bigg(\int_{\mathbb{R}_+^{1+n}} \big|w\big(t^2,x\big)\big|^p\,d\mu(t,x)\bigg)^{1/p} \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty \\ \Leftrightarrow\quad &\sup_{r>0,\,x\in\mathbb{R}^n} \frac{(\mu((0,r)\times B(x,r)))^{1/p}}{H_{n-\alpha}(B(x,r))} < \infty. \end{aligned} \tag{1.5}$$

Especially, if $p = (n+1)/(n-\alpha)$ and $d\mu(t,x) = dt\,dx$, then $H_{n-\alpha}(B(x,r)) \approx r^{n-\alpha}$ and hence the supremum condition in (1.5) is satisfied. An application of the mean-value property of $w(t^2,x)$ (cf. [12]) yields the decay of temperature:

$$\big|w\big(t^2,x\big)\big| \lesssim t^{\alpha-n}\|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty. \tag{1.6}$$

While working on $\mathbb{R}^n$, the boundary of $\mathbb{R}_+^{1+n}$, we derive the following trace assertion.

Theorem 1.4. *Let $\alpha \in (0,n)$, $p \in [1,\infty)$ and ν be a nonnegative Radon measure on $\mathbb{R}^n$. Then*

$$\bigg(\int_{\mathbb{R}^n} |f|^p\,d\nu\bigg)^{1/p} \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty \quad \Leftrightarrow \quad \sup_{r>0,\,x\in\mathbb{R}^n} \frac{(\nu(B(x,r)))^{1/p}}{H_{n-\alpha}(B(x,r))} < \infty. \tag{1.7}$$

This extends Maz'ya–Shaposhinkova's [15, p. 24, Theorem 1] from $\|\nabla^k f\|_{L^1}$ to $\|f\|_{\dot{\Lambda}_\alpha^{1,1}}$. Note that (1.7) may be regarded as the extreme case of (1.5). It is not surprising that the limit of (1.6) as $t \to 0$ is trivial: $\|f\|_{L^\infty} < \infty$, $f \in C_0^\infty$. Additionally, if $p = n/(n-\alpha)$ and $d\nu(x) = dx$, then (1.7) implies Herz's fractional Sobolev-type inequality in [11, Theorem 4] (see [17, p. 26, Theorem 1.4.5] for the higher order derivative setting):

$$\left(\int_{\mathbb{R}^n} \left|f(x)\right|^{n/(n-\alpha)} dx\right)^{(n-\alpha)/n} \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty.$$

In order to prove the foregoing four theorems, we turn to consider their equivalent forms. The details are presented in the forthcoming four sections. On the basis of our argument and the fundamental observation that the heat equation is sometimes treated as an intermediate type between the Poisson equation and the wave equation, we conclude this section by making a few remarks upon Theorem 1.3. As with the solution $u(t,x) = P_t * f(x)$, where $P_t(z) = \pi^{-(n+1)/2}\Gamma((n+1)/2)t(t^2+|z|^2)^{-(n+1)/2}$ is the Poisson kernel, to the Dirichlet problem for Poisson equation:

$$\begin{cases} (\partial_t^2 + \Delta_x)u(t,x) = 0, & (t,x) \in \mathbb{R}_+^{1+n}, \\ u(0,x) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

we can similarly derive the a priori estimate of Carleson type for $u(t,x)$ associated with $p \in [1,\infty)$ and a nonnegative Radon measure μ on $\mathbb{R}_+^{1+n}$ as follows:

$$\left(\int_{\mathbb{R}_+^{1+n}} \left|u(t,x)\right|^p d\mu(t,x)\right)^{1/p} \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty$$

$$\Leftrightarrow \quad \sup_{r>0,\, x\in\mathbb{R}^n} \frac{(\mu((0,r)\times B(x,r)))^{1/p}}{H_{n-\alpha}(B(x,r))} < \infty.$$

This, along with the mean-value property of $u(t,x)$, produces the decay estimate:

$$\left|u(t,x)\right| \lesssim t^{\alpha-n}\|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty.$$

On the other hand, concerning the solution $v(t,x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot y}(\cos t|y|)\hat{f}(y)\,dy$, where $\hat{f}(y) = \int_{\mathbb{R}^n} e^{-iy\cdot z} f(z)\,dz$ is the Fourier transform of f, to the linear Cauchy problem for the wave equation:

$$\begin{cases} (\partial_t^2 - \Delta_x)v(t,x) = 0, & (t,x) \in \mathbb{R}^{1+n}, \\ \partial_t v(0,x) = 0, & x \in \mathbb{R}^n, \\ v(0,x) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

we are unfortunately unable to work out a wave version of (1.5) except posting up the following conjecture for $\alpha = (n+1)/2 < n$, $p \in [1,\infty)$ and a nonnegative Radon measure μ on $\mathbb{R}^{1+n}$:

$$\left(\int_{\mathbb{R}^{1+n}} \left|v(t,x)\right|^p \, d\mu(t,x)\right)^{1/p} \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty$$

$$\Leftrightarrow \quad \sup_{r>0,\, x\in\mathbb{R}^n} \frac{(\mu((-r,r)\times B(x,r)))^{1/p}}{H_{n-\alpha}(B(x,r))} < \infty,$$

due to the dispersive estimate (see, e.g., [18, p. 47]):

$$\left|v(t,x)\right| \lesssim |t|^{\alpha-n} \|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty.$$

2. Proof of Theorem 1.1

In order to show Theorem 1.1, we recall several more notations and introduce a lemma. For a function f on $\mathbb{R}^n$, and a natural number k, we write $\Delta_h^k f$ for the kth difference:

$$\Delta_h^k f(x) = \begin{cases} \Delta_h^1 \Delta_h^{k-1} f(x), & k > 1, \\ f(x+h) - f(x), & k = 1. \end{cases}$$

With this convention, we define the homogeneous Besov space $\dot{\Lambda}_\alpha^{p,q} = \dot{\Lambda}_\alpha^{p,q}(\mathbb{R}^n)$, $1 \leqslant p, q < \infty$, to be the completion of all functions $f \in C_0^\infty$ with

$$\|f\|_{\dot{\Lambda}_\alpha^{p,q}} = \left(\int_{\mathbb{R}^n} \left(\frac{\|\Delta_h^k f\|_{L^p}^q}{|h|^{n+\alpha q}}\right) dh\right)^{1/q} < \infty,$$

where $\alpha \in (0,\infty)$, $k = 1 + [\alpha]$. In particular, if $\alpha \in (0,1)$, then $\|f\|_{\dot{\Lambda}_\alpha^{p,q}}$ can be replaced by

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left|f(x+h) - f(x)\right|^p dx\right)^{q/p} |h|^{-(n+\alpha q)} \, dh\right)^{1/q}.$$

Note that the character of $\dot{\Lambda}_\alpha^{p,q}$ relies heavily upon $\alpha - n/p$, the degree of homogeneity/smoothness (cf. [11, Lemma 1.2]):

$$\left\|f(t\cdot)\right\|_{\dot{\Lambda}_\alpha^{p,q}} = t^{\alpha-n/p} \|f\|_{\dot{\Lambda}_\alpha^{p,q}}, \quad f \in C_0^\infty, \ t > 0.$$

So, although $\dot{\Lambda}_\alpha^{p,q}$ is contained in the space of tempered distributions modulo polynomials (cf. [11, Corollary]), it is seen from [11, Proposition 1.2] that $\dot{\Lambda}_{p/n}^{p,1} \subset \mathbf{C}_0$, and if $\alpha p < n$ then $\dot{\Lambda}_\alpha^{p,q}$ is a subspace of the class of distributions f with $\phi \in C_0^\infty \Rightarrow f * \phi \in \mathbf{C}_0$, where $\mathbf{C}_0$ is the completion of C_0^∞ with respect to L^∞-norm.

At all events, $\dot{\Lambda}_\alpha^{p,q}$ is employed to define the homogeneous Besov capacity of a compact set $K \subset \mathbb{R}^n$:

$$\operatorname{cap}\left(K; \dot{\Lambda}_\alpha^{p,q}\right) = \inf\left\{\|f\|_{\dot{\Lambda}_\alpha^{p,q}}^p \colon f \in C_0^\infty, \ f \geqslant 0 \text{ on } \mathbb{R}^n, \ f \geqslant 1 \text{ on } K\right\}.$$

This definition stretches to any set $E \subseteq \mathbb{R}^n$ via $\operatorname{cap}(E; \dot{\Lambda}_\alpha^{p,q}) = \sup_{K \subset E} \operatorname{cap}(K; \dot{\Lambda}_\alpha^{p,q})$, where the supremum is taken over all compact subsets K of E. Using this definition, we obtain a practical comparability result (see also [3] or [4]):

$$\operatorname{cap}\bigl(\cdot\,; \dot{\Lambda}_\alpha^{1,1}\bigr) \approx H_{n-\alpha}(\cdot), \quad \alpha \in (0,n). \tag{2.1}$$

In the above and below, whenever $\beta \in (0,n)$, the β-dimensional Hausdorff capacity $H_\beta(E)$ of a set $E \subseteq \mathbb{R}^n$ is defined by $\inf \sum_j r_j^\beta$, where the infimum is over all countable coverings of E by open balls B_j with radius r_j.

Lemma 2.1. *Let $\beta \in (0,n)$. If f is a nonnegative lower semi-continuous function on $\mathbb{R}^n$, then*

$$\int_0^\infty H_\beta\bigl(\bigl\{x \in \mathbb{R}^n\colon\ f(x) > t\bigr\}\bigr)\,dt \approx \sup\biggl\{\int_{\mathbb{R}^n} f\,d\nu\colon\ \nu \in \mathbf{L}_+^{1,\beta},\ |||\nu|||_\beta \leqslant 1\biggr\}, \tag{2.2}$$

where $\mathbf{L}_+^{1,\beta}$ is the Morrey space of all nonnegative Radon measures ν on $\mathbb{R}^n$ for which

$$|||\nu|||_\beta = \sup_{r>0,\, x\in\mathbb{R}^n} r^{-\beta}\nu\bigl(B(x,r)\bigr) < \infty.$$

Proof. By the definition of the Choquet integral with respect to the Hausdorff capacity, we know that the left-hand side of (2.2) is equal to $\int_{\mathbb{R}^n} f\,dH_\beta$. So (2.2) follows from [2, Corollary]. □

Owing to (2.1), Theorem 1.1 is essentially the same as the following strong-type estimate for $\operatorname{cap}(\cdot\,; \Lambda_\alpha^{1,1})$.

Theorem 2.2. *Let $\alpha \in (0,n)$. Then*

$$\int_0^\infty \operatorname{cap}\bigl(\bigl\{x \in \mathbb{R}^n\colon\ \bigl|f(x)\bigr| > t\bigr\}; \dot{\Lambda}_\alpha^{1,1}\bigr)\,dt \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty. \tag{2.3}$$

Proof. To prove (2.3), we write

$$I_\beta * f(z) = \int_{\mathbb{R}^n} |z-y|^{\beta-n} f(y)\,dy, \quad z \in \mathbb{R}^n,$$

for the Riesz potential with order $\beta \in (0,n)$ of a function f defined on $\mathbb{R}^n$.

Next, suppose $\alpha = \beta + \gamma$ where $\beta, \gamma \in (0,n)$. From [11, Proposition 6.1] it follows that $\dot{\Lambda}_\alpha^{1,1} = I_\beta * \dot{\Lambda}_\gamma^{1,1}$. In other words, $f \in \dot{\Lambda}_\alpha^{1,1}$ if and only if there is a $g \in \dot{\Lambda}_\gamma^{1,1}$ such that $f = I_\beta * g$ and $\|f\|_{\dot{\Lambda}_\alpha^{1,1}} \approx \|g\|_{\dot{\Lambda}_\gamma^{1,1}}$. This observation, together with (2.1) and (2.2), suggests that (2.3) is equivalent to

$$\sup\biggl\{\int_{\mathbb{R}^n} |I_\beta * g|\,d\nu\colon\ \nu \in \mathbf{L}_+^{1,n-\beta-\gamma},\ |||\nu|||_{n-\beta-\gamma} \leqslant 1\biggr\} \lesssim \|g\|_{\dot{\Lambda}_\gamma^{1,1}}, \quad g \in C_0^\infty. \tag{2.4}$$

Thus, it is enough to verify (2.4). Let now $\nu \in \mathbf{L}_+^{1,n-\beta-\gamma}$ and $g \in C_0^\infty$. Then it turns out from Fubini's theorem and some elementary calculation that for any open ball $B(x,r) \subset \mathbb{R}^n$,

$$
\begin{aligned}
I_\beta * \nu\big(B(x,r)\big) &= \int_{B(x,r)} \left(\int_{\mathbb{R}^n} |y-z|^{\beta-n}\, d\nu(y) \right) dz \\
&= \int_{B(x,r)} \left(\int_{B(z,2r)} + \int_{\mathbb{R}^n \setminus B(z,2r)} \right) |y-z|^{\beta-n}\, d\nu(y)\, dz \\
&\lesssim \int_{B(x,r)} \left(\int_{B(x,3r)} |y-z|^{\beta-n}\, d\nu(y) \right) dz \\
&\quad + \int_{B(x,r)} \left(\sum_{j=1}^\infty \int_{2^j r \leqslant |y-z| < 2^{j+1} r} |y-z|^{\beta-n}\, d\nu(y) \right) dz \\
&\lesssim \int_{B(x,3r)} \left(\int_{B(x,r)} |y-z|^{\beta-n}\, dz \right) d\nu(y) \\
&\quad + \int_{B(x,r)} \sum_{j=1}^\infty \big(2^j r\big)^{\beta-n} \nu\big(B\big(z, 2^{j+1} r\big)\big)\, dz \\
&\lesssim \int_{B(x,r)} \left(\int_{|y-z|<4r} |y-z|^{\beta-n}\, dz \right) d\nu(y) + r^{n-\gamma} \| \nu \|_{n-\beta-\gamma} \\
&\lesssim r^\beta \nu\big(B(x,r)\big) + r^{n-\gamma} \| \nu \|_{n-\beta-\gamma} \\
&\lesssim r^{n-\gamma} \| \nu \|_{n-\beta-\gamma}. \qquad (2.5)
\end{aligned}
$$

That is to say: $I_\beta * \nu \in \mathbf{L}_+^{1,n-\gamma}$. This implies that $I_\beta * \nu$ and $(I_\beta * \nu)\operatorname{sgn} g$ belong to the dual space $[\dot{\Lambda}_\gamma^{1,1}]^* = \dot{\Lambda}_{-\gamma}^{\infty,\infty}$ (i.e., the homogeneous version of [6, p. 87, Theorem 4.1.3(d)] where $p = q = 1$) which consists of those Radon measures μ on $\mathbb{R}^n$ with

$$
\|\mu\|_{\dot{\Lambda}_{-\gamma}^{\infty,\infty}} = \sup\left\{ \left| \int_{\mathbb{R}^n} f\, d\mu \right| :\ f \in C_0^\infty,\ \|f\|_{\dot{\Lambda}_\gamma^{1,1}} \leqslant 1 \right\} \approx \sup_{r>0,\, x\in\mathbb{R}^n} r^{\gamma-n} |\mu|\big(B(x,r)\big) < \infty
$$

(i.e., the homogeneous variation of [6, p. 90, Remark 2] where $p = q = \infty$). Accordingly,

$$
\big\|(I_\beta * \nu)\operatorname{sgn} g\big\|_{\dot{\Lambda}_{-\gamma}^{\infty,\infty}} \lesssim \|I_\beta * \nu\|_{\dot{\Lambda}_{-\gamma}^{\infty,\infty}} \lesssim \| \nu \|_{n-\beta-\gamma}.
$$

Therefore, a further use of Fubini's theorem yields

$$
\int_{\mathbb{R}^n} |I_\beta * g|\, d\nu \leqslant \int_{\mathbb{R}^n} I_\beta * \nu(x) \big|g(x)\big|\, dx \leqslant \|g\|_{\dot{\Lambda}_\gamma^{1,1}} \big\|(I_\beta * \nu)\operatorname{sgn} g\big\|_{\dot{\Lambda}_{-\gamma}^{\infty,\infty}} \lesssim \|g\|_{\dot{\Lambda}_\gamma^{1,1}} \| \nu \|_{n-\beta-\gamma},
$$

and so (2.4). This proves Theorem 2.2/1.1. $\square$

Remark 2.3. It seems interesting to determine the range of those triples (α, p, q) for which

$$\int_0^\infty \big(\mathrm{cap}\big(\{x \in \mathbb{R}^n\colon \ |f(x)| > t\}; \dot{\Lambda}_\alpha^{p,q}\big)\big)^s \, dt^{sp} \lesssim \|f\|^{sp}_{\dot{\Lambda}_\alpha^{p,q}}, \quad f \in C_0^\infty. \tag{2.6}$$

Here $s = \max\{q/p, 1\}$; see also [7].

(2.3) tells us that (2.6) is valid for $(\alpha, p, q) \in (0, n) \times \{1\} \times \{1\}$. At the same time, if $\alpha = n/p$ and $q = 1$ then (2.6) is true too (cf. [5]). In fact, since $\dot{\Lambda}^{p,1}_{n/p} \subset \mathbf{C}_0$, we have that if supp f denotes the compact support of $f \in C_0^\infty$ then there exists a constant $\kappa > 0$ such that $\sup_{x \in \operatorname{supp} f} |f(x)| \leqslant \kappa \|f\|_{\dot{\Lambda}^{p,1}_{n/p}}$. As a result, we derive $s = 1$ and

$$\int_0^\infty \mathrm{cap}\big(\{x \in \mathbb{R}^n\colon \ |f(x)| > t\}; \dot{\Lambda}^{p,1}_{n/p}\big)\, dt^p \leqslant \kappa^p \,\mathrm{cap}\big(\operatorname{supp} f; \dot{\Lambda}^{p,1}_{n/p}\big) \|f\|^p_{\dot{\Lambda}^{p,1}_{n/p}}.$$

In addition, (2.6) holds with $\alpha p \leqslant n$ and $(p, q) \in (1, \infty) \times (1, \infty)$ as proved in [1,8,14,19]. Furthermore, following [8] and [19], we are about to see that (2.6) keeps true for other two cases: $(p, q) \in \{1\} \times (1, \infty)$ and $(p, q) \in (1, \infty) \times \{1\}$, with an extra constraint $0 < \alpha < \min\{1, n/p\}$. To be more specific, we choose a C^∞-smooth function G on the real line $\mathbb{R}^1$ such that $G(t) = 0$ respectively 1 when $t \leqslant 0$ respectively $t \geqslant 1$. Set $G_j(t) = 2^j G(2^{2-j} t - 1)$. Let $f \in C_0^\infty$. Then $G_j(|f|) \in C_0^\infty$ and $2^{-j} G_j(|f|) = 1$ on $E_j = \{x \in \mathbb{R}^n\colon \ |f(x)| > 2^{j-1}\}$. This fact, plus the definition of $\mathrm{cap}(\cdot\,; \dot{\Lambda}_\alpha^{p,q})$, implies $\mathrm{cap}(E_j; \dot{\Lambda}_\alpha^{p,q}) \leqslant 2^{-jp} \|G_j(|f|)\|^p_{\dot{\Lambda}_\alpha^{p,q}}$, but also

$$\begin{aligned}
&Q(f; \alpha, p, q, s) \\
&\quad = \int_0^\infty \big(\mathrm{cap}\big(\{x \in \mathbb{R}^n\colon \ |f(x)| > t\}; \dot{\Lambda}_\alpha^{p,q}\big)\big)^s \, dt^{sp} \\
&\quad \approx \sum_{j=-\infty}^{\infty} \big(\mathrm{cap}\big(E_j; \dot{\Lambda}_\alpha^{p,q}\big) 2^{pj}\big)^s \lesssim \sum_{j=-\infty}^{\infty} \big\|G_j\big(|f|\big)\big\|^{ps}_{\dot{\Lambda}_\alpha^{p,q}} \\
&\quad \approx \sum_{j=-\infty}^{\infty} \bigg(\int_{\mathbb{R}^n} \bigg(\int_{\mathbb{R}^n} \big|G_j\big(|f|\big)(x+h) - G_j\big(|f|\big)(x)\big|^p \, dx \bigg)^{q/p} |h|^{-(n+\alpha q)} \, dh \bigg)^{sp/q}.
\end{aligned} \tag{2.7}$$

Since $G_j'(t) = 2G'(2^{2-j} t - 1)$, from the construction of G and the mean-value theorem it turns out that (cf. [19])

$$\sum_{j=-\infty}^{\infty} \big|G_j\big(|f|\big)(x+h) - G_j\big(|f|\big)(x)\big|^p \lesssim \big|f(x+h) - f(x)\big|^p. \tag{2.8}$$

Assuming $0 < \alpha < \min\{1, n/p\}$, we handle two situations below:

Case 1: $(p, q) \in \{1\} \times (1, \infty)$. This implies $q > p = 1$ and $s = q$. Using (2.7) and (2.8), we get

$$
\begin{aligned}
&Q(f;\alpha,p,q,s)\\
&\lesssim \sum_{j=-\infty}^{\infty}\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|G_j\big(|f|\big)(x+h)-G_j\big(|f|\big)(x)\right|^p dx\right)^{q/p}|h|^{-(n+\alpha q)}\,dh\right)^{sp/q}\\
&=\int_{\mathbb{R}^n}\sum_{j=-\infty}^{\infty}\left(\int_{\mathbb{R}^n}\left|G_j\big(|f|\big)(x+h)-G_j\big(|f|\big)(x)\right|^p dx\right)^{q}|h|^{-(n+\alpha q)}\,dh\\
&\leqslant\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\sum_{j=-\infty}^{\infty}\left|G_j\big(|f|\big)(x+h)-G_j\big(|f|\big)(x)\right| dx\right)^{q}|h|^{-(n+\alpha q)}\,dh\\
&\lesssim\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|f(x+h)-f(x)\right| dx\right)^{q}|h|^{-(n+\alpha q)}\,dh\approx\|f\|^{sp}_{\dot{\Lambda}^{p,q}_{\alpha}},
\end{aligned}
$$

and so (2.6).

Case 2: $(p,q)\in(1,\infty)\times\{1\}$. This yields $p>q=1$ and $s=1$. Then we apply the Minkowski inequality for the sequence space $l^{p/q}$, (2.7) and (2.8) to produce

$$
\begin{aligned}
&\big(Q(f;\alpha,p,q,s)\big)^{q/p}\\
&\lesssim\left(\sum_{j=-\infty}^{\infty}\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|G_j\big(|f|\big)(x+h)-G_j\big(|f|\big)(x)\right|^p dx\right)^{sq/p}|h|^{-(n+\alpha q)}\,dh\right)^{p/q}\right)^{q/p}\\
&\lesssim\int_{\mathbb{R}^n}\left(\sum_{j=-\infty}^{\infty}\int_{\mathbb{R}^n}\left|G_j\big(|f|\big)(x+h)-G_j\big(|f|\big)(x)\right|^p dx\right)^{1/p}|h|^{-(n+\alpha)}\,dh\\
&\lesssim\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\sum_{j=-\infty}^{\infty}\left|G_j\big(|f|\big)(x+h)-G_j\big(|f|\big)(x)\right|^p dx\right)^{1/p}|h|^{-(n+\alpha)}\,dh\\
&\lesssim\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|f(x+h)-f(x)\right|^p dx\right)^{1/p}|h|^{-(n+\alpha)}\,dh\approx\|f\|^{sq}_{\dot{\Lambda}^{p,q}_{\alpha}},
\end{aligned}
$$

and so (2.6).

Perhaps, it is worth mentioning that the above approach appears less possible to show that (2.6) is still valid for the remaining cases:

(i) $1\leqslant\alpha\leqslant n$, $(p,q)\in\{1\}\times(1,\infty)$;
(ii) $1\leqslant\alpha<n/p$, $(p,q)\in(1,\infty)\times\{1\}$.

3. Proof of Theorem 1.2

To prove Theorem 1.2, we need an auxiliary result. For the sake of convenience, **M** is used to denote the set of all Radon measures on $\mathbb{R}^n$. Further, $\mathbf{M}_+(E)$ refers to those nonnegative elements in **M** with support in the set $E\subseteq\mathbb{R}^n$.

Lemma 3.1. *Given $0 < \beta, \gamma, \beta + \gamma < n$ and $\phi \in B_0$, let*

$$R_{\beta,\gamma}(\phi) = \inf\bigl\{\|g\|_{\dot{\Lambda}^{1,1}_{\gamma}}\colon\ g \in C_0^{\infty},\ g \geqslant 0,\ I_\beta * g \geqslant |\phi|\bigr\}$$

and

$$S_{\beta,\gamma}(\phi) = \sup\left\{\int\limits_{\mathbb{R}^n} |\phi|\,d\nu\colon\ \nu \in \mathbf{M}_+(\operatorname{supp}\phi),\ \|I_\beta * \nu\|_{\dot{\Lambda}^{\infty,\infty}_{-\gamma}} \leqslant 1\right\}.$$

Then

$$R_{\beta,\gamma}(\phi) = S_{\beta,\gamma}(\phi). \tag{3.1}$$

Proof. To prove (3.1), set two classes:

$$\mathbf{M}_\phi = \left\{\nu \in \mathbf{M}^+\bigl(\operatorname{supp}(\phi)\bigr)\colon \int\limits_{\mathbb{R}^n} |\phi|\,d\nu = 1\right\}$$

and

$$\mathbf{N}_\gamma = \bigl\{g \in C_0^{\infty}\colon\ g \geqslant 0,\ \|g\|_{\dot{\Lambda}^{1,1}_{\gamma}} \leqslant 1\bigr\}.$$

Consider two functionals based on $\mathbf{M}_\phi$ and $\mathbf{N}_\gamma$:

$$A_{\beta,\gamma}(\phi) = \left(\sup_{g \in \mathbf{N}_\gamma} \inf_{\nu \in \mathbf{M}_\phi} \int\limits_{\mathbb{R}^n} I_\beta * g\,d\nu\right)^{-1}$$

and

$$B_{\beta,\gamma}(\phi) = \left(\inf_{\nu \in \mathbf{M}_\phi} \sup_{g \in \mathbf{N}_\gamma} \int\limits_{\mathbb{R}^n} I_\beta * g\,d\nu\right)^{-1}.$$

Clearly, $\mathbf{M}_\phi$ is a compact subset in the vague topology of $\mathbf{M}$ and that both $\mathbf{M}_\phi$ and $\mathbf{N}_\gamma$ are convex. These facts plus the linearity and continuity of $\int_{\mathbb{R}^n} I_\beta * g\,d\nu$ insure that Fan's Minimax Theorem (cf. [10]) applies and yields

$$A_{\beta,\gamma}(\phi) = B_{\beta,\gamma}(\phi). \tag{3.2}$$

If one can prove

$$R_{\beta,\gamma}(\phi) = A_{\beta,\gamma}(\phi) \quad \text{and} \quad S_{\beta,\gamma}(\phi) = B_{\beta,\gamma}(\phi), \tag{3.3}$$

then (3.1) follows from (3.2) and hence the proof is complete.

In the sequel, we are about to verify (3.3). Regarding the former equality in (3.3), we make the following consideration. If $R_{\beta,\gamma}(\phi) < \infty$ then for any $\epsilon > 0$ there is a nonnegative function $f_\epsilon \in C_0^\infty$ such that $I_\alpha * f_\epsilon \geqslant |\phi|$ on $\operatorname{supp}\phi$ and $\|f_\epsilon\|_{\dot\Lambda_\gamma^{1,1}} < R_{\beta,\gamma}(\phi) + \epsilon$. So

$$\left\| \frac{f_\epsilon}{R_{\beta,\gamma}(\phi) + \epsilon} \right\|_{\dot\Lambda_\gamma^{1,1}} \leqslant 1$$

and

$$\int_{\mathbb{R}^n} I_\alpha * \left(\frac{f_\epsilon}{R_{\beta,\gamma}(\phi) + \epsilon} \right) d\nu \geqslant \frac{1}{R_{\beta,\gamma}(\phi) + \epsilon}, \quad \nu \in \mathbf{M}_\phi.$$

Accordingly,

$$R_{\beta,\gamma}(\phi) + \epsilon \geqslant \left(\int_{\mathbb{R}^n} I_\alpha * \left(\frac{f_\epsilon}{R_{\beta,\gamma}(\phi) + \epsilon} \right) d\nu \right)^{-1}, \quad \nu \in \mathbf{M}_\phi.$$

This implies $R_{\beta,\gamma}(\phi) + \epsilon \geqslant A_{\beta,\gamma}(\phi)$ and then $R_{\beta,\gamma}(\phi) \geqslant A_{\beta,\gamma}(\phi)$.

If $A_{\beta,\gamma}(\phi) < \infty$ then for any $\epsilon > 0$ there exists a $g_\epsilon \in \mathbf{N}_\gamma$ such that

$$\left(\inf_{\nu \in \mathbf{M}_\phi} \int_{\mathbb{R}^n} I_\alpha * g_\epsilon \, d\nu \right)^{-1} < A_{\beta,\gamma}(\phi) + \epsilon.$$

Hence

$$1 \leqslant \inf_{\nu \in \mathbf{M}_\phi} \int_{\mathbb{R}^n} I_\alpha * \big(g_\epsilon \cdot \big(A_{\beta,\gamma}(\phi) + \epsilon \big) \big) \, d\nu.$$

Now, fix $x \in \operatorname{supp}\phi$ and choose $d\nu = |\phi(x)|^{-1} \, d\delta_x$ where $d\delta_x$ is the Dirac measure attached to x. Then $\int_{\mathbb{R}^n} |\phi| \, d\nu = 1$ and the last inequality yields $|\phi(x)| \leqslant I_\alpha * (g_\epsilon \cdot (A_{\beta,\gamma}(\phi) + \epsilon))(x)$. Since $\|g_\epsilon\|_{\dot\Lambda_\gamma^{1,1}} \leqslant 1$, we further get

$$R_{\beta,\gamma}(\phi) \leqslant \big\| g_\epsilon \cdot \big(A_{\beta,\gamma}(\phi) + \epsilon \big) \big\|_{\dot\Lambda_\gamma^{1,1}} \leqslant A_{\beta,\gamma}(\phi) + \epsilon$$

and then $R_{\beta,\gamma}(\phi) \leqslant A_{\beta,\gamma}(\phi)$.

To check the latter equality in (3.3), notice first that $B_{\beta,\gamma}(\phi) \geqslant 0$ since $0 \in \mathbf{N}_\gamma$. Next, note that $B_{\beta,\gamma}(\phi) < \infty$, for if not then there would be a sequence of measures $\{\nu_j\}$ in $\mathbf{M}_\phi$ such that $\lim_{j\to\infty} \sup_{g \in \mathbf{N}_\gamma} \int_{\mathbb{R}^n} I_\beta * g \, d\nu_j = 0$ and hence, by the duality $[\dot\Lambda_\gamma^{1,1}]^* = \dot\Lambda_{-\gamma}^{\infty,\infty}$, there exists $\lim_{j\to\infty} \|I_\beta * \nu_j\|_{\dot\Lambda_{-\gamma}^{\infty,\infty}} = 0$, which in turn implies

$$\lim_{j\to\infty} \|\!|\nu_j\|\!|_{n-(\beta+\gamma)} = 0.$$

Consequently, there is a subsequence $\{\nu_{j_k}\}$ converging vaguely to a measure $\nu \in \mathbf{M}_\phi$. The measure ν must satisfy $\int_{\mathbb{R}^n} |\phi|\,d\nu = 1$. But the convergence in $\mathbf{L}_+^{1,n-(\beta+\gamma)}$ yields $\nu(B) = 0$ for all open balls $B \subset \mathbb{R}^n$ and then $\int_{\mathbb{R}^n} |\phi|\,d\nu = 0$, a contradiction.

Thanks to $B_{\beta,\gamma}(\phi) < \infty$ and $[\dot{\Lambda}_\gamma^{1,1}]^* = \dot{\Lambda}_{-\gamma}^{\infty,\infty}$, we see that for any $\epsilon > 0$ there is a measure $\nu \in \mathbf{M}_\phi$ such that

$$B_{\beta,\gamma}(\phi) - \epsilon < \left(\sup_{g \in \mathbf{N}_\gamma} \int_{\mathbb{R}^n} I_\beta * g\,d\nu \right)^{-1} = \|I_\beta * \nu\|^{-1}_{\dot{\Lambda}_{-\gamma}^{\infty,\infty}}.$$

Taking $\nu' = \|I_\beta * \nu\|^{-1}_{\dot{\Lambda}_{-\gamma}^{\infty,\infty}} \nu$, we get

$$B_{\beta,\gamma}(\phi) - \epsilon < \|I_\beta * \nu\|^{-1}_{\dot{\Lambda}_{-\gamma}^{\infty,\infty}} = \int_{\mathbb{R}^n} |\phi|\,d\nu',$$

and thus $B_{\beta,\gamma}(\phi) \leqslant S_{\beta,\gamma}(\phi)$.

To establish the reverse of the last inequality, assume $S_{\beta,\gamma}(\phi) < \infty$. The three conditions: $\nu \in \mathbf{M}^+(\operatorname{supp}\phi)$ with $\|I_\beta * \nu\|_{\dot{\Lambda}_{-\gamma}^{\infty,\infty}} \leqslant 1$, $\nu' = (\int_{\mathbb{R}^n} |\phi|\,d\nu)^{-1}\nu$ and $g \in \mathbf{N}_\gamma$ imply

$$\int_{\mathbb{R}^n} I_\beta * g\,d\nu' = \int_{\mathbb{R}^n} g(x) I_\beta * \nu'(x)\,dx \leqslant \left(\int_{\mathbb{R}^n} |\phi|\,d\nu \right)^{-1},$$

due to the duality $[\dot{\Lambda}_\gamma^{1,1}]^* = \dot{\Lambda}_{-\gamma}^{\infty,\infty}$ once again. Therefore, $\int_{\mathbb{R}^n} |\phi|\,d\nu \leqslant B_{\beta,\gamma}(\phi)$ and consequently, $S_{\beta,\gamma}(\phi) \leqslant B_{\beta,\gamma}(\phi)$. We are done. $\square$

To prove Theorem 1.2, we need to introduce the Choquet space $L^1(\operatorname{cap}(\cdot\,; \dot{\Lambda}_\alpha^{1,1}))$ which consists of all $\operatorname{cap}(\cdot\,; \dot{\Lambda}_\alpha^{1,1})$-quasi-continuous functions f on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} |f|\,d\operatorname{cap}\big(\cdot\,; \dot{\Lambda}_\alpha^{1,1}\big) = \int_0^\infty \operatorname{cap}\big(\big\{x \in \mathbb{R}^n\colon\ \big|f(x)\big| > t\big\}; \dot{\Lambda}_\alpha^{1,1}\big)\,dt < \infty.$$

Moreover, we find it necessary to review $\tilde{H}_\beta(\cdot) \approx H_\beta(\cdot)$. Note that the dyadic Hausdorff capacity $\tilde{H}_\beta(\cdot)$ is determined via requiring that the ball-coverings in the definition of $H_\beta(\cdot)$ are replaced by the dyadic cube-coverings and the sum there, by the corresponding sum of the side lengths of the cubes raised to the βth power. Using this comparability result, we are now in a position to show the equivalent version of Theorem 1.2 as follows.

Theorem 3.2. *Let $\alpha \in (0, n)$. Then*

$$\int_{\mathbb{R}^n} |f|\,d\operatorname{cap}\big(\cdot\,; \dot{\Lambda}_\alpha^{1,1}\big) \approx \inf\big\{\|\phi\|_{\dot{\Lambda}_\alpha^{1,1}}\colon\ \phi \in C_0^\infty,\ \phi \geqslant |f|\big\} \tag{3.4}$$

for every lower semi-continuous function $f \in B_0$.

Proof. By Theorem 2.2, we get that for $\phi \in C_0^\infty$ with $\phi \geqslant |f|$,

$$\int_{\mathbb{R}^n} |f|\, d\,\mathrm{cap}\big(\cdot\,;\dot{\Lambda}_\alpha^{1,1}\big) \leqslant \int_{\mathbb{R}^n} |\phi|\, d\,\mathrm{cap}\big(\cdot\,;\dot{\Lambda}_\alpha^{1,1}\big) \lesssim \|\phi\|_{\dot{\Lambda}_\alpha^{1,1}}.$$

This just proves one part of (3.4).

As to another part, let $\alpha = \beta + \gamma$ where $\beta, \gamma \in (0, n)$. Then $\dot{\Lambda}_\alpha^{1,1} = I_\beta * \dot{\Lambda}_\gamma^{1,1}$, and hence the right-hand side of (3.4) is comparable to $R_{\beta,\gamma}(f)$. Note that

$$\big\| I_\beta * \nu \big\|_{\dot{\Lambda}_{-\gamma}^{\infty,\infty}} \approx \sup_{r>0,\, x\in\mathbb{R}^n} r^{\gamma-n} I_\beta * \nu\big(B(x,r)\big) \approx |||\nu|||_{n-(\beta+\gamma)}, \quad \nu \in \mathbf{L}_+^{n-(\beta+\gamma)},$$

thanks to (2.5) and the following simple estimate

$$I_\beta * \nu\big(B(x,r)\big) \geqslant \int_{B(x,r)} \bigg(\int_{B(z,r)} \frac{dy}{|y-z|^{n-\beta}} \bigg) d\nu(z) \gtrsim r^\beta \nu\big(B(x,r)\big).$$

So, combining (2.1) and Lemmas 3.1 and 2.1 we have

$$\begin{aligned}
R_{\beta,\gamma}(f) &= S_{\beta,\gamma}(f) \\
&\approx \sup\bigg\{ \int_{\mathbb{R}^n} |f|\, d\mu\colon\ \mu \in \mathbf{M}^+(\operatorname{supp} f),\ |||\mu|||_{n-(\beta+\gamma)} \leqslant 1 \bigg\} \\
&\lesssim \int_0^\infty H_{n-\alpha}\big(\big\{x \in \mathbb{R}^n\colon\ \big|f(x)\big| > t\big\}\big)\, dt \\
&\approx \int_0^\infty \mathrm{cap}\big(\big\{x \in \mathbb{R}^n\colon\ \big|f(x)\big| > t\big\};\, \dot{\Lambda}_\alpha^{1,1}\big)\, dt \approx \int_{\mathbb{R}^n} |f|\, d\mathrm{cap}\big(\cdot\,;\dot{\Lambda}_\alpha^{1,1}\big).
\end{aligned}$$

This completes the proof of Theorem 3.2/1.2. □

4. Proof of Theorem 1.3

To prove Theorem 1.3, let

$$M(f)(x) = \sup_{r>0} \big|B(x,r)\big|^{-1} \int_{B(x,r)} \big|f(y)\big|\, dy$$

be the Hardy–Littlewood maximal function of a locally Lebesgue integrable function f on $\mathbb{R}^n$, and write

$$T(E) = \big\{(t,x) \in \mathbb{R}_+^{1+n}\colon\ B(x,t) \subseteq E\big\}$$

for the tent based on a set $E \subseteq \mathbb{R}^n$.

Lemma 4.1. *Let $\alpha \in (0,n)$. Then*

$$\int_0^\infty \operatorname{cap}\big(\{x \in \mathbb{R}^n\colon\ M(f)(x) > \lambda\}; \dot{\Lambda}_\alpha^{1,1}\big)\,d\lambda \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty. \tag{4.1}$$

Proof. As proved in [2], we have that for $\beta \in (0,n)$,

$$\int_0^\infty H_\beta\big(\{x \in \mathbb{R}^n\colon\ M(f)(x) > \lambda\}\big)\,d\lambda \lesssim \int_0^\infty H_\beta\big(\{x \in \mathbb{R}^n\colon\ |f(x)| > \lambda\}\big)\,d\lambda, \quad f \in C_0^\infty.$$

This, together with (2.1) and (2.3), yields

$$\begin{aligned}
\int_0^\infty \operatorname{cap}\big(\{x \in \mathbb{R}^n\colon\ M(f)(x) > \lambda\}; \dot{\Lambda}_\alpha^{1,1}\big)\,d\lambda &\approx \int_0^\infty H_{n-\alpha}\big(\{x \in \mathbb{R}^n\colon\ M(f)(x) > \lambda\}\big)\,d\lambda \\
&\lesssim \int_0^\infty \operatorname{cap}\big(\{x \in \mathbb{R}^n\colon\ |f(x)| > \lambda\}; \dot{\Lambda}_\alpha^{1,1}\big)\,d\lambda \\
&\lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty,
\end{aligned}$$

as desired. □

Using Lemma 4.1, (2.1) and $H_{n-\alpha}(B(x,r)) \approx r^{n-\alpha}$, we derive the following result which covers Theorem 1.3.

Theorem 4.2. *Let $\alpha \in (0,n)$, $p \in [1,\infty)$ and μ be a nonnegative Radon measure on $\mathbb{R}_+^{1+n}$. Then the following statements are equivalent*:

(i) $\int_0^\infty (\mu(\{(t,x) \in \mathbb{R}_+^{1+n}\colon\ |W_{t^2} * f(x)| > \lambda\}))^{1/p}\,d\lambda \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}$, $f \in C_0^\infty$.

(ii) $(\int_{\mathbb{R}_+^{1+n}} |W_{t^2} * f(x)|^p\,d\mu(t,x))^{1/p} \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}$, $f \in C_0^\infty$.

(iii) $\sup_{\lambda>0} \lambda(\mu(\{(t,x) \in \mathbb{R}_+^{1+n}\colon\ |W_{t^2} * f(x)| > \lambda\}))^{1/p} \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}$, $f \in C_0^\infty$.

(iv) $(\mu(T(O)))^{1/p} \lesssim \operatorname{cap}(O; \dot{\Lambda}_\alpha^{1,1})$, *open* $O \subset \mathbb{R}^n$.

(v) $\sup_{r>0,\, x\in\mathbb{R}^n} r^{\alpha-n}(\mu(T(B(x,r))))^{1/p} < \infty$.

Proof. We prove that (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (i) and (iv) ⇔ (v), thereby establishing their equivalence.

(i) ⇒ (ii) ⇒ (iii). For $\lambda > 0$ and $f \in C_0^\infty$, let $D_\lambda(f) = \{(t,x) \in \mathbb{R}_+^{1+n}\colon\ |W_{t^2} * f(x)| > \lambda\}$. It is clear that $\mu(D_\lambda(f))$ decreases with increasing λ. So, it follows that

$$\frac{d}{d\lambda}\bigg(\int_0^\lambda \big(\mu\big(D_s(f)\big)\big)^{1/p}\,ds\bigg)^p \geqslant p\mu\big(D_\lambda(f)\big)\lambda^{p-1}.$$

This estimate yields that for any $s > 0$,

$$\big(s^p \mu\big(D_s(f)\big)\big)^{1/p} \leqslant \left(p \int_0^\infty \mu\big(D_\lambda(f)\big)\lambda^{p-1}\, d\lambda\right)^{1/p} \leqslant \int_0^\infty \big(\mu\big(D_\lambda(f)\big)\big)^{1/p}\, d\lambda,$$

giving the required implications.

(iii) $\Rightarrow$ (iv). Assume that (iii) holds. Let $O \subset \mathbb{R}^n$ be an open set. Given a compact $K \subset O$. If $f \in C_0^\infty$ satisfies $f \geqslant 0$ on $\mathbb{R}^n$ and $f \geqslant 1$ on K, then there exists a constant $c > 0$ (depending only on n) such that $(t, x) \in T(K)$ implies

$$\big|W_{t^2} * f(x)\big| \geqslant \int_K W_{t^2}(x-y)\, dy \geqslant \int_{B(x,t)} W_{t^2}(x-y)\, dy \geqslant c.$$

This estimate, in association with (iii), yields

$$\big(\mu\big(T(K)\big)\big)^{1/p} \leqslant \big(\mu\big(\big\{(t,x) \in \mathbb{R}_+^{1+n}\colon\ \big|W_{t^2} * f(x)\big| > c/2\big\}\big)\big)^{1/p} \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}.$$

Thus $\mu(T(K)) \lesssim (\mathrm{cap}(K; \dot{\Lambda}_\alpha^{1,1}))^p$ and in consequence

$$\mu\big(T(O)\big) = \sup_{K \subset O} \mu\big(T(K)\big) \lesssim \Big(\sup_{K \subset O} \mathrm{cap}\big(K; \dot{\Lambda}_\alpha^{1,1}\big)\Big)^p \approx \big(\mathrm{cap}\big(O; \dot{\Lambda}_\alpha^{1,1}\big)\big)^p.$$

Therefore, (iv) follows.

(iv) $\Rightarrow$ (i). Given $f \in C_0^\infty$, let $N(W)(x) = \sup_{|y-x|<t} |W_{t^2} * f(y)|$ be the nontangential maximal function of $W_{t^2} * f(y)$ at $x \in \mathbb{R}^n$. Clearly, $N(W)$ is lower semi-continuous and so $\{x \in \mathbb{R}^n\colon N(W)(x) > \lambda\}$ is open for any $\lambda > 0$. Notice that

$$\big|W_{t^2} * f(x-y)\big| \leqslant c\left(1 + \frac{|y|^2}{t^2}\right) M(f)(x), \quad x, y \in \mathbb{R}^n,$$

where $c > 0$ is a constant depending only on n; see [12, Lemma 3.1]. Thus, $N(W) \leqslant 2cM(f)$. So, if (iv) is true for every open set $O \subset \mathbb{R}^n$ then the last inequality implies

$$\begin{aligned}
\mu\big(D_\lambda(f)\big) &\leqslant \mu\big(T\big(\big\{x \in \mathbb{R}^n\colon N(W)(x) > \lambda\big\}\big)\big) \\
&\leqslant \mu\big(T\big(\big\{x \in \mathbb{R}^n\colon 2cM(f)(x) > \lambda\big\}\big)\big) \\
&\lesssim \big(\mathrm{cap}\big(\big\{x \in \mathbb{R}^n\colon 2cM(f)(x) > \lambda\big\}; \dot{\Lambda}_\alpha^{1,1}\big)\big)^p.
\end{aligned}$$

This, together with (4.1), implies

$$\int_0^\infty \big(\mu\big(D_\lambda(f)\big)\big)^{1/p}\, d\lambda \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty.$$

Thus, (i) is true.

The proof ends up with showing (iv) $\Leftrightarrow$ (v). As a matter of fact, it is enough to prove (v) $\Rightarrow$ (iv). Given an open set $O \subset \mathbb{R}^n$ and any open ball covering $\bigcup_j B(x_j, r_j) \supseteq O$. Since $O = \bigcup_{k=1}^{\infty} O \cap B(0,k)$, we may assume without loss of generality that O is bounded. Then there exists a dyadic cube sequence $\{I_j^{(1)}\}$ in $\mathbb{R}^n$ such that

$$O \subseteq \bigcup_j I_j^{(1)} \quad \text{and} \quad \left|B(x_j, r_j)\right| \approx \left|I_j^{(1)}\right|$$

namely, the volume of $B(x_j, r_j)$ is comparable to the volume of $I_j^{(1)}$. Furthermore, by [9, Lemma 4.1] there exist a dyadic cube sequence $\{I_j^{(2)}\}$ and its $5\sqrt{n}$-expansion $\{I_j^{(3)}\}$ (i.e., the center respectively side-length of $I_j^{(3)}$ is the same as $I_j^{(2)}$'s respectively $5\sqrt{n}$ times the side-length of $I_j^{(2)}$) such that

$$\bigcup_j I_j^{(2)} = \bigcup_j I_j^{(1)}, \qquad \sum_j \left|I_j^{(2)}\right|^{(n-\alpha)/n} \leqslant \sum_j \left|I_j^{(1)}\right|^{(n-\alpha)/n} \quad \text{and} \quad T(O) \subseteq \bigcup_j T\big(I_j^{(3)}\big).$$

This existence tells us that if (v) is true then

$$\begin{aligned}
\mu\big(T(O)\big) &\leqslant \sum_j \mu\big(T\big(I_j^{(3)}\big)\big) \\
&\lesssim \sup_{r>0,\, x\in\mathbb{R}^n} r^{p(\alpha-n)} \mu\big(T\big(B(x,r)\big)\big) \sum_j \left|I_j^{(3)}\right|^{p(n-\alpha)/n} \\
&\lesssim \sup_{r>0,\, x\in\mathbb{R}^n} r^{p(\alpha-n)} \mu\big(T\big(B(x,r)\big)\big) \sum_j \left|I_j^{(2)}\right|^{p(n-\alpha)/n} \\
&\lesssim \sup_{r>0,\, x\in\mathbb{R}^n} r^{p(\alpha-n)} \mu\big(T\big(B(x,r)\big)\big) \Big(\sum_j \left|I_j^{(2)}\right|^{(n-\alpha)/n}\Big)^p \\
&\lesssim \sup_{r>0,\, x\in\mathbb{R}^n} r^{p(\alpha-n)} \mu\big(T\big(B(x,r)\big)\big) \Big(\sum_j \left|I_j^{(1)}\right|^{(n-\alpha)/n}\Big)^p \\
&\lesssim \sup_{r>0,\, x\in\mathbb{R}^n} r^{p(\alpha-n)} \mu\big(T\big(B(x,r)\big)\big) \Big(\sum_j \left|B(x_j, r_j)\right|^{(n-\alpha)/n}\Big)^p.
\end{aligned}$$

By the definition of $H_{n-\alpha}(\cdot)$ and (2.1), we finally obtain

$$\mu\big(T(O)\big) \lesssim \sup_{r>0,\, x\in\mathbb{R}^n} r^{p(\alpha-n)} \mu\big(T\big(B(x,r)\big)\big) \big(\mathrm{cap}\big(O; \dot{\Lambda}_\alpha^{1,1}\big)\big)^p,$$

and thus (iv). Now, the proof of Theorem 4.2/1.3 is complete. □

As an interesting aside, we derive the following iso-capacitary inequality (4.2) and its analytic form (4.3) attached to the heat kernel $W_{t^2}(z)$, which appear not to have been observed until now.

Corollary 4.3. *Let $\alpha \in (0,n)$, $\beta \in (0,1]$, $\gamma > 0$ and $\gamma + n\beta \geqslant n - \alpha$. If $d\mu_{\beta,\gamma}(t,x) = t^{\gamma-1}|x|^{n(\beta-1)}\,dt\,dx$, then*

$$\big(\mu_{\beta,\gamma}\big(T(O)\big)\big)^{(n-\alpha)/(\gamma+n\beta)} \lesssim \operatorname{cap}\big(O; \dot{\Lambda}_\alpha^{1,1}\big), \quad \textit{open } O \subset \mathbb{R}^n. \tag{4.2}$$

Equivalently,

$$\bigg(\int\limits_{\mathbb{R}_+^{1+n}} \big|W_{t^2} * f(x)\big|^{(\gamma+n\beta)/(n-\alpha)}\, d\mu_{\beta,\gamma}(t,x)\bigg)^{(n-\alpha)/(\gamma+n\beta)} \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}, \quad f \in C_0^\infty. \tag{4.3}$$

Proof. This assertion follows from the case $p = (\gamma + n\beta)/(n-\alpha)$ and $\mu = \mu_{\beta,\gamma}$ of Theorem 4.2(v). $\square$

5. Proof of Theorem 1.4

To prove Theorem 1.4, we recall that for $p \in [1,\infty)$ and a nonnegative measure $\nu \in \mathbf{M}$, $L_\nu^{p,1} = L_\nu^{p,1}(\mathbb{R}^n)$ and $L_\nu^p = L_\nu^p(\mathbb{R}^n)$ denote the Lorentz space and the Lebesgue space of all functions f on $\mathbb{R}^n$ for which

$$\|f\|_{L_\nu^{p,1}} = \int\limits_0^\infty \big(\nu\big(\big\{x \in \mathbb{R}^n\colon\ \big|f(x)\big| > t\big\}\big)\big)^{1/p}\,dt \quad \text{and} \quad \|f\|_{L_\nu^p} = \bigg(\int\limits_{\mathbb{R}^n} |f|^p\,d\nu\bigg)^{1/p}$$

are finite respectively. Moreover, we use $L_\nu^{p,\infty} = L_\nu^{p,\infty}(\mathbb{R}^n)$ as the class of all ν-measurable functions f on $\mathbb{R}^n$ with

$$\|f\|_{L_\nu^{p,\infty}} = \sup_{t>0} t\big(\nu\big(\big\{x \in \mathbb{R}^n\colon\ \big|f(x)\big| > t\big\}\big)\big)^{1/p} < \infty.$$

The proof of Theorem 1.4 will be finished as long as the following result is demonstrated.

Theorem 5.1. *Let $\alpha \in (0,n)$, $p \in [1,\infty)$ and $\nu \in \mathbf{M}$ be nonnegative. Then the following statements are equivalent:*

(i) $\|f\|_{L_\nu^{p,1}} \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}$, $f \in C_0^\infty$.
(ii) $\|f\|_{L_\nu^{p}} \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}$, $f \in C_0^\infty$.
(iii) $\|f\|_{L_\nu^{p,\infty}} \lesssim \|f\|_{\dot{\Lambda}_\alpha^{1,1}}$, $f \in C_0^\infty$.
(iv) $(\nu(K))^{1/p} \lesssim \operatorname{cap}(K; \dot{\Lambda}_\alpha^{1,1})$, *compact* $K \subset \mathbb{R}^n$.
(v) $\nu \in \mathbf{L}_+^{1,p(n-\alpha)}$.

Proof. In what follows, for $t > 0$ and $f \in C_0^\infty$ let $E_t(f) = \{x \in \mathbb{R}^n\colon\ |f(x)| > t\}$.

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Since $0 < t_1 < t_2$ implies $0 \leqslant \nu(E_{t_2}(f)) \leqslant \nu(E_{t_1}(f))$, we can conclude

$$\frac{d}{dt}\bigg(\int\limits_0^t \big(\nu\big(E_s(f)\big)\big)^{1/p}\,ds\bigg)^p \geqslant p\nu\big(E_t(f)\big)t^{p-1}.$$

This implies

$$\left(s^p \nu\big(E_s(f)\big)\right)^{1/p} \leqslant \left(p \int\limits_0^\infty \nu\big(E_t(f)\big) t^{p-1}\, dt\right)^{1/p} \leqslant \int\limits_0^\infty \big(\nu\big(E_t(f)\big)\big)^{1/p}\, dt, \quad s>0,$$

and then establishes the desired implications.

(iii) ⇒ (iv). Let (iii) be valid. Given a compact set $K \subset \mathbb{R}^n$. Then for any nonnegative function $f \in C_0^\infty$ obeying $f \geqslant 1$ on K, we have $\nu(K) \leqslant \nu(E_1(f)) \lesssim \|f\|^p_{\dot{\Lambda}^{1,1}_\alpha}$, thereby deriving (iv).

(iv) ⇒ (i). Assuming (iv), by virtue of (2.3) we get

$$\|f\|_{L^{p,1}_\nu} \lesssim \int\limits_0^\infty \operatorname{cap}\big(E_t(f); \dot{\Lambda}^{1,1}_\alpha\big)\, dt \lesssim \|f\|_{\dot{\Lambda}^{1,1}_\alpha}, \quad f \in C_0^\infty,$$

and (i) right away.

To close the argument, it suffices to prove (iv) ⇔ (v). Obviously, (iv) implies (v) because of $H_{n-\alpha}(B(x,r)) \approx r^{n-\alpha}$. Conversely, if (v) holds, then for $K \subset \mathbb{R}^n$, a compact set covered by a countable open balls $\{B_j\}$ with radii $\{r_j\}$, one has that for $p \geqslant 1$,

$$\big(\nu(K)\big)^{1/p} \leqslant \sum_j \big(\nu(B_j)\big)^{1/p} \lesssim |||\nu|||^{1/p}_{p(n-\alpha)} \sum_j r_j^{n-\alpha}.$$

This implies that $(\nu(K))^{1/p} \lesssim |||\nu|||^{1/p}_{p(n-\alpha)} H_{n-\alpha}(K)$, and so that (iv) holds owing to (2.1). Therefore, the proof of Theorem 5.1/1.4 is complete. □

Theorem 5.1 leads to the following iso-capacitary inequality on $\mathbb{R}^n$ (5.1) for which the equivalent analytic representation (5.2) generalizes Herz's inequality in [11, Theorem 5] and the Hardy-type inequality in [16, Theorem 2].

Corollary 5.2. *Let $\alpha \in (0,n)$ and $(n-\alpha)/n \leqslant \beta \leqslant 1$. If $d\nu_\beta(x) = |x|^{n(\beta-1)}\, dx$, then*

$$\big(\nu_\beta(K)\big)^{(n-\alpha)/(n\beta)} \lesssim \operatorname{cap}\big(K; \dot{\Lambda}^{1,1}_\alpha\big), \quad \textit{compact } K \subset \mathbb{R}^n. \tag{5.1}$$

Equivalently,

$$\|f\|_{L^{(n\beta)/(n-\alpha)}_{\nu_\beta}} \lesssim \|f\|_{\dot{\Lambda}^{1,1}_\alpha}, \quad f \in C_0^\infty. \tag{5.2}$$

Proof. The result follows from Theorem 5.1(v) with $p = (n\beta)/(n-\alpha)$ and $\nu = \nu_\beta$. □

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