DISCRETE WEIGHTED MEAN METHODS

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ABSTRACT: Discrete weighted mean methods of summability are defined. Their basic regularity and abelian properties are developed and it is shown that each strictly includes its corresponding (J_p) method.

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1. Introduction

Throughout this paper $\sum_{n=0}^{\infty} a_n$ is a series of real or complex numbers and $\{s_n\}$ represents its associated sequence of partial sums. The sequence $\{p_k\}_{k=0}^{\infty}$ is nonnegative with $p_0 > 0$ and satisfies $P_n := \sum_{k=0}^{n} p_k \to \infty$. Assume that the power series $p(x) := \sum_{k=0}^{\infty} p_k x^k$ has radius of convergence 1 and define $t_n := \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k$ and $p_s(x) := \frac{1}{p(x)} \sum_{k=0}^{\infty} p_k s_k x^k$.

We assume that the real sequence $\{\lambda_n\}$ satisfies $1 \le \lambda_0 < \lambda_1 < \cdots \to \infty$ and define the sequence $\{x_n\}$, associated with $\{\lambda_n\}$, by $x_n = 1 - \frac{1}{\lambda_n}$.

Weighted mean and power series methods are defined as follows.

Definition 1.1. *If $t_n \to s$ as $n \to \infty$ then we say that $\{s_n\}$ is limitable to s by the weighted mean method M_p and write $s_n \to s(M_p)$.*

Definition 1.2. *If $p_s(x)$ exists for each $x \in (0,1)$ and $\lim_{x \to 1^-} p_s(x) = s$ then we say that $\{s_n\}$ is limitable to s by the power series method (P) and write $s_n \to s(P)$.*

Weighted mean methods, also called $(\overline{N}, p_n)$ methods in the literature, and power series methods, called (J_p) methods, have been studied extensively. It is known (see [4]) that both are regular and that $s_n \to s(M_p)$ implies $s_n \to s(P)$ (see [5]).

If $p_k = 1$ for all k then the corresponding weighted mean and power series methods are the $(C,1)$ method of Cesàro and ordinary Abel summability, (A), respectively.

We define discrete methods corresponding to (M_p) and (P) as follows.

Definition 1.3. *We say that $\{s_n\}$ is limitable to s by the discrete weighted mean method, (M_{P_λ}), and write $s_n \to s(M_{P_\lambda})$ if $\tau_n := t_{[\lambda_n]} = \dfrac{1}{P_{[\lambda_n]}} \displaystyle\sum_{k=0}^{[\lambda_n]} p_k s_k \to s$ as $n \to \infty$ where $[\cdot]$ denotes the greatest integer function.*

Definition 1.4. *Suppose that $p_s(x_n)$ exists for all n. If $\lim_{n\to\infty} p_s(x_n) = s$ then s_n is limitable to s by the discrete power series method (P_λ) and we write $s_n \to s(P_\lambda)$.*

Discrete methods have been investigated in [1], [2] and [6]. Note that (M_{P_λ}) includes (M_p) and (P_λ) includes (P) in the sense that $s_n \to s(M_p)$ or $s_n \to s(P)$ implies $s_n \to s(M_{P_\lambda})$ or $s_n \to s(P_\lambda)$ respectively. Consequently, (M_{P_λ}) and (P_λ) inherit regularity from the underlying weighted mean or power series method.

Regularity of (M_{P_λ}) can also be shown directly. In fact, τ_n can be expressed as a nonnegative matrix method, $\tau_n = \displaystyle\sum_{k=0}^{\infty} c_{n,k} s_k$, where $c_{n,k}$ is $\dfrac{p_k}{P_{[\lambda_n]}}$ for $0 \le k \le [\lambda_n]$ and is zero otherwise. This infinite matrix clearly satisfies the regularity requirements of the general theorem in [4]. The regularity of (P_λ) was established directly in [6].

2. Abelian Results

The main theorems of this paper establish abelian results between different discrete weighted mean methods and between (M_{P_λ}) and (P_λ). They use the following notation. The "greatest integer range" of the sequence $\{\lambda_n\}$ is denoted by $E(\lambda) := \{[\lambda_n] : n \ge 0\}$.

The following result includes theorem $1(i)$ of [1].

Theorem 2.1.

1. $(M_{P_\lambda}) \subseteq (M_{P_\mu})$ *if $E(\mu) \setminus E(\lambda)$ is finite.*
2. *Suppose that $p_k > 0$ for all k. If $(M_{P_\lambda}) \subseteq (M_{P_\mu})$ then $E(\mu) \setminus E(\lambda)$ is finite.*

Proof. For part (1) suppose that $E(\mu) \setminus E(\lambda)$ is finite. Then there exists an integer N such that $\{[\mu_n] : n \ge N\} \subseteq E(\lambda)$. That is, there is an increasing sequence $\{j_n\}_{n=N}^{\infty}$ such that $j_n \to \infty$ and $[\mu_n] = [\lambda_{j_n}]$ for $n \ge N$.

If $s_n \to s(M_{P_\lambda})$ then

$$\begin{aligned}\frac{1}{P_{[\mu_n]}}\sum_{k=0}^{[\mu_n]} p_k s_k &= \frac{1}{P_{[\lambda_{jn}]}}\sum_{k=0}^{[\lambda_{jn}]} p_k s_k \\ &\to s.\end{aligned}$$

That is, $s_n \to s(M_{P_\mu})$.

For the second part suppose, by way of contradiction, that $(M_{P_\lambda}) \subseteq (M_{P_\mu})$ but that $E(\mu) \setminus E(\lambda)$ is infinite.

There there exists a strictly increasing sequence $\{[\mu_{n_j}]\}_{j=1}^{\infty}$ such that $[\mu_{n_j}] \notin E(\lambda)$.

Recall that $t_n := \dfrac{1}{P_n}\displaystyle\sum_{k=0}^{n} p_k s_k$ and define a sequence $\{t_n\}$ as follows.

$$t_n = \begin{cases} 0 & \text{if } n \neq [\mu_{n_j}], \\ (-1)^j & \text{if } n = [\mu_{n_j}]. \end{cases}$$

Recover the sequence $\{s_n\}$ using $P_n t_n - P_{n-1}t_{n-1} = p_n s_n$ with $t_{-1} = P_{-1} = 0$. Then $s_n \to 0(M_{P_\lambda})$ since $t_{[\lambda_n]} = 0$ for all n but $\{s_n\}$ is not limitable (M_{P_μ}). This completes the proof.

Corollary 2.2.

(1) (M_{P_λ}) *is equivalent to* (M_{P_μ}) *(in the sense that each includes the other) if the symmetric difference* $E(\lambda)\Delta E(\mu)$ *is finite.*
(2) *Suppose that* $p_k > 0$ *for all* k. *If* (M_{P_λ}) *is equivalent to* (M_{P_μ}) *then* $E(\lambda)\Delta E(\mu)$ *is finite.*

The observation earlier that (M_{P_λ}) includes (M_p) also follows from part 1 of Theorem 2.1.

Corollary 2.3. $(M_p) \subseteq (M_{P_\mu})$ *for any* $\{\mu_n\}$.

Proof. Since $E(\mu) \setminus \{1, 2, 3, \cdots\}$ is empty, the result follows with a suitable choice of the sequence $\{\lambda_n\}$ in theorem 2.1. For example, either $\lambda_n = n + 1$ for $n \geq 0$ or $\lambda_0 = 1$ and $\lambda_n = n + \frac{1}{2}$ for $n \geq 1$ will work.

And, reversing the role of the set $\{1, 2, 3, \cdots\}$ gives

Corollary 2.4. $(M_{P_\lambda}) \subseteq (M_p)$ *if* $\{1, 2, 3, \cdots\} \setminus E(\lambda)$ *is finite.*

The second main abelian result is the following.

Theorem 2.5.

(1) $(M_{P_\lambda}) \subseteq (P_\lambda)$ *if* $\{1, 2, 3, \cdots\} \setminus E(\lambda)$ *is finite.*
(2) *Suppose that* $p_k > 0$ *for all* k. *If* $(M_{P_\lambda}) \subseteq (P_\lambda)$ *then the set* $\{1, 2, 3, \cdots\} \setminus E(\lambda)$ *is finite.*

Proof. If $\{1, 2, 3, \cdots\} \setminus E(\lambda)$ is finite then the chain of inclusions

$$(M_{P_\lambda}) \subseteq (M_p) \subseteq (P) \subseteq (P_\lambda)$$

establishes (1).

On the other hand, suppose that $\{1, 2, 3, \cdots\} \setminus E(\lambda)$ is infinite and that $\{p_k\}$ is positive. Then there is a sequence of positive integers $\{n_j\}_{j=0}^{\infty}$ such that $n_j \notin E(\lambda)$ and $n_{j+1} - n_j \geq 2$.

With $t_n := \dfrac{1}{P_n} \sum_{k=0}^{n} p_k s_k$ representing the (M_p)-transform of the sequence $\{s_n\}$, define $\{t_n\}$ as follows.

$$t_n = \begin{cases} \frac{n_j!}{P_{n_j}} & \text{if } n = n_j, \\ 0 & \text{otherwise} \end{cases}$$

Then, $s_k \to 0(M_{P_\lambda})$.

But $p_{n_j} s_{n_j} = n_j!$ since we always have $P_n t_n - P_{n-1} t_{n-1} = p_n s_n$. Hence, $\sum_{n=0}^{\infty} p_n s_n x^n$ diverges for all $x \in (0, 1)$. This implies that $p_s(x_n)$ does not exist for any n and, hence, $\{s_n\}$ is not limitable (P_λ). This completes the proof of (2).

Note that the same proof gives

Corollary 2.6.

(1) $(M_{P_\lambda}) \subseteq (P)$ *if* $\{1, 2, 3, \cdots\} \setminus E(\lambda)$ *is finite.*
(2) *Suppose that* $p_k > 0$. *If* $(M_{P_\lambda}) \subseteq (P)$ *then* $\{1, 2, 3, \cdots\} \setminus E(\lambda)$ *is finite.*

In [6] the following was shown.

Theorem 2.7. *If* $p_k > 0$ *for all* $k \geq 0$ *then* $(P) \subset (P_\lambda)$ *strictly.*

Combining results gives the following observations.

Theorem 2.8. *If* $p_n > 0$ *for all* $n \geq 0$ *and* $\{1, 2, 3, \cdots\} \setminus E(\lambda)$ *is finite then* $(M_{P_\lambda}) \subset (P_\lambda)$ *strictly.*

Theorem 2.9 *If* $p_n > 0$ *for all* $n \geq 0$ *then* $(M_p) \subset (P_\lambda)$ *strictly.*

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