MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

FALL 2005

Pure Mathematics 3370
Worksheet on Gaussian Integers

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1. Compute a gcd of $\alpha = 26 + 7i$ and $\beta = -59 - 17i$, by copying the method for rational integers. Write the gcd in the form $\alpha\lambda + \beta\sigma$.

2. Let α and β be Gaussian integers. If $\alpha \mid \beta$, prove that $N(\alpha) \mid N(\beta)$. Is the converse true?

3. (a) Find a gcd of $\alpha = -172 + 210i$ and $\beta = 624 - 52i$.

   (b) Factor α and β completely into primes and hence check your answer in part (a).

4. Let $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\}$. Then the norm map N is defined in exactly the same way in this set of "integers", namely, for $\alpha = a + b\sqrt{-2}$, $N(\alpha) = \alpha\overline{\alpha} = a^2 + 2b^2$. State and prove a Division Algorithm for $\mathbb{Z}[\sqrt{-2}]$.

5. Prove that you cannot have a Division Algorithm in the "rings" $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Z}[\sqrt{-6}]$ and $\mathbb{Z}[\sqrt{-10}]$ by examining the factorizations $3\cdot 7 = (1+2\sqrt{-5})(1-2\sqrt{-5})$, $2\cdot 3 = -\sqrt{-6}\sqrt{-6}$ and $2 \cdot 5 = -\sqrt{-10}\sqrt{-10}$. (You should first show that the set of units in these three rings is $\{\pm 1\}$).

   The norm map N can be defined in the setting $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$ where d is a square free positive integer greater than 1. For $\alpha = a + b\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$, $N(\alpha) = \alpha\overline{\alpha} = (a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - db^2$. Prove that $N(\alpha\beta) = N(\alpha)N(\beta)$ for $\alpha, \beta \in \mathbb{Z}[\sqrt{d}]$. Show that $\mathbb{Z}[\sqrt{10}]$ does not have a Division Algorithm by examining the factorization $2 \cdot 5 = (\sqrt{10})^2$.