MATH 4310/ MATH 3111

Course Notes
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1 Chapter 1

1.1 Topology of $\mathbb{C}$

Open balls (disks) $B(z_0, r)$ in the complex plane are specified by a centre $z_0 \in \mathbb{C}$ and a radius $r > 0$,

$$B(z_0, r) = \{z \in \mathbb{C} : \ |z - z_0| < r\},$$

where

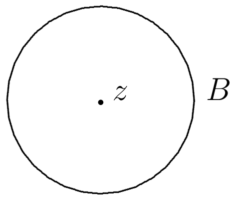
$$|z| = \sqrt{[\mathrm{Re}\, z]^2 + [\mathrm{Im}\, z]^2}$$

is the *modulus* of z. We denote by $\mathcal{B}$ the collection of all open balls,

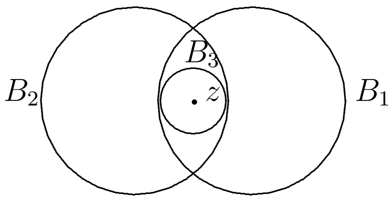
$$\mathcal{B} \ = \ \{B(z_0, r) : \ z_0 \in \mathbb{C}, r > 0\}.$$

It is easy to see that the following two properties hold:

(T1) For all $z \in \mathbb{C}$ there exists a $B \in \mathcal{B}$ such that $z \in B$.



(T2) If $z \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there exists a $B_3 \in \mathcal{B}$ such that $B_3 \subset (B_1 \cap B_2)$ and $z \in B_3$.



A collection of sets $\mathcal{B}$ satisfying (T1) and (T2) is called a *basis for a topology*. The *topology $\mathcal{T}$ generated by $\mathcal{B}$* is the collection of sets defined as follows:

$$U \in \mathcal{T} \quad \Longleftrightarrow \quad \text{for all } x \in U \text{ there exists a } B_x \in \mathcal{B} \text{ such that } x \in B_x \subset U.$$

A set U is called *open* if and only if $U \in \mathcal{T}$. Let U be an open set and, for all $x \in U$, let B_x be as above. Then $U = \cup_{x \in U} B_x$. Conversely, suppose $U = \cup_{\alpha \in I} B_\alpha$ with $B_\alpha \in \mathcal{B}$ for all α in the index set I. Then for all $x \in U$ there exists an $\alpha_x \in I$ such that $x \in B_{\alpha_x} \subset U$. Thus, U is open if and only if U is a union of basis elements. Note that in proving this fact we did not use any properties of complex numbers – it is true in all topological spaces. The topology of $\mathbb{C}$ generated by all open balls is simply called the topology (or Euclidean topology) of $\mathbb{C}$.

An open ball around z (with centre z) is called a *neighbourhood* of z. A sequence of complex numbers z_n is said to *converge* to z, written $z_n \to z$ or $\lim_{n\to\infty} z_n = z$ if, given any neighbourhood U of z, there is an $N = N(U) < \infty$, such that $z_n \in U$ whenever $n > N$. Equivalently, $z_n \to z$ if and only if for all $\epsilon > 0$ there exists $N = N(\epsilon)$ such that if $n > N$, then $|z_n - z| < \epsilon$.

Example 1. If $z_n = x_n + \mathrm{i}y_n \to z = x + \mathrm{i}y$, then $\overline{z_n} \to \overline{z}$. Indeed, since $x_n \to x$ and $y_n \to y$, we have $\overline{z_n} = x_n - \mathrm{i}y_n \to x - \mathrm{i}y = \overline{z}$. (This means that the map $z_n \mapsto \overline{z}$ is continuous).

A subset $U \subset \mathbb{C}$ is called *connected* if and only if each pair of points z_1 and z_2 in U can be joined by a polygonal line (a path of straight line segments) lying in U. An open connected set is called a *domain*.

1.2 Complex Derivative and Analyticity

Let $f : \mathbb{C} \to \mathbb{C}$ be a complex function. We say that f is *continuous* at z_0 if $f(z_0)$ is defined, and $\lim_{z\to z_0} f(z) = f(z_0)$. In other words, f is continuous at z_0 if $f(z_0)$ is defined, and if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \epsilon$.

Example 2. $f(z) = z + \mathrm{i}$ is continuous at $z_0 = \mathrm{i}$. Indeed, $|f(z) - f(z_0)| = |z + \mathrm{i} - 2\mathrm{i}| = |z - \mathrm{i}|$, so we can take $\delta = \epsilon$.

Example 3. $\mathrm{Arg}(z)$, the principal value of the argument, taking values in $(-\pi, \pi]$, is not continuous at any point on the negative real axis $(-\infty, 0]$. It is continuous everywhere else.

The *derivative* of a function $f : \mathbb{C} \to \mathbb{C}$ at a point z_0 is defined by the following limit, if it exists

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z_0)}{\Delta z}.$$

Note that $\Delta z \to 0$ means that the complex number Δz approaches the origin, but it is not specified on which path. It is understood in the definition of the derivative that the limit exists for Δz approaching zero in any possible way, and that the value of the limit is independent of how the origin is approached. Two particular ways of realizing $\Delta z = \Delta x + \mathrm{i}\Delta y \to 0$ are: $\Delta x = 0$, $\Delta y \to 0$ and $\Delta y \to 0$, $\Delta x = 0$. They lead to the Cauchy-Riemann equations (see the next section). If the derivative of f at z_0 exists, then we say f *is differentiable at the point* z_0.

We say that f is *analytic at the point* z_0 if there is a neighbourhood U of z_0 s.t. $f'(z)$ exists for all $z \in U$. "Analytic" is sometimes called "holomorphic"; the distinction between these two words will become important when considering multi-valued functions later on (holomorphic is then used for single-valued functions only). If a function is analytic at every point in a domain then we say it is *analytic in that domain.* A function that is analytic throughout $\mathbb{C}$ is called an *entire function.* It is important to note that analyticity has an equivalent characterization in terms of power series expansions (details will be given later, see Theorems 19, 20). If a function is not analytic at a point z_0, but is analytic at some point in every neighbourhood of z_0, then z_0 is called a *singular point*, or a *singularity* of f.

Examples 4.

1. Every polynomial is entire.
2. $f(z) = 1/z$ is analytic at all points except the origin, which is a singularity of f.
3. The function $f(z) = \overline{z}$ is not analytic at any point and f has no singular points.

The usual rules for derivatives of sums, products and quotients yield the result: *Sums, products, compositions of analytic functions are analytic. The quotient of two analytic functions is analytic except at points where the denominator vanishes.*

1.3 Cauchy-Riemann Equations

Suppose that f is differentiable at z_0. We write $f(z) = u(x,y) + \mathrm{i}v(x,y)$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$ are the real and imaginary parts of the function f, viewed as functions of $\mathbb{R} \times \mathbb{R} \cong \mathbb{C}$. The difference quotient takes the form

$$\begin{aligned} &\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta z} + \mathrm{i}\frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta z}, \end{aligned}$$

where $z_0 = x_0 + \mathrm{i}y_0$ and $\Delta z = \Delta x + \mathrm{i}\Delta y$. We know that the left hand side has the limit $f'(z_0)$ as $\Delta z \to 0$, that is, as $\Delta x \to 0$ and $\Delta y \to 0$. Choosing $\Delta z = \Delta x$ (i.e., $\Delta y = 0$), we get

$$f'(z_0) = \lim_{\Delta x \to 0} \left[\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + \mathrm{i}\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right].$$

Therefore, the limits of the real part and the imaginary part of the right hand side must exist individually, and we obtain

$$f'(z_0) = \partial_x u(x_0, y_0) + \mathrm{i}\partial_x v(x_0, y_0).$$

Similarly, we can take $\Delta z = \mathrm{i}\Delta y$ (that is, $\Delta x = 0$), and we obtain

$$f'(z_0) = -\mathrm{i}\partial_y u(x_0, y_0) + \partial_y v(x_0, y_0).$$

Consequently, since $f'(z_0)$ does not depend on how Δz approaches zero, it follows that the above two expressions for $f'(z_0)$ must be equal. Hence (equating individually the real and imaginary parts) we see that u and v satisfy

$$\begin{aligned} \partial_x u(x_0, y_0) &= \partial_y v(x_0, y_0) \\ \partial_y u(x_0, y_0) &= -\partial_x v(x_0, y_0). \end{aligned}$$

The latter two equations are called the *Cauchy-Riemann Equations*. They are necessarily satisfied if f is differentiable at $z_0 = x_0 + \mathrm{i}y_0$. We have just shown the following result.

Theorem 1. *Suppose that $f(z) = u(x,y) + \mathrm{i}v(x,y)$ is differentiable at $z_0 = x_0 + \mathrm{i}y_0$. Then the partial derivatives of u, v at (x_0, y_0) exist and satisfy the Cauchy-Riemann equations*

$$\partial_x u(x_0, y_0) = \partial_y v(x_0, y_0) \quad \textit{and} \quad \partial_y u(x_0, y_0) = -\partial_x v(x_0, y_0).$$

Example 5. For $f(z) = \overline{z}$ we have $u(x,y) = x$, $v(x,y) = -y$. Since $\partial_x u = 1$ and $\partial_y v = -1$, the Cauchy-Riemann equations are not satisfied at any point $z = x + \mathrm{i}y$. It follows that f is nowhere differentiable (and in particular, nowhere analytic).

A sufficient condition for differentiability is obtained by imposing continuity of the partial derivatives of u and v, as shows the following result.

Theorem 2. *Suppose that* $f(z) = u(x, y) + \mathrm{i}v(x, y)$ *is defined in a neighbourhood of* $z_0 = x_0 + \mathrm{i}y_0$*, and that* $\partial_x u$*,* $\partial_y u$*,* $\partial_x v$*,* $\partial_y v$ *exist in this neighbourhood and are continuous at* (x_0, y_0)*. If* u *and* v *satisfy the Cauchy-Riemann equations at* (x_0, y_0)*, then* $f'(z_0)$ *exists.*

Proof. Let us write $\Delta z = \alpha + \mathrm{i}\beta$ with $\alpha, \beta \in \mathbb{R}$. Then

$$f(z_0 + \Delta z) - f(z_0) = u(x_0 + \alpha, y_0 + \beta) - u(x_0, y_0) + \mathrm{i}[v(x_0 + \alpha, y_0 + \beta) - v(x_0, y_0)].$$

We use the mean value theorem (for real functions of a single variable) to arrive at

$$\begin{aligned} & u(x_0 + \alpha, y_0 + \beta) - u(x_0, y_0) \\ &\quad = u(x_0 + \alpha, y_0 + \beta) - u(x_0, y_0 + \beta) + u(x_0, y_0 + \beta) - u(x_0, y_0) \\ &\quad = \alpha \partial_x u(x_0 + \theta\alpha, y_0 + \beta) + \beta \partial_y u(x_0, y_0 + \psi\beta), \end{aligned}$$

for some $0 < \theta, \psi < 1$. Next, since $\partial_x u$ and $\partial_y u$ are continuous at (x_0, y_0), we can write

$$\begin{aligned} \partial_x u(x_0 + \theta\alpha, y_0 + \beta) &= \partial_x u(x_0, y_0) + \epsilon_1 \\ \partial_y u(x_0, y_0 + \psi\beta) &= \partial_y u(x_0, y_0) + \epsilon_2, \end{aligned}$$

where ϵ_1, $\epsilon_2 \to 0$ as α, $\beta \to 0$. One can proceed in the same way with $v(x_0 + \alpha, y_0 + \beta) - v(x_0, y_0)$. It follows that

$$\begin{aligned} f(z_0 + \Delta z) - f(z_0) &= \alpha \partial_x u(x_0, y_0) + \beta \partial_y u(x_0, y_0) + \alpha\epsilon_1 + \beta\epsilon_2 \\ &\quad + \mathrm{i}[\alpha \partial_x v(x_0, y_0) + \beta \partial_y v(x_0, y_0) + \alpha\eta_1 + \beta\eta_2], \end{aligned}$$

where η_1, $\eta_2 \to 0$ as α, $\beta \to 0$. Since the Cauchy-Riemann equations are satisfied, we obtain

$$\begin{aligned} & f(z_0 + \Delta z) - f(z_0) \\ &\quad = (\alpha + \mathrm{i}\beta)\partial_x u(x_0, y_0) + (\beta - \mathrm{i}\alpha)\partial_y u(x_0, y_0) + \alpha(\epsilon_1 + \mathrm{i}\eta_1) + \beta(\epsilon_2 + \mathrm{i}\eta_2). \end{aligned}$$

Dividing both sides by $\Delta z = \alpha + \mathrm{i}\beta$ we get

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \partial_x u(x_0, y_0) - \mathrm{i}\partial_y u(x_0, y_0) + \frac{\alpha(\epsilon_1 + \mathrm{i}\eta_1) + \beta(\epsilon_2 + \mathrm{i}\eta_2)}{\Delta z}$$

Finally, we need to estimate the last quotient. To do so, we observe that

$$\begin{aligned} \left| \frac{\alpha(\epsilon_1 + \mathrm{i}\eta_1) + \beta(\epsilon_2 + \mathrm{i}\eta_2)}{\sqrt{\alpha^2 + \beta^2}} \right| &\le \frac{\max\{|\alpha|, |\beta|\}}{\sqrt{\alpha^2 + \beta^2}} \big[|\epsilon_1| + |\epsilon_2| + |\eta_1| + |\eta_2|\big] \\ &\le |\epsilon_1| + |\epsilon_2| + |\eta_1| + |\eta_2|, \end{aligned}$$

so this quotient tends to zero as α, $\beta \to 0$. This completes the proof of the Theorem. □

1.4 Elementary Functions

We introduce in this section a few elementary complex functions and discuss some of their properties.

1.4.1 Exponential Function

For $y \in \mathbb{R}$ we define $\mathrm{e}^{\mathrm{i}y}$ as

$$\mathrm{e}^{\mathrm{i}y} = \cos y + \mathrm{i}\sin y.$$

This is the *Euler formula.* The exponential function is defined for $z = x + \mathrm{i}y \in \mathbb{C}$ by

$$\mathrm{e}^z = \mathrm{e}^x \mathrm{e}^{\mathrm{i}y},$$

so that by Euler's formula, $\mathrm{e}^z = \mathrm{e}^x[\cos y + \mathrm{i}\sin y]$. In particular, for $y = 0$, e^z reduces to the familiar real function e^x. The following properties are easily derived from the definition and from the properties of the real exponential function:

1. $\mathrm{e}^{z_1+z_2} = \mathrm{e}^{z_1}\mathrm{e}^{z_2}$, for all $z_1, z_2 \in \mathbb{C}$
2. $\mathrm{e}^z \neq 0$ and $\mathrm{e}^{-z} = 1/\mathrm{e}^z$, for all $z \in \mathbb{C}$
3. e^z is entire and $\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^z) = \mathrm{e}^z$ for all $z \in \mathbb{C}$
4. $\mathrm{e}^{z+2\pi\mathrm{i}} = \mathrm{e}^z$, for all $z \in \mathbb{C}$ (periodic function with period $2\pi\mathrm{i}$)

The first point can be shown using the properties of the real exponential function and trigonometric identities. The second point follows from the first one. To prove 3. one can first show that e^z is everywhere differentiable, using Theorem 2, and then one can choose a specific easy path (say $\Delta z = \Delta x \in \mathbb{R}$) to calculate the derivative. Finally, the periodicity follows directly from the definition and the periodicity of the trigonometric functions.

1.4.2 Logarithms

The logarithmic function is naturally introduced as the "inverse" of the exponential function. Let $z \neq 0$ be given and let us try to solve the equation $\mathrm{e}^w = z$ for w. We have seen above that $\mathrm{e}^w \neq 0$, so the equation we propose to solve has no solution for $z = 0$. Since the exponential function is periodic of period $2\pi\mathrm{i}$, there will be multiple solutions and hence log is naturally a multi-valued function.

We set $w = u + \mathrm{i}v$ and $z = r\mathrm{e}^{\mathrm{i}\theta}$, with $u, v \in \mathbb{R}$, $r > 0$ and $-\pi < \theta \leq \pi$ (i.e., θ is the principal argument of z, $\mathrm{Arg}(z)$). The equation $\mathrm{e}^w = z$ is equivalent to $\mathrm{e}^u\mathrm{e}^{\mathrm{i}v} = r\mathrm{e}^{\mathrm{i}\theta}$, from which it follows that

$$\mathrm{e}^u = r, \quad \text{and} \quad \mathrm{e}^{\mathrm{i}v} = \mathrm{e}^{\mathrm{i}\theta}.$$

Hence $u = \ln(r)$ and $v = \theta + 2\pi n$ for some $n \in \mathbb{Z}$. It follows that the complex numbers $w = \ln(r) + \mathrm{i}(\theta + 2\pi n)$, $n \in \mathbb{Z}$, are exactly the solutions of the equation $\mathrm{e}^w = z$. The non-uniqueness of the solution is due to the periodicity of the exponential function.

We would like to keep the simple identity $\mathrm{e}^{\ln(x)} = x$ also in the complex case (in which ln is generally written as log to distinguish the natural logarithm in the real and the complex case). We define the logarithm to be the *multiple-valued function*

$$\log(z) = \ln(|z|) + \mathrm{i}(\mathrm{Arg}(z) + 2\pi n), \quad n \in \mathbb{Z}.$$

Each value $n \in \mathbb{Z}$ defines a *branch* of the multiple-valued function. For fixed n, we get a single-valued function. By construction, we have the desired relation

$$\mathrm{e}^{\log z} = z.$$

This identity is to be understood as a property of the multiple-valued function log, i.e., it holds for all branches. The *principal value* (or principal branch) of $\log(z)$ is obtained by setting $n = 0$ and denoted by Log,

$$\mathrm{Log}(z) = \ln|z| + \mathrm{i}\mathrm{Arg}(z).$$

In the real case, we have $\ln(\mathrm{e}^x) = x$. Does this generalize to the complex case? We have

$$\log(\mathrm{e}^z) = \log(\mathrm{e}^x\mathrm{e}^{\mathrm{i}y}) = \ln\big(|\mathrm{e}^x\mathrm{e}^{\mathrm{i}y}|\big) + \mathrm{i}\big(\mathrm{Arg}(\mathrm{e}^x\mathrm{e}^{\mathrm{i}y}) + 2\pi n\big) = z + 2\pi\mathrm{i}n, \quad n \in \mathbb{Z}.$$

Here we have used that $\mathrm{Arg}(\mathrm{e}^x\mathrm{e}^{\mathrm{i}y}) = \mathrm{Arg}(\mathrm{e}^{\mathrm{i}y}) = y + 2\pi\mathrm{i}k$ for some $k \in \mathbb{Z}$ (chosen so that $y + 2\pi\mathrm{i}k \in (-\pi, \pi])$. Hence the relation $\ln(\mathrm{e}^x) = x$ does not generalize to the complex case: $\log(\mathrm{e}^z)$ is a multiple-valued function taking values $z + 2\pi\mathrm{i}n$, $n \in \mathbb{Z}$.

Example 6. $\mathrm{Log}(-1-\mathrm{i}) = \ln(\sqrt{2}) + \mathrm{i}\mathrm{Arg}(-1-\mathrm{i}) = \frac{1}{2}\ln(2) - \frac{3\pi\mathrm{i}}{4}$.

The logarithm has been constructed as the solution $z = \log w$ of the equation $\mathrm{e}^z = w$, for $w \neq 0$. On the other hand, in real analysis, we are accustomed to simply applying the ln to solve $\mathrm{e}^y = x$, obtaining the unique solution $y = \ln x$. The next example shows how to apply the same strategy in the complex case.

Example 7. Consider the equation $\mathrm{e}^z = -1$. Fix a branch of the logarithm, say n_0, and, in an attempt to solve the equation for z, apply the logarithm to both sides of the equation: $\log \mathrm{e}^z = \log(-1) = \mathrm{i}\pi + 2\pi\mathrm{i}n_0$. Set $z = x + \mathrm{i}y$, then $\log \mathrm{e}^z = \ln|\mathrm{e}^x| + \mathrm{i}\mathrm{Arg}(\mathrm{e}^{\mathrm{i}y}) + 2\pi\mathrm{i}n_0$. There exists exactly one $k(y) \in \mathbb{Z}$ s.t. $\mathrm{Arg}(\mathrm{e}^{\mathrm{i}y}) = y + 2\pi k(y)$ (this $k(y)$ is taken exactly such that $y + 2\pi k(y) \in (-\pi, \pi])$. Therefore we obtain $\log \mathrm{e}^z = z + 2\pi\mathrm{i}[k(y) + n_0] = \mathrm{i}\pi + 2\pi\mathrm{i}n_0$. Equating the real and imaginary parts gives $x = 0$ and $y = \pi[1 - 2k(y)]$. The latter is an implicit equation for y. A solution in the interval $(-\pi, \pi]$ exists if and only if $y = \pi$ has a solution in this interval (because on this interval, $k(y) = 0$). So $y = \pi$ is the only solution in $(-\pi, \pi]$. A solution in $(\pi, 3\pi]$ exists if and only if $y = 3\pi$ has a solution in this interval (since there $k(y) = -1$). Hence $y = 3\pi$ is the only solution in this interval. Proceeding in this way we see that the solutions to the implicit equation are exactly $y \in \{(2n+1)\pi \ : \ n \in \mathbb{Z}\}$. This shows that $\mathrm{e}^z = -1 \Leftrightarrow z \in \{\mathrm{i}(2n+1)\pi \ : \ n \in \mathbb{Z}\}$. Of course, this set of solutions coincides with the multiple values of the logarithm of -1.

Consider a fixed branch of the logarithm (n fixed) and simply denote the corresponding single-valued function by log. Where is log analytic? It is clear that $\log(z)$ is not continuous on the negative real axis $(-\infty, 0]$ because $\mathrm{Arg}(z)$ is not continuous there. Consequently, log cannot be analytic at any point on $(-\infty, 0]$. We now show that log is analytic on $\mathbb{C}\backslash(-\infty, 0]$, by applying Theorem 2.

In order to verify the Cauchy-Riemann equations for the logarithm (and for other functions exhibiting certain symmetries), it is useful to pass to *polar coordinates* $(r, \theta) \in \mathbb{R}_+ \times$

$(-\pi, \pi]$, defined by $r = |z|$, $\theta = \mathrm{Arg}(z)$. The relation between the polar and euclidean coordinates are

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

In polar coordinates, we have

$$\log(z) = \ln(r) + \mathrm{i}(\theta + 2\pi n), \quad n \in \mathbb{Z},$$

a simple function of the variables r, θ (expressed in euclidean coordinates, the form is more complicated). We now write the Cauchy-Riemann equations in polar coordinates. Let $f(x, y)$ be a function and set $g(r, \theta) = f(r\cos\theta, r\sin\theta)$. We take the partial derivatives w.r.t. r and θ and obtain

$$\begin{array}{rcl} \partial_r g(r,\theta) & = & \cos\theta\ \partial_x f(r\cos\theta, r\sin\theta) + \sin\theta\ \partial_y f(r\cos\theta, r\sin\theta), \\ \partial_\theta g(r,\theta) & = & -r\sin\theta\ \partial_x f(r\cos\theta, r\sin\theta) + r\cos\theta\ \partial_y f(r\cos\theta, r\sin\theta). \end{array}$$

These two equations are usually written in a shorter way as

$$\partial_r = \cos\theta\ \partial_x + \sin\theta\ \partial_y \quad \text{and} \quad \partial_\theta = -r\sin\theta\ \partial_x + r\cos\theta\ \partial_y.$$

It is easy to solve this system for ∂_x and ∂_y,

$$\partial_x = \cos\theta\ \partial_r - \frac{\sin\theta}{r}\ \partial_\theta \quad \text{and} \quad \partial_y = \sin\theta\ \partial_r + \frac{\cos\theta}{r}\ \partial_\theta.$$

It follows that the Cauchy-Riemann equations for the function $f(z) = u(r, \theta) + \mathrm{i}v(r, \theta)$ in polar coordinates take the form

$$\begin{array}{rcl} \cos\theta\ \partial_r u(r,\theta) - \dfrac{\sin\theta}{r}\partial_\theta u(r,\theta) & = & \sin\theta\ \partial_r v(r,\theta) + \dfrac{\cos\theta}{r}\partial_\theta v(r,\theta) \\ \sin\theta\ \partial_r u(r,\theta) + \dfrac{\cos\theta}{r}\partial_\theta u(r,\theta) & = & -\cos\theta\ \partial_r v(r,\theta) + \dfrac{\sin\theta}{r}\partial_\theta v(r,\theta). \end{array}$$

For a fixed branch $f = \log$, we have $u(r, \theta) = \ln(r)$ and $v(r, \theta) = \theta + 2\pi n$. Clearly, the partial derivatives $\partial_r u = 1/r$, $\partial_\theta v = 1$, $\partial_\theta u = \partial_r v = 0$ are continuous on $\mathbb{C}\backslash(-\infty, 0]$ and satisfy the Cauchy-Riemann equations. This shows analyticity of $\log(z)$.

Let $z \in \mathbb{C}\backslash(-\infty, 0]$. For any fixed branch (fixed $n \in \mathbb{N}$), log is differentiable at z, since it is even analytic at z. What is the derivative $\frac{\mathrm{d}}{\mathrm{d}z}\log(z)$? By differentiating the relation $\mathrm{e}^{\log(z)} = z$ we obtain (chain rule) $\mathrm{e}^{\log(z)}\frac{\mathrm{d}}{\mathrm{d}z}\log(z) = 1$ and hence $\frac{\mathrm{d}}{\mathrm{d}z}\log(z) = 1/\mathrm{e}^{\log(z)} = 1/z$, i.e., for any fixed branch, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\log(z) = \frac{1}{z} \qquad \text{for all } z \in \mathbb{C}\backslash(-\infty, 0]. \tag{1}$$

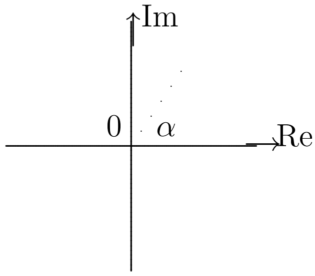
A bit more about branches. Let $\arg(z)$ be the multiple-valued argument function, $\arg(z) = \mathrm{Arg}(z) + 2\pi\mathrm{i}n$, $n \in \mathbb{Z}$, where $\mathrm{Arg}(z) \in (-\pi, \pi]$. We see from the above definition of the logarithm that

$$\log(z) = \ln|z| + \mathrm{i}\arg(z).$$

The convention that Arg takes values in $(-\pi, \pi]$ is arbitrary. It may be changed to $\mathrm{Arg}(z) \in (-\alpha, \alpha + 2\pi]$, for any $\alpha \in \mathbb{R}$. One may then define the (multi-valued) logarithm associated to this choice of Arg by

$$\log(z) = \ln|z| + \mathrm{i}(\theta + 2\pi \mathrm{i} n), \qquad n \in \mathbb{Z},\ \theta = \mathrm{Arg}(z) \in (-\alpha, \alpha + 2\pi].$$

Any specific choice of fixed n gives a *single valued* function for this logarithm.



We may use the Cauchy-Riemann equations to prove that $\log(z)$ is analytic in the domain $\{z \in \mathbb{C} :\ \mathrm{Arg}(z) \in (-\alpha, \alpha + 2\pi)\}$. On the ray $\mathrm{Arg}(z) = \alpha + 2\pi$, log is not analytic (indeed, $\mathrm{Arg}(z)$ is not even continuous there).

The concept of branches we encountered in the above discussion of the logarithm is introduced in more generality for multiple-valued functions. A *branch* of a multiple-valued function f is any single valued function F which is analytic in some domain at each point of which the value $F(z)$ is one of the values of $f(z)$. A *branch cut* is a portion of a line (or curve) that is introduced in order to define a branch F of a multiple-valued function f. Points on a branch cut are singular points of F, and any point that is common to all possible branch cuts of f is called a *branch point.*

In the above example of the logarithm, a branch cut is the ray $\mathrm{Arg}(z) = \alpha$, and the point zero is a branch point.

1.4.3 Complex Exponentials

For $z \neq 0$ and $c \in \mathbb{C}$ we define the multiple-valued function

$$z^c = \mathrm{e}^{c \log(z)}.$$

Multiple-valuedness is inherited from the log.

Example 8. We have $\mathrm{i}^{\mathrm{i}} = \mathrm{e}^{\mathrm{i} \log(\mathrm{i})} = \mathrm{e}^{\mathrm{i}[\ln(1) + \mathrm{i}\frac{\pi}{2} + 2\pi \mathrm{i} n]} = \mathrm{e}^{-\frac{\pi}{2}[1+4n]}$, $n \in \mathbb{Z}$.

It follows directly from the properties of the exponential function that $z^{-c} = \mathrm{e}^{-c\log(z)} = 1/e^{c\log(z)} = 1/z^c$.

Let us fix a branch of the logarithm. Then z^c is a single-valued function, analytic on $\mathbb{C}\backslash(-\infty, 0]$ (since the logarithm is analytic there and the exponential function is entire). What is the derivative of z^c on that domain? To find the derivative, we use the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}z} z^c = \frac{\mathrm{d}}{\mathrm{d}z} \mathrm{e}^{c\log(z)} = \frac{c}{z} \mathrm{e}^{c\log(z)} = c\mathrm{e}^{-\log(z)} \mathrm{e}^{c\log(z)} = c\mathrm{e}^{(c-1)\log(z)} = cz^{c-1}.$$

The principal value of z^c is denoted by $\mathrm{P.V.}\, z^c$. It is the single-valued function defined by

$$\mathrm{P.V.}\, z^c = \mathrm{e}^{c\mathrm{Log}(z)}.$$

Example 9. The principal value of $z^{2/3}$ is P.V. $z^{2/3} = r^{2/3}\cos(2\Theta/3) + \mathrm{i}r^{2/3}\sin(2\Theta/3)$, where $\Theta = \mathrm{Arg}(z)$.

We know the relations $\ln x^\alpha = \alpha \ln x$ and $(x^\alpha)^\beta = x^{\alpha\beta}$ for $x > 0$, $\alpha, \beta \in \mathbb{R}$. Do they generalize to the complex case? To investigate the first relation, consider a fixed branch n_0 of the logarithm, and let $a = \alpha + \mathrm{i}\beta$, $\alpha, \beta \in \mathbb{R}$. By using the definition of the logarithm, is not hard to see that for any $z \neq 0$,

$$\log z^a = a \log z + 2\pi\mathrm{i}[k(z, a, n_0) + n_0],$$

where $k(z, a, n_0) \in \mathbb{Z}$ is the unique integer chosen exactly so that

$$\beta \ln|z| + \alpha \mathrm{Arg} z + 2\pi\alpha n_0 + 2\pi k \in (-\pi, \pi].$$

Note that if $z \in (0, \infty)$, $a \in \mathbb{R}$ and if we consider the principal branch $n_0 = 0$, then $k = 0$ and we recover the habitual formula from real analysis. By now we feel that $(x^\alpha)^\beta = x^{\alpha\beta}$ is not going to hold in the complex case, since it is based on the property $\ln x^\alpha = \alpha \ln x$. In the complex case, we have for $a, b, z \in \mathbb{C}$, $z \neq 0$,

$$(z^a)^b = \mathrm{e}^{b \log z^a} = \mathrm{e}^{b[a \log z + 2\pi\mathrm{i}k(z,a,n_0) + 2\pi\mathrm{i}n_0]} = z^{ab}\mathrm{e}^{2\pi\mathrm{i}b[k(z,a,n_0)+n_0]}.$$

We conclude that $(z^a)^m = z^{am}$ for any $z \neq 0$, $a \in \mathbb{C}$ and $m \in \mathbb{Z}$, and for any branch.

1.4.4 Trigonometric Functions

We define for $z \in \mathbb{C}$

$$\sin(z) = \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}} \quad \text{and} \quad \cos(z) = \frac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2}.$$

Both are entire functions since $\mathrm{e}^{\pm\mathrm{i}z}$ are. For real numbers $z = x \in \mathbb{R}$, the definition yields $\sin(x) = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}} = \frac{1}{2\mathrm{i}}(\mathrm{i}\sin(x) + \mathrm{i}\sin(x))$. It thus coincides with the real function sin for real arguments. The same holds for the cosine. The complex trigonometric functions are examples of analytic continuations: they are analytic functions which coincide with given functions when restricted to the real line (the given functions being the real trigonometric functions in this example).

As usual, we set

$$\tan(z) = \frac{\sin(z)}{\cos(z)} \quad \text{and} \quad \cot(z) = \frac{\cos(z)}{\sin(z)}.$$

The latter functions are defined for all z for which the denominators do not vanish. They are analytic at all points in their domain of definition.

1.4.5 Hyperbolic Functions

In analogy with real hyperbolic functions, we define

$$\sinh(z) = \frac{\mathrm{e}^z - \mathrm{e}^{-z}}{2}, \quad \cosh(z) = \frac{\mathrm{e}^z + \mathrm{e}^{-z}}{2}, \quad \tanh(z) = \frac{\sinh(z)}{\cosh(z)}, \quad \coth(z) = \frac{\cosh(z)}{\sinh(z)}.$$

The first two functions are defined for all $z \in \mathbb{C}$ and are entire. The latter two are defined for all z where the denominators do not vanish, and they are analytic in their domains of definition.

Example 10. Solve $z = \sinh(w)$ for w. We write $z = (\mathrm{e}^{\mathrm{i}w} - \mathrm{e}^{-\mathrm{i}w})/2$ as $2z = \zeta - 1/\zeta$, where $\zeta = \mathrm{e}^{\mathrm{i}w}$. It follows that $\zeta^2 - 2z\zeta - 1 = 0$. The solutions to this quadratic equation are

$$\zeta = \frac{2z + (4z^2 + 4)^{1/2}}{2} = z + (1 + z^2)^{1/2}.$$

(Note that $(1 + z^2)^{1/2}$ is a double-valued function.) So, by taking the log on both sides we obtain

$$\log(\mathrm{e}^{\mathrm{i}w}) = \mathrm{i}w + 2\pi \mathrm{i}n = \log[z \pm (1 + z^2)^{1/2}], \quad n \in \mathbb{Z}.$$

Consequently, $\sinh^{-1}(z) = -\mathrm{i}\log[z \pm (1 + z^2)^{1/2}] + 2\pi n$, $n \in \mathbb{Z}$.

1.5 Integration

Let $[a, b]$ be an interval in $\mathbb{R}$, and let $u, v : [a, b] \to \mathbb{R}$ be piecewise continuous functions. Define $w : [a, b] \to \mathbb{C}$ by $w(t) = u(t) + \mathrm{i}v(t)$. The (Riemann) integral over the complex valued function w is defined by

$$\int_a^b w(t)\mathrm{d}t = \int_a^b u(t)\mathrm{d}t + \mathrm{i}\int_a^b v(t)\mathrm{d}t.$$

The usual properties of (real) Riemann integrals hold for $\int_a^b w(t)\mathrm{d}t$. We want to define $\int_{\mathcal{C}} f(z)\mathrm{d}z$, where $\mathcal{C}$ is a curve (a "contour") in the complex plane. The "contour integral" should reduce to the usual integral for the special case $\mathcal{C} = [a, b]$.

An *arc* $\mathcal{C}$ is a set of points $\{z(t) = x(t) + \mathrm{i}y(t) : \ t \in [a, b]\}$ in the complex plane, where x and y are real continuous functions, and where $[a, b]$ is a real interval. $\mathcal{C}$ is a *simple arc* or a *Jordan arc* if it does not cross itself, that is, if $z(t_1) = z(t_2) \Rightarrow t_1 = t_2$. An arc $\mathcal{C}$ is *smooth* if $z'(t)$ exists and is continuous and nonzero for all $t \in [a, b]$. (The slope of the tangent of the curve is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'(t)}{x'(t)}$; we allow for infinite slope or zero slope, but not for both $x'(t) = y'(t) = 0$).

A *contour* is an arc consisting of a finite number of smooth arcs joined end to end. When only the initial and final values of $z(t)$ are the same, the contour is called a *simple closed contour*.

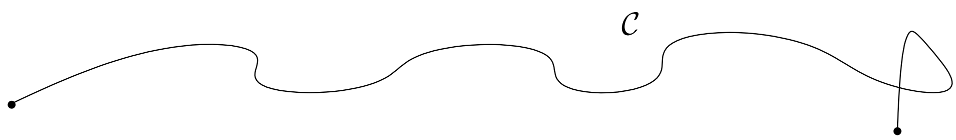


Figure 1: Arc

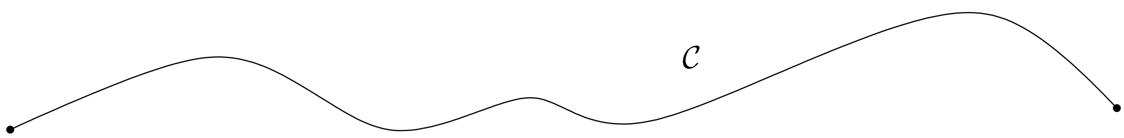


Figure 2: Smooth Simple Arc

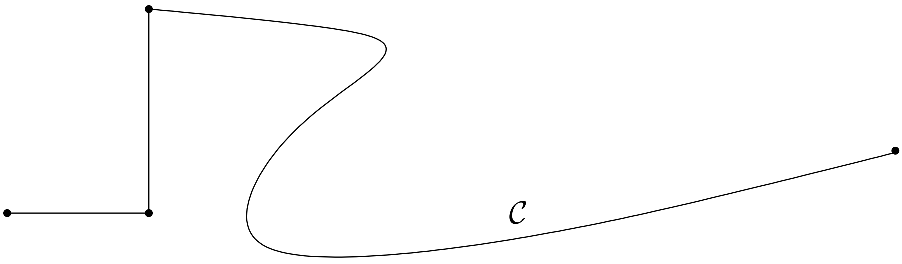


Figure 3: Contour

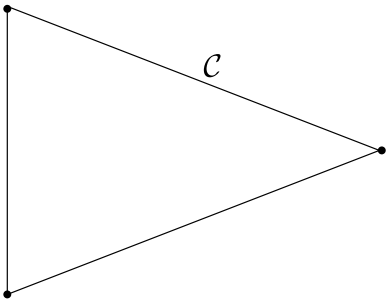


Figure 4: Simple Closed Contour

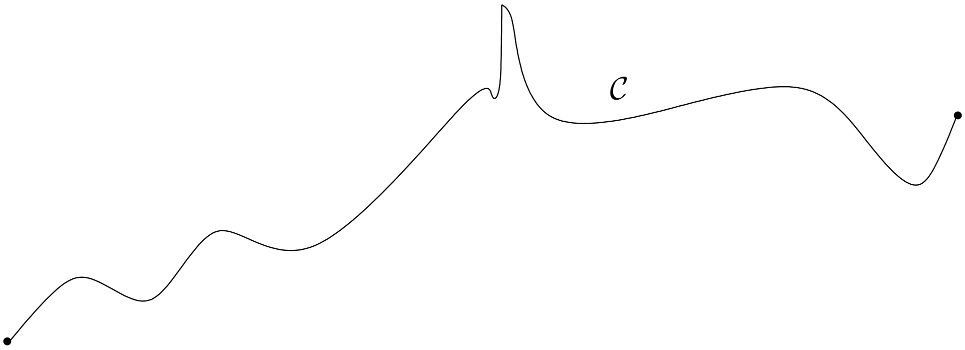


Figure 5: Simple Arc

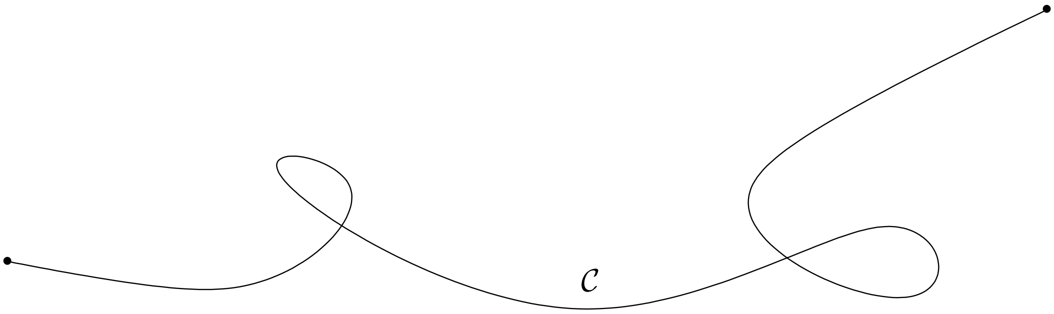


Figure 6: Smooth Arc

It is a general convention to take the direction of increasing parameter to be *anti-clockwise* for all closed contours. Let $\mathcal{C}$ be a contour represented by $z(t) = x(t) + \mathrm{i}y(t)$, $t \in [a, b]$, and denote the endpoints by $z(a) = z_1$ and $z(b) = z_2$. Let $f : \mathbb{C} \to \mathbb{C}$ be defined and piecewise continuous on $\mathcal{C}$, i.e., $f(z(t))$ is piecewise continuous on $[a, b]$. We define the complex (definite) line integral of f along $\mathcal{C}$ by

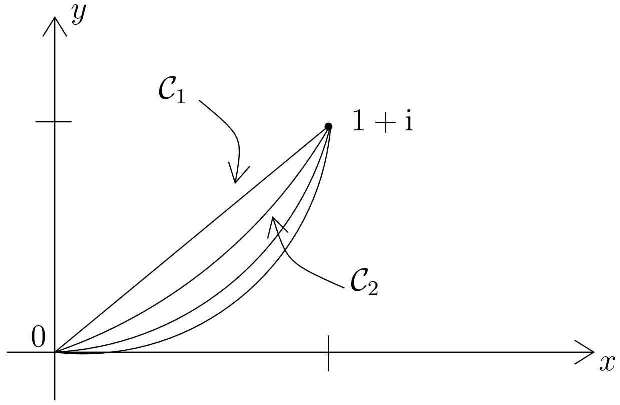
$$\int_{\mathcal{C}} f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t.$$

We point out that the right hand side is independent of the choice of parametrization of $\mathcal{C}$. Indeed, without loss of generality, let $\mathcal{C}$ be a smooth simple arc, parameterized as well by $\zeta(s)$, $s \in [c, d] \subset \mathbb{R}$, with $\zeta(c) = z_1$ and $\zeta(d) = z_2$. Then we have

$$\int_c^d f(\zeta(s))\zeta'(s)\mathrm{d}s = \int_c^d f(z(\tau(s)))z'(\tau(s))\tau'(s)\mathrm{d}s = \int_a^b f(z(t))z'(t)\mathrm{d}t,$$

where $\tau : [c, d] \to [a, b]$ is defined by $\zeta(s) = z(\tau(s))$, so that $\zeta'(s) = z'(\tau(s))\tau'(s)$.

Example 11. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = x + \mathrm{i}ky$, for some $k \in \mathbb{Z}$. Let $\mathcal{C}_n$ be the contour joining 0 to $1 + \mathrm{i}$ given by $y = x^n$, $n \in \mathbb{N}$. Calculate $\int_{\mathcal{C}_n} f(z)\mathrm{d}z$.



Parametrizing $\mathcal{C}_n$ as $z_n(t) = t + \mathrm{i}t^n$, $t \in [0, 1]$, we have

$$\begin{aligned}
\int_{\mathcal{C}_n} f(z)\mathrm{d}z &= \int_0^1 f(z_n(t))z_n'(t)\mathrm{d}t \\
&= \int_0^1 (t + \mathrm{i}kt^n)(1 + \mathrm{i}nt^{n-1})\mathrm{d}t \\
&= \int_0^1 (t - nkt^{2n-1})\mathrm{d}t + \mathrm{i}\int_0^1 (nt^n + kt^n)\mathrm{d}t \\
&= \frac{1}{2}(1-k) + \mathrm{i}\frac{n+k}{n+1}.
\end{aligned}$$

We see that the value of the integral depends on the path joining the endpoints 0 and $1 + \mathrm{i}$ (i.e., on n), except if $k = 1$. This is no accident: f is analytic (actually entire) if and only if $k = 1$. This can easily be check by applying Theorem 2. We shall examine in detail the relation between path-dependence of integrals and analyticity of the integrand below in Theorem 3.

Lemma 1. *Let f and g be piecewise continuous functions on a contour $\mathcal{C}$.*

1. *For all $\alpha, \beta \in \mathbb{C}$ we have*
$$\int_{\mathcal{C}} [\alpha f(z) + \beta g(z)] \mathrm{d}z = \alpha \int_{\mathcal{C}} f(z) \mathrm{d}z + \beta \int_{\mathcal{C}} g(z) \mathrm{d}z.$$

2. *If $\mathcal{C}$ consists of a contour $\mathcal{C}_1$ joining z_1 to z_2 and a contour $\mathcal{C}_2$ joining z_2 to z_3, then*
$$\int_{\mathcal{C}} f(z) \mathrm{d}z = \int_{\mathcal{C}_1} f(z) \mathrm{d}z + \int_{\mathcal{C}_2} f(z) \mathrm{d}z.$$

3. *Let $\mathcal{C}$ be given by $z(t)$, $t \in [a, b]$, and let $-\mathcal{C}$ be the same path, but taken in reverse order. That is, $-\mathcal{C}$ is given by $\zeta(t) = z(-t)$, $t \in [-b, -a]$. Then we have*
$$\int\limits_{-\mathcal{C}} f(z) \mathrm{d}z = -\int\limits_{\mathcal{C}} f(z) \mathrm{d}z.$$

The proof of the Lemma follows directly from the definition of the integral. It follows from 2. that $\int_{\mathcal{C}} f(z) \mathrm{d}z$ is independent of the choice of the initial point of $\mathcal{C}$ for any closed contour $\mathcal{C}$. The next result is very useful.

Lemma 2. *Let f be piecewise continuous on a contour $\mathcal{C}$ given by $z(t)$, $t \in [a, b]$, having length $L = \int_a^b |z'(t)| \mathrm{d}t$. Then*
$$\left| \int_{\mathcal{C}} f(z) \mathrm{d}z \right| \leq L \sup_{z \in \mathcal{C}} |f(z)|.$$

Remark that since f is piecewise continuous on the closed set $\mathcal{C}$, f is bounded on $\mathcal{C}$, so the supremum is finite.

Proof. First we notice that for $g : [a, b] \to \mathbb{C}$ we have $\left| \int_a^b g(t) \mathrm{d}t \right| \leq \int_a^b |g(t)| \mathrm{d}t$. To show this we write the integral in polar coordinates, $\int_a^b g(t) \mathrm{d}t = r \mathrm{e}^{\mathrm{i}\theta}$, where $\theta \in \mathbb{R}$ and $r \geq 0$. Then
$$r = \int_a^b \mathrm{e}^{-\mathrm{i}\theta} g(t) \mathrm{d}t = \mathrm{Re} \int_a^b \mathrm{e}^{-\mathrm{i}\theta} g(t) \mathrm{d}t = \int_a^b \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\theta} g(t)) \mathrm{d}t.$$

Now $\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\theta} g(t)) \leq |\mathrm{e}^{-\mathrm{i}\theta} g(t)| = |g(t)|$, so the desired formula follows from the monotonicity of real (Riemann) integrals. Therefore,
$$\begin{aligned} \left| \int_{\mathcal{C}} f(z) \mathrm{d}z \right| &= \left| \int_a^b f(z(t)) z'(t) \mathrm{d}t \right| \\ &\leq \int_a^b |f(z(t))| \, |z'(t)| \mathrm{d}t \\ &\leq \sup_{t \in [a,b]} |f(z(t))| \int_a^b |z'(t)| \mathrm{d}t \\ &= L \sup_{z \in \mathcal{C}} |f(z)|. \end{aligned}$$

□

Example 12. It is not true that $\left|\int_{\mathcal{C}} f(z)\mathrm{d}z\right| \le \int_{\mathcal{C}} |f(z)|\mathrm{d}z$. (Note that the r.h.s. is in general a complex number, so the inequality does not even make sense, in general.) For instance, let $f(z) = 1/z$, $\mathcal{C} = \{z \in \mathbb{C} : \ |z| = 1\}$. Then, parametrizing the circle as $z(t) = \mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$, we obtain

$$\int\limits_{\mathcal{C}} \frac{\mathrm{d}z}{z} = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}t}\mathrm{e}^{\mathrm{i}t}\mathrm{i}\mathrm{d}t = 2\pi\mathrm{i},$$

but on the other hand,

$$\int\limits_{\mathcal{C}} |f(z)|\mathrm{d}z = \int\limits_0^{2\pi} \mathrm{i}\mathrm{e}^{\mathrm{i}t}\mathrm{d}t = \mathrm{i}\frac{\mathrm{e}^{2\pi\mathrm{i}} - 1}{\mathrm{i}} = 0.$$

Let F be analytic on a domain D. We say that F is the *anti-derivative* of f on D if $f(z) = F'(z)$ on D.

Theorem 3. *Suppose that f is continuous on a domain D. Then the following are equivalent*

(1) *f has an anti-derivative F on D.*

(2) *Take any points $\zeta_1, \zeta_2 \in D$. The integral of f along any contour linking ζ_1 and ζ_2, lying inside D, has the same value.*

(3) *The integral of f around any closed contour lying inside D is zero.*

If the integrals of contours linking ζ_1 and ζ_2 are independent of the paths, as it is the case in the theorem, then we simply write $\int_{\zeta_1}^{\zeta_2} \mathrm{d}z$ for that integral.

Proof. We show (1) ⇒ (2) ⇒ (3) ⇒ (2) ⇒ (1).

To show (1) ⇒ (2) we let $\mathcal{C}$ be a contour, $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n$, where the $\mathcal{C}_j$ are smooth arcs, linking points z_j to z_{j+1} as in Figure 7, so that

$$\int_{\mathcal{C}} f(z)\mathrm{d}z = \sum_{j=1}^{n} \int_{\mathcal{C}_j} f(z)\mathrm{d}z,$$

and $z_1 = \zeta_1$, $z_{n+1} = \zeta_2$.

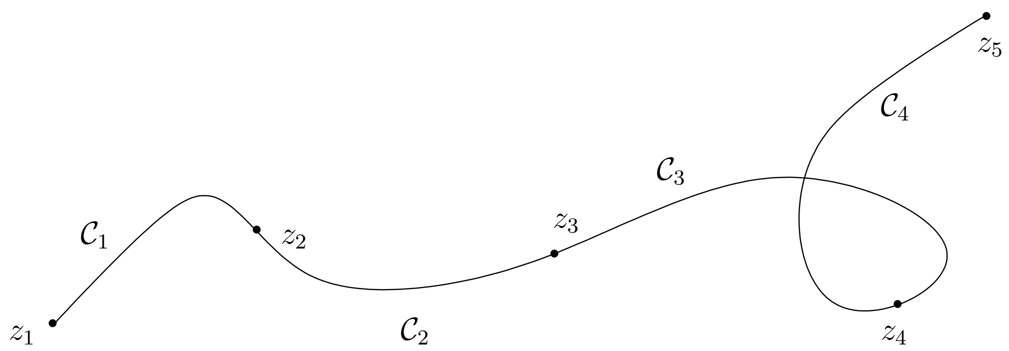


Figure 7: The contour $\mathcal{C} = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n$

Let us analyze one smooth arc $\mathcal{C}_j$, represented by $z(t)$, $t \in [a,b]$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t),$$

from which it follows that

$$\int_{\mathcal{C}_j} f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}F(z(t))\mathrm{d}t.$$

Let us decompose F into real and imaginary parts, $F(z) = U(z) + \mathrm{i}V(z)$, $U,V : \mathbb{C} \to \mathbb{R}$. Then $\frac{\mathrm{d}}{\mathrm{d}t}F(z(t)) = \frac{\mathrm{d}}{\mathrm{d}t}U(z(t)) + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}V(z(t))$, so by the fundamental theorem of calculus,

$$\int_{\mathcal{C}_j} f(z)\mathrm{d}z = \\ U(z(b)) - U(z(a)) + \mathrm{i}V(z(b)) - \mathrm{i}V(z(a)) = F(z(b)) - F(z(a)) = F(z_{j+1}) - F(z_j).$$

Consequently we obtain

$$\int_{\mathcal{C}} f(z)\mathrm{d}z = \sum_{j=1}^{n}[F(z_{j+1}) - F(z_j)] = F(z_{n+1}) - F(z_1).$$

This shows that the value of the integral is just the difference of the values the anti-derivative takes at the end points of the contour. In particular, it does not depend on the path linking z_1 to z_{n+1}.

To show (2) $\Rightarrow$ (3) we pick two points z_1, z_2 on the closed contour $\mathcal{C}$ (see Figure 8).

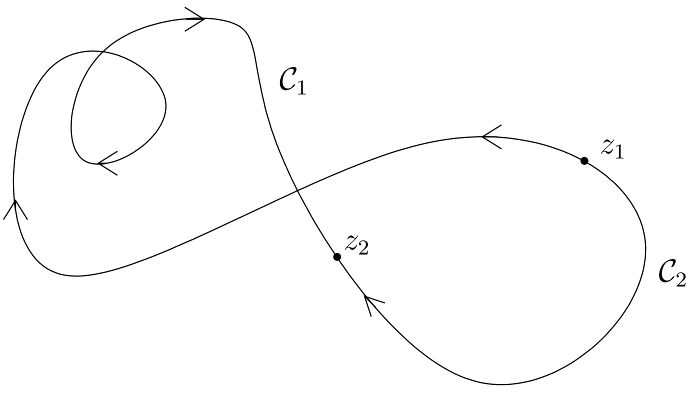


Figure 8: Dividing $\mathcal{C}$ into two contours

We define two contours $\mathcal{C}_1$ and $\mathcal{C}_2$, each linking z_1 to z_2, such that $\mathcal{C}_1 \cup (-\mathcal{C}_2) = \mathcal{C}$. Then

$$\int_{\mathcal{C}} f(z)\mathrm{d}z = \int_{\mathcal{C}_1} f(z)\mathrm{d}z - \int_{\mathcal{C}_2} f(z)\mathrm{d}z.$$

Now the right hand side equals zero by (2).

To show (3) $\Rightarrow$ (2) we let z_1, z_2 be two points in D, and $\mathcal{C}_1$, $\mathcal{C}_2 \subset D$ be contours linking z_1 and z_2. The contour $\mathcal{C} = \mathcal{C}_1 \cup (-\mathcal{C}_2)$ is a closed contour lying inside D, so we have

$$\int_{\mathcal{C}} f(z)\mathrm{d}z = 0 = \int_{\mathcal{C}_1} f(z)\mathrm{d}z - \int_{\mathcal{C}_2} f(z)\mathrm{d}z,$$

which means that (2) holds.

To prove (2) $\Rightarrow$ (1) we fix $z_0 \in D$ and define $F(z) = \int_{z_0}^{z} f(\zeta)\mathrm{d}\zeta$ for $z \in D$. We now show that F is an anti-derivative of f, that is, F is analytic in D and $F'(z) = f(z)$, for all $z \in D$. Fix $z \in D$ and take Δz so small that the ball of radius $|\Delta z|$ around z is in D (this is possible since D is open). We have

$$F(z+\Delta z) - F(z) = \int_{z}^{z+\Delta z} f(\zeta)\mathrm{d}\zeta,$$

where we may choose the path in the integral to be the straight line linking z and $z + \Delta z$. Since $\int_z^{z+\Delta z} \mathrm{d}\zeta = \Delta z$ we have $f(z) = \frac{1}{\Delta z}\int_z^{z+\Delta z} f(z)\mathrm{d}\zeta$, and hence

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(\zeta) - f(z)]\mathrm{d}\zeta.$$

The function f is continuous at z, so for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|\zeta - z| < \delta$, then $|f(\zeta) - f(z)| < \epsilon$. Therefore,

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| \leq \left|\frac{1}{\Delta z}\right| \epsilon |\Delta z| = \epsilon,$$

whenever $|\Delta z| < \delta$. This means that F is differentiable at z, and that $F'(z) = f(z)$, for all $z \in D$. $\square$

The proof of the previous theorem yields also the following result.

Theorem 4 (Fundamental theorem of calculus)**.** *Let $\mathcal{C}$ be a contour lying inside a domain D, and suppose that f is continuous on D and has an anti-derivative F on D. Let z_1, z_2 denote the endpoints of $\mathcal{C}$. Then we have*

$$\int_{\mathcal{C}} f(z)\mathrm{d}z = F(z_2) - F(z_1).$$

Corollary 1. *Suppose f is a function satisfying $f'(z) = 0$ for all z in a domain D. Then f is constant on D.*

Proof. The function f' is continuous on D and has the anti-derivative f on D. Thus by Theorem 4 we have, for any $z_1, z_2 \in D$, $\int_{\mathcal{C}} f'(z)\mathrm{d}z = f(z_2) - f(z_1)$, where $\mathcal{C}$ is any contour linking z_1 to z_2 and lying in D. Since the integral is zero ($f'(z) = 0$) we have $f(z_1) = f(z_2)$ for all $z_1, z_2 \in D$. $\square$

Example 13. Evaluate $\int_{\mathcal{C}} z^{-2}\mathrm{d}z$, where $\mathcal{C} = \{z \in \mathbb{C} : \ |z| = 1\}$. Let D be the annulus $D = \{r\mathrm{e}^{\mathrm{i}\theta} : \ 0 < a < r < b < 2, \ \theta \in \mathbb{R}\}$. On this domain, $f(z) = z^{-2}$ has the anti-derivative $F(z) = -1/z$. Moreover, $\mathcal{C} \subset D$ and f is continuous on D. Thus, by the theorem, $\int_{\mathcal{C}} z^{-2}\mathrm{d}z = 0$. We can verify this explicitly:

$$\int_0^{2\pi} \mathrm{e}^{-2\mathrm{i}t}\mathrm{i}\mathrm{e}^{\mathrm{i}t}\mathrm{d}t = \mathrm{i}\frac{\mathrm{e}^{2\pi\mathrm{i}} - 1}{\mathrm{i}} = 0.$$

Example 14. Evaluate $\int_{\mathcal{C}} z^{-1}\mathrm{d}z$, where $\mathcal{C} = \{z \in \mathbb{C} : \ |z| = 1\}$. It is easy to evaluate the integral directly, it has the value $2\pi\mathrm{i}$. Thus $\int_{\mathcal{C}} z^{-1}\mathrm{d}z \neq 0$. This is so because there is no domain containing $\mathcal{C}$, and throughout which the integrand is the continuous derivative of an analytic function. Indeed, the derivative of $\mathrm{Log}(z)$ is $1/z$, but Log fails to be analytic along the branch cut given by $z = x \in (-\infty, 0]$. (Note that any branch cut must contain the origin which is a branch point.) However, *we can still use the fundamental theorem of calculus to evaluate the integral*! The trick is to "open" slightly the contour in order to avoid the branch cut (in this example the negative real axis, see Figure 9). Formally, we may set

$$\mathcal{C}_\epsilon = \{z \in \mathbb{C} : \ z = \mathrm{e}^{\mathrm{i}\theta}, \theta \in (-\pi + \epsilon, \pi - \epsilon)\}.$$

Then we have $\int_{\mathcal{C}} \mathrm{d}z/z = \lim_{\epsilon \to 0} \int_{\mathcal{C}_\epsilon} \mathrm{d}z/z$, since the integrand is bounded on $\mathcal{C}$.

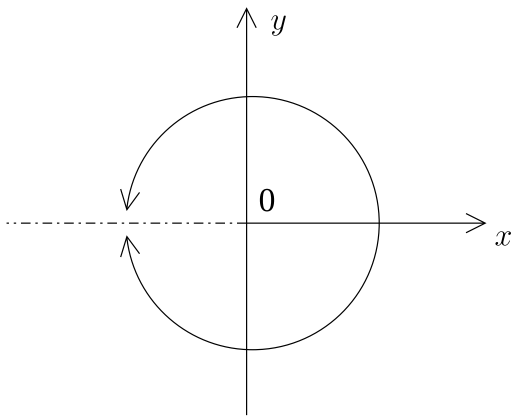


Figure 9: Opening the contour

For any $\epsilon > 0$ we can find a domain D_ϵ containing $\mathcal{C}_\epsilon$, but not intersecting the negative real axis. Now, on D_ϵ the function f has the anti-derivative Log, and thus we obtain

$$\int_{\mathcal{C}} \frac{\mathrm{d}z}{z} = \lim_{\theta \uparrow \pi} \mathrm{Log}(z)\Big|_{z=\mathrm{e}^{-\mathrm{i}\theta}}^{z=\mathrm{e}^{\mathrm{i}\theta}} = \lim_{\theta \uparrow \pi} \big(\ln|z| + \mathrm{i}\mathrm{Arg}(z)\big)\Big|_{z=\mathrm{e}^{-\mathrm{i}\theta}}^{z=\mathrm{e}^{\mathrm{i}\theta}} = 2\pi\mathrm{i}.$$

1.6 The Cauchy-Goursat Theorem

We have seen in the previous section that if f is continuous on a domain D and has an anti-derivative on that domain, then $\int_{\mathcal{C}} f(z)\mathrm{d}z = 0$ for all closed contours $\mathcal{C}$ lying inside D.

We can arrive at the conclusion $\int_{\mathcal{C}} f(z)\mathrm{d}z = 0$ also if we assume that f is analytic in the interior and on a simple closed curve $\mathcal{C}$, and if f' is continuous there too. To see this, we recall Green's Theorem. Let $\mathcal{C}$ be a simple closed curve in $\mathbb{R}^2$ and denote by R the closed region consisting of all points interior to and on $\mathcal{C}$. If P and Q are continuously differentiable functions from R to the real numbers, then

$$\int_{\mathcal{C}} P\mathrm{d}x + Q\mathrm{d}y = \int_R [\partial_x Q - \partial_y P]\mathrm{d}x\mathrm{d}y.^1$$

We represent $f(z) = u(z) + \mathrm{i}v(z)$ (real and imaginary parts) and write $z'(t) = x'(t) + \mathrm{i}y'(t)$. Then

$$\begin{aligned}\int_{\mathcal{C}} f(z)\mathrm{d}z &= \int_a^b f(z(t))z'(t)\mathrm{d}t \\ &= \int_a^b [u(z(t))x'(t) - v(z(t))y'(t)]\mathrm{d}t + \mathrm{i}\int_a^b [u(z(t))y'(t) + v(z(t))x'(t)]\mathrm{d}t.\end{aligned}$$

Now $\int_a^b u(x(t), y(t))x'(t)\mathrm{d}t = \int_{\mathcal{C}} u(x, y)\mathrm{d}x$ and similarly for the other integrals. Hence

$$\begin{aligned}\int_{\mathcal{C}} f(z)\mathrm{d}z &= \int_{\mathcal{C}} [u(x, y)\mathrm{d}x - v(x, y)\mathrm{d}y] + \mathrm{i}\int_{\mathcal{C}} [v(x, y)\mathrm{d}x + u(x, y)\mathrm{d}y] \\ &= \int_R [-\partial_x v(x, y) - \partial_y u(x, y)]\mathrm{d}x\mathrm{d}y + \mathrm{i}\int_R [\partial_x u(x, y) - \partial_y v(x, y)]\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where we used Green's theorem in the last step. The r.h.s. is zero by the Cauchy-Riemann equations. Thus, we have the following result, known as the Cauchy Theorem.

Suppose that f is analytic at all points interior to and on a simple closed contour $\mathcal{C}$, and that f' is continuous in that same region. Then we have $\int_{\mathcal{C}} f(z)\mathrm{d}z = 0$.

The next theorem gives the same assertion without the assumption that f' is continuous.

Theorem 5 (Cauchy-Goursat Theorem)**.** *Suppose that f is analytic at all points interior to and on a simple closed contour $\mathcal{C}$. Then*

$$\int_{\mathcal{C}} f(z)\mathrm{d}z = 0.$$

We start with a preparatory result which will be useful in the proof of the Cauchy-Goursat Theorem.

Lemma 3. *Let $\mathcal{C}$ be a simple closed contour and denote by R the closed region consisting of points lying interior to and on $\mathcal{C}$. Let f be analytic on R. Then, given any $\epsilon > 0$, we can cover R with finitely many squares (and partial squares), indexed by $j = 1, 2, \dots, n$, such that in each one there is a point z_j for which we have*

$$\left|\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)\right| < \epsilon,$$

for all others points z in the same square (partial square).

[1]Recall that if $\mathcal{C}$ is represented by $(x(t), y(t))$ with $t \in [a, b]$, then $\int_{\mathcal{C}} P\mathrm{d}x = \int_a^b P(x(t), y(t))x'(t)\mathrm{d}t$.

By a partial square we mean one that does not lie entirely inside $\mathcal{C}$, but is "cut" by $\mathcal{C}$, see Figure 10.

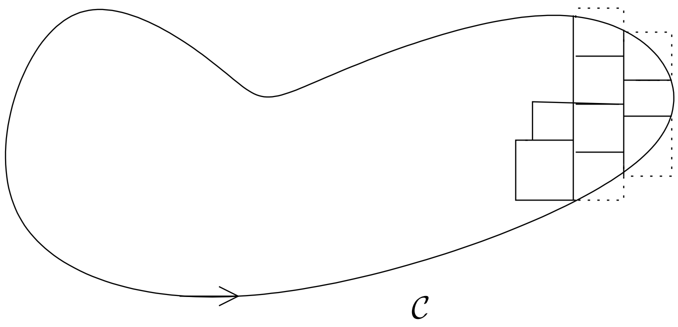


Figure 10: Squares and partial squares

Proof. Start off with a subdivision of R in squares and partial squares. Suppose that in a given square (or partial square), there is no point z_j satisfying the inequality. Denote this (partial) square by σ_0. Now we subdivide σ_0 into four smaller squares by cutting each side in two (Figure 11).

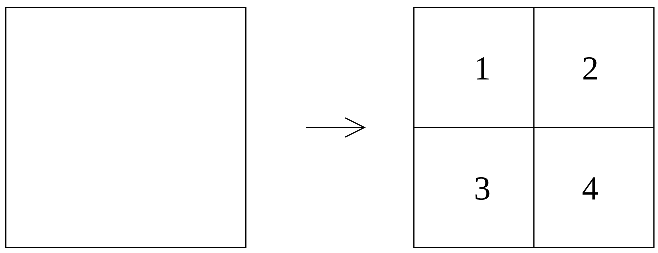


Figure 11: Subdivisions of a square

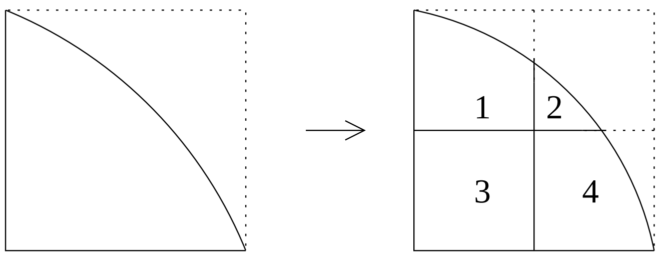
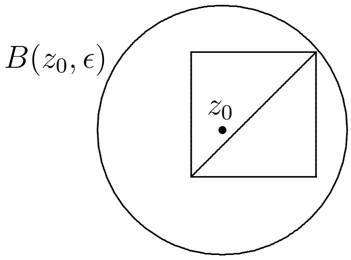


Figure 12: Subdivisions of a partial square

If σ_0 is a partial square, then we subdivide it as indicated in Figure 12. If all four new subregions contain points z_j such that the inequality of the lemma is satisfied in each new (partial) square, then we stop the treatment of σ_0. Suppose that one of the four new subregions does not have a point z_j satisfying the inequality. Call this subregion (square or partial square) σ_1. We subdivide σ_1 in the same fashion we did σ_0. Like this we get a sequence of subregions $\sigma_0, \sigma_1, \sigma_2, \ldots$. We now show that after a finite number of steps in this process, there must be a z_j satisfying the inequality.

Suppose that there is a an infinite sequence of σ_k. The set $\cap_{k=0}^{\infty}\sigma_k$ is not empty, as the σ_k are decreasing and compact. Let $z_0 \in \cap_{k=0}^{\infty}\sigma_k$. Since $\sigma_k \subset R$, the intersection is a subset of R, and so $z_0 \in R$.



The function f is analytic at z_0 (since $z_0 \in R$), so $f'(z_0)$ exists, which means that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $0 \neq |z - z_0| < \delta$ then

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon.$$

However, $\{z \in \mathbb{C} : \; |z - z_0| < \delta\} = B(z_0, \delta)$ contains a square σ_k with index k large enough, so that the diagonal of σ_k is less than δ. Thus, z_0 serves as the point z_j in the lemma for the square (or partial square) σ_k.

We thus conclude that there is no infinite sequence $\sigma_0 \supset \cdots \supset \sigma_k \supset \sigma_{k+1} \supset \cdots$ with the property that none of the σ_k contain a point z_j satisfying the inequality of the lemma. $\square$

Proof of the Cauchy-Goursat Theorem. Fix ϵ and consider a cover of R satisfying the property of Lemma 3. On the j^{th} (partial) square, we define the function

$$\delta_j(z) = \begin{cases} \frac{f(z)-f(z_j)}{z-z_j} - f'(z_j), & z \neq z_j, \\ 0, & z = z_j, \end{cases}$$

where z_j is as in Lemma 3. In particular, we have $|\delta_j(z)| < \epsilon$. Clearly, $\delta_j(z)$ is continuous and $\lim_{z \to z_j} \delta_j(z) = 0$.

Denote by $\mathcal{C}_j$, $j = 1, 2, \ldots, n$, the boundary of the square (or partial square) being the j^{th} element in the cover of R. For all $z \in \mathcal{C}_j$ we have

$$f(z) = f(z_j) + (z - z_j)\delta_j(z) + (z - z_j)f'(z_j),$$

so

$$\int_{\mathcal{C}_j} f(z)\mathrm{d}z \;=\; [f(z_j) - z_j f'(z_j)] \int_{\mathcal{C}_j} \mathrm{d}z + f'(z_j) \int_{\mathcal{C}_j} z\mathrm{d}z + \int_{\mathcal{C}_j} (z - z_j)\delta_j(z)\mathrm{d}z.$$

The anti-derivatives of 1 and z are z and $z^2/2$, respectively (on all of $\mathbb{C}$), 1 and z are continuous, and $\mathcal{C}_j$ is a closed contour, so we have

$$\int_{\mathcal{C}_j} \mathrm{d}z = 0 = \int_{\mathcal{C}_j} z\mathrm{d}z.$$

Consequently,

$$\int_{\mathcal{C}_j} f(z)\mathrm{d}z = \int_{\mathcal{C}_j} (z - z_j)\delta_j(z)\mathrm{d}z.$$

The sum of all the integrals $\int_{\mathcal{C}_j} f(z)\mathrm{d}z$, $j = 1, \ldots, n$, is just equal to $\int_{\mathcal{C}} f(z)\mathrm{d}z$, because the integration over all sides of squares and partial squares inside $\mathcal{C}$ cancel each other out (see also Figure 13). Note that this is the reason for covering R with (non-overlapping) squares and partial squares (triangles would be possible too, for instance).

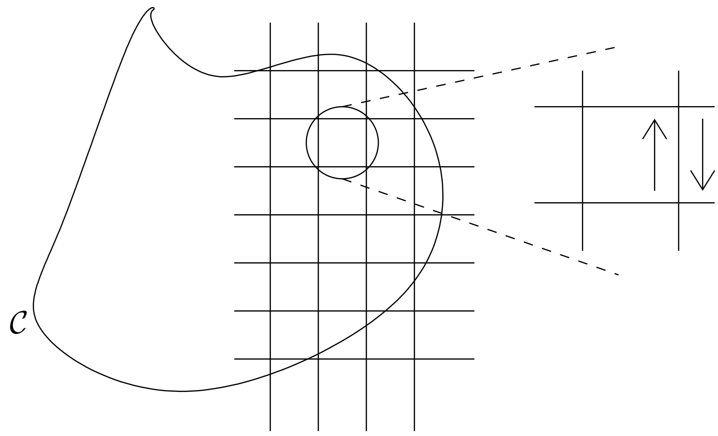


Figure 13: Cancellation of integrals on sides of squares inside $\mathcal{C}$

Therefore, we have

$$\int_{\mathcal{C}} f(z)\mathrm{d}z = \sum_{j=1}^{n} \int_{\mathcal{C}_j} f(z)\mathrm{d}z = \sum_{j=1}^{n} \int_{\mathcal{C}_j} (z - z_j)\delta_j(z)\mathrm{d}z.$$

Denote by s_j the length of a side of the square (or partial square) indexed by j. Every z in that square satisfies $|z - z_j| \leq \sqrt{2}s_j$ and so we get

$$\left|\int_{\mathcal{C}} f(z)\mathrm{d}z\right| \leq \sum_{j=1}^{n} \left|\int_{\mathcal{C}_j} (z - z_j)\delta_j(z)\mathrm{d}z\right| \leq \sum_{j=1}^{n} \sqrt{2}s_j\epsilon L(\mathcal{C}_j),$$

where we used Lemma 2, and where $L(\mathcal{C}_j)$ is the length of the contour $\mathcal{C}_j$. If $\mathcal{C}_j$ is a square, then $L(\mathcal{C}_j) = 4s_j$. If $\mathcal{C}_j$ is a partial square, then $L(\mathcal{C}_j) \leq 4s_j + L_j$, where L_j is the length of the part of $\mathcal{C}$ which is part of $\mathcal{C}_j$. In any event, we have $L(\mathcal{C}_j) \leq 4s_j + L_j$ and it follows that

$$\left|\int_{\mathcal{C}} f(z)\mathrm{d}z\right| < \sqrt{2}\epsilon \sum_{j=1}^{n} (4s_j^2 + s_jL_j).$$

Let S be a large enough number so that the entire curve $\mathcal{C}$ lies inside an (open) square of side length S, and such that all squares of the initial covering lie also inside this square. Comparing areas, we obtain the inequality $\sum_{j=1}^{n} s_j^2 \leq S^2$. Moreover, $\sum_{j=1}^{n} s_jL_j \leq S\sum_{j=1}^{n} L_j = SL(\mathcal{C})$, where $L(\mathcal{C})$ is the length of $\mathcal{C}$. So we finally arrive at the estimate

$$\left|\int_{\mathcal{C}} f(z)\mathrm{d}z\right| \leq \sqrt{2}\big(4S^2 + SL(\mathcal{C})\big)\epsilon.$$

Since ϵ is arbitrary, the left hand side must vanish. This completes the proof of the Cauchy-Goursat theorem. $\square$

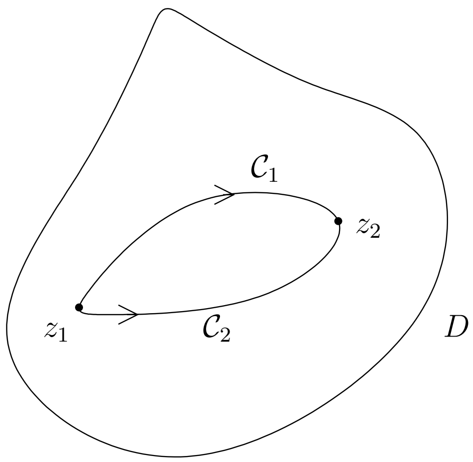
A domain D is called *simply connected* if every simple closed contour within it only encloses points of D. The unit disk $\{z \in \mathbb{C} : \ |z| < 1\}$ is simply connected, while the punctured disk $\{z \in \mathbb{C} : \ 0 \neq |z| < 1\}$ is not.

An easy consequence of the Cauchy-Goursat theorem is the following useful result.

Corollary 2. *Let D be a simply connected domain, z_1, $z_2 \in D$, and let $\mathcal{C}$ be a contour linking z_1 and z_2, lying entirely in D. If f is analytic on D, then $\int_{\mathcal{C}} f(z)\mathrm{d}z = \int_{z_1}^{z_2} f(z)\mathrm{d}z$ is independent of $\mathcal{C}$. In particular, f has an anti-derivative on D.*

The condition that D be simply connected is crucial for this result to hold (see example 15 below).

Proof of Corollary 2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be non-intersecting contours linking z_1 and z_2, and such that $\mathcal{C}_1$, $\mathcal{C}_2 \subset D$.



Then $\mathcal{C} = \mathcal{C}_2 \cup (-\mathcal{C}_1)$ is a simple closed contour lying in D, and f is analytic inside and on $\mathcal{C}$. By the Cauchy-Goursat theorem, we have $\int_{\mathcal{C}} f(z)\mathrm{d}z = 0$, so $\int_{\mathcal{C}_1} f(z)\mathrm{d}z = \int_{\mathcal{C}_2} f(z)\mathrm{d}z$. Finally, since f is analytic on D it is continuous on D, so the existence of an anti-derivative follows from Theorem 3. (In fact the anti-derivative can be expressed as $F(z) = \int_{z_0}^{z} f(\zeta)\mathrm{d}\zeta$, where the integral is over any contour linking a fixed base point z_0 to z, lying in D.) $\square$

Example 15. Let $D = \{\frac{1}{2} < |z| < 2\}$ and $f(z) = 1/z$. Then f is analytic in D. We introduce the unit circle $\mathcal{C}$ and the arcs $\mathcal{C}_1$ and $\mathcal{C}_2$, both linking $-\mathrm{i}$ to i (see Figure 14).

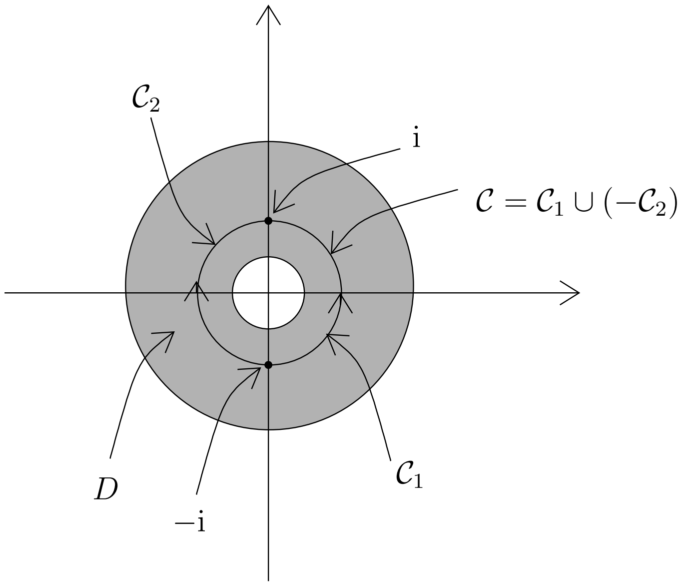


Figure 14: The annulus D is not simply connected

By direct calculation we find $\int_{|z|=1} \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i}$ and therefore $\int_{\mathcal{C}_1} \frac{\mathrm{d}z}{z} - \int_{\mathcal{C}_2} \frac{\mathrm{d}z}{z} = 2\pi\mathrm{i}$. Therefore, the integrals over $\mathcal{C}_1$ and $\mathcal{C}_2$ do not have the same values.

Example 16. (Application of the Cauchy-Goursat theorem.) Consider the contour $\mathcal{C}$ consisting of two half-circles and two intervals, as represented in Figure 15, and let $f(z) = \mathrm{e}^{\mathrm{i}z}/z$.

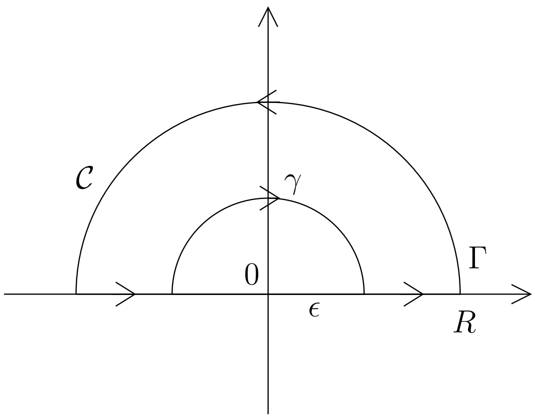


Figure 15: Contour $\mathcal{C}$

We know from the Cauchy-Goursat theorem that $\int_{\mathcal{C}} f(z)\mathrm{d}z = 0$. Parameterize the big half-circle Γ as $R\mathrm{e}^{\mathrm{i}\theta}$, $0 \le \theta \le \pi$. Then

$$\left|\int_\Gamma f(z)\mathrm{d}z\right| \le \int_0^\pi \left|\mathrm{e}^{\mathrm{i}R\mathrm{e}^{\mathrm{i}\theta}}\right| \mathrm{d}\theta = \int_0^\pi \mathrm{e}^{-R\sin(\theta)}\mathrm{d}\theta \overset{R\to\infty}{\longrightarrow} 0.$$

We parametrize γ similarly and obtain

$$\int_\gamma f(z)\mathrm{d}z = -\int_0^\pi \mathrm{i}\mathrm{e}^{\mathrm{i}\epsilon\mathrm{e}^{\mathrm{i}\theta}}\mathrm{d}\theta \overset{\epsilon\to 0}{\longrightarrow} -\mathrm{i}\pi.$$

Since the integral over the whole contour vanishes, we have

$$0 = \int_{-R}^{-\epsilon} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\mathrm{d}x + \int_{\epsilon}^{R} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\mathrm{d}x - \mathrm{i}\pi + T(R,\epsilon),$$

where $|T(R,\epsilon)| \to 0$ as $R\to\infty$ and $\epsilon \to 0$. We take the imaginary part of the last equation to arrive at

$$\int_{-R}^{-\epsilon} \frac{\sin(x)}{x}\mathrm{d}x + \int_{\epsilon}^{R} \frac{\sin(x)}{x}\mathrm{d}x = \pi - \mathrm{Im}T(R,\epsilon).$$

Finally we take $R\to\infty$ and $\epsilon\to 0$, and obtain

$$\int_0^\infty \frac{\sin(x)}{x}\mathrm{d}x = \frac{\pi}{2}.$$

The last integral is understood as the limit of $\int_\epsilon^R \sin(x)/x\,\mathrm{d}x$ as $R\to\infty$ and $\epsilon\to 0$ (independently).

Another simple but important consequence of the Cauchy-Goursat theorem is the following "deformation lemma."

Lemma 4. (Deformation Lemma.) *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be (positively oriented) simple closed contours, such that $\mathcal{C}_2$ is in the interior of $\mathcal{C}_1$. If f is analytic in the closed region consisting of those contours and all points between them, then*

$$\int_{\mathcal{C}_1} f(z)\mathrm{d}z = \int_{\mathcal{C}_2} f(z)\mathrm{d}z.$$

This result says in particular that any simple closed contour enclosing just one singular point of the integrand can be replaced by a circle.

Proof. We introduce two additional contours and obtain two simple closed contours $\mathcal{C}_1'$ and $\mathcal{C}_2'$ such that f is analytic inside and on each of $\mathcal{C}_1'$, $\mathcal{C}_2'$, see Figure 16.

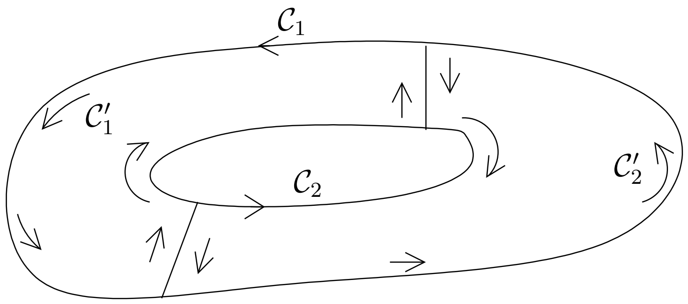


Figure 16: Contours $\mathcal{C}_j$, $\mathcal{C}_j'$

We have

$$\int_{\mathcal{C}_1'\cup\mathcal{C}_2'} f(z)\mathrm{d}z = \int_{\mathcal{C}_1\cup(-\mathcal{C}_2)} f(z)\mathrm{d}z = \int_{\mathcal{C}_1} f(z)\mathrm{d}z - \int_{\mathcal{C}_2} f(z)\mathrm{d}z.$$

However, by the Cauchy-Goursat theorem, the left hand side is zero. □

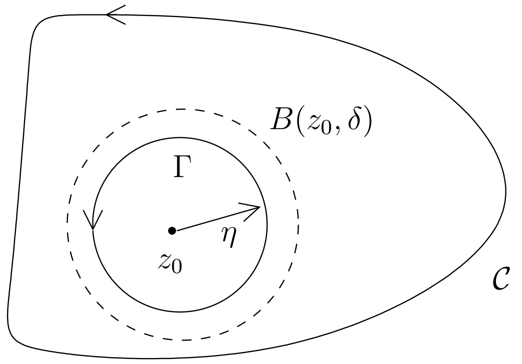
2 Cauchy's Integral Formula

This important formula expresses the value of an analytic function at a point z in terms of the integral on a contour around z.

Theorem 6. *Let $\mathcal{C}$ be a simple closed contour (positively oriented) and suppose that f is analytic inside and on $\mathcal{C}$. If z_0 is any point interior to $\mathcal{C}$, then*

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \frac{f(z)}{z - z_0} \mathrm{d}z.$$

Proof. The function f is continuous at z_0 (in fact, it is differentiable there). Given any $\epsilon > 0$ there exists a $\delta' > 0$ such that if $|z - z_0| < \delta'$ then $|f(z) - f(z_0)| < \epsilon$. Choose $\delta \leq \delta'$ such that $B(z_0, \delta)$ is contained in the interior of $\mathcal{C}$.



By Lemma 4, we have

$$\int_{\mathcal{C}} \frac{f(z)}{z - z_0} \mathrm{d}z = \int_{\Gamma} \frac{f(z)}{z - z_0} \mathrm{d}z,$$

where $\Gamma = \{z \in \mathbb{C} : \ |z| = \eta\}$, for any $\eta < \delta$. Since $\int_\Gamma (z - z_0)^{-1} \mathrm{d}z = 2\pi\mathrm{i}$ we have

$$\int_{\mathcal{C}} \frac{f(z)}{z - z_0} \mathrm{d}z - 2\pi\mathrm{i} f(z_0) = \int_{\mathcal{C}} \frac{f(z)}{z - z_0} \mathrm{d}z - f(z_0) \int_{\Gamma} \frac{\mathrm{d}z}{z - z_0} = \int_{\Gamma} \frac{f(z) - f(z_0)}{z - z_0} \mathrm{d}z.$$

For all $z \in \Gamma$ we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \frac{\epsilon}{|z - z_0|} = \frac{\epsilon}{\eta}.$$

It follows that

$$\left| \int_{\mathcal{C}} \frac{f(z)}{z - z_0} \mathrm{d}z - 2\pi\mathrm{i} f(z_0) \right| < \frac{\epsilon}{\eta} L(\Gamma) = 2\pi\epsilon.$$

Since $\epsilon > 0$ is arbitrary the result follows. $\square$

Example 17. Consider the unit circle $\mathcal{C}$, parameterized as $z(\theta) = \mathrm{e}^{\mathrm{i}\theta}$, $-\pi < \theta \leq \pi$. By the Cauchy integral formula we have $\int_{\mathcal{C}} \frac{\mathrm{e}^{az}}{z} \mathrm{d}z = 2\pi\mathrm{i}$, for all $a \in \mathbb{C}$. On the other hand, $\int_{\mathcal{C}} \frac{\mathrm{e}^{az}}{z} \mathrm{d}z = \mathrm{i} \int_{-\pi}^{\pi} \mathrm{e}^{a(\cos(\theta) + \mathrm{i}\sin(\theta))} \mathrm{d}\theta$, so by taking the imaginary part we obtain, for all $a \in \mathbb{R}$,

$$\int_0^\pi \mathrm{e}^{a\cos(\theta)} \cos(a \sin(\theta)) \mathrm{d}\theta = \pi.$$

By formally differentiating the equality

$$f(z) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

we get

$$f'(z) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta,$$

and continuing this procedure,

$$f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta.$$

The next result shows that this formal procedure leads to a correct result.

Theorem 7 (Cauchy Integral Formula for Derivatives)**.** *Let $\mathcal{C}$ be a simple closed contour (positively oriented) and suppose that f is analytic inside and on $\mathcal{C}$. If z is any point inside $\mathcal{C}$, then $f^{(n)}(z)$ exists and is given by*

$$f^{(n)}(z) = \frac{n!}{2\pi\mathrm{i}} \int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta,$$

for all $n \in \mathbb{N}$.

An immediate corollary is the following very important result.

Corollary 3. *If a function f is analytic at a point, then its derivatives of all orders at this point exist and are also analytic at this point.*

Proof of Corollary 3. Suppose f is analytic at z_0. This means that there is an $\epsilon > 0$ s.t. f is differentiable at all points inside $B(z_0, \epsilon)$. Let $\mathcal{C} = \{z \in \mathbb{C} : \ |z - z_0| = \epsilon/2\}$. Then f is analytic inside and on $\mathcal{C}$, hence by the Cauchy Integral Formula, $f^{(n)}(z)$ exists for all $z \in B(z_0, \epsilon/2)$ and all $n \in \mathbb{N}$. $\square$

The following is a (partial) converse to the Cauchy-Goursat theorem.

Theorem 8 (Morera's Theorem)**.** *Let f be continuous on a domain D. If $\int_{\mathcal{C}} f(z)\mathrm{d}z = 0$ for every closed contour $\mathcal{C}$ lying in D, then f is analytic on D.*

Proof of Theorem 8. Since f is continuous and the integral over every closed contour lying in D vanishes, f has an anti-derivative F on D. This means there exists an F analytic on D such that $F'(z) = f(z)$, for all $z \in D$. By the previous corollary f is analytic at each point of D. $\square$

Proof of Theorem 7. We consider first the case $n = 1$. Take $h \in \mathbb{C}$ such that both z and $z + h$ lie inside $\mathcal{C}$. Then

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \frac{f(\zeta)}{h} \left[\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z} \right] \mathrm{d}\zeta \\ &= \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} \mathrm{d}\zeta \equiv \frac{1}{2\pi\mathrm{i}} I_1. \end{aligned}$$

Our goal is to show that

$$\lim_{h\to 0} I_1 = I_2 \equiv \int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta-z)^2}\mathrm{d}\zeta.$$

We have

$$\begin{aligned}
|I_1 - I_2| &= \left|\int_{\mathcal{C}} \frac{f(\zeta)}{\zeta - z}\left[\frac{1}{\zeta-z-h} - \frac{1}{\zeta-z}\right]\mathrm{d}\zeta\right| \\
&= \left|\int_{\mathcal{C}} \frac{f(\zeta)}{\zeta-z}\frac{h}{(\zeta-z-h)(\zeta-z)}\mathrm{d}\zeta\right| \\
&\le |h| L(\mathcal{C}) \sup_{\zeta\in\mathcal{C}}\left(\left|\frac{f(\zeta)}{(\zeta-z)^2}\right|\frac{1}{|\zeta-z-h|}\right).
\end{aligned}$$

Let $d = \mathrm{dist}(z,\mathcal{C}) = \min\{|z-\zeta| :\ \zeta\in\mathcal{C}\}$ be the distance of z to the contour $\mathcal{C}$. Since z is in the interior of $\mathcal{C}$, we have $d>0$. Now for $\zeta\in\mathcal{C}$, $|\zeta-z-h|\ge|\zeta-z|-|h|>\frac{d}{2}$, provided $|h|<\frac{d}{2}$. Therefore,

$$|I_1-I_2|\le 2|h|L(\mathcal{C})d^{-3}\sup_{\zeta\in\mathcal{C}}|f(\zeta)|,$$

provided $|h|<\frac{d}{2}$. It follows that $\lim_{h\to0}|I_1-I_2|=0$.

We use an induction argument to show that the formula is correct for all n. Assume that it is correct for $n=1,2,\dots,k$. Thus we have

$$\begin{aligned}
\frac{f^{(k)}(z+h)-f^{(k)}(z)}{h} &= \frac{k!}{2\pi\mathrm{i}}\int_{\mathcal{C}}\frac{f(\zeta)}{h}\left[\frac{1}{(\zeta-z-h)^{k+1}}-\frac{1}{(\zeta-z)^{k+1}}\right]\mathrm{d}\zeta \\
&= \frac{k!}{2\pi\mathrm{i}}\int_{\mathcal{C}}\frac{f(\zeta)}{h}\frac{(\zeta-z)^{k+1}-(\zeta-z-h)^{k+1}}{(\zeta-z-h)^{k+1}(\zeta-z)^{k+1}}\mathrm{d}\zeta \\
&\equiv \frac{k!}{2\pi\mathrm{i}}I_1.
\end{aligned}$$

Our goal is to show that the Cauchy integral formula holds for $k+1$, which is equivalent to showing $\lim_{h\to0}|I_1-I_2|=0$, where

$$I_2\equiv(k+1)\int_{\mathcal{C}}\frac{f(\zeta)}{(\zeta-z)^{k+2}}\mathrm{d}z.$$

In order to expand the difference in the second fraction of the integrand defining I_1, we note that for all x,ϵ,

$$(x+\epsilon)^N-x^N = x\left[(1+\epsilon/x)(x+\epsilon)^{N-1}-x^{N-1}\right]=\epsilon(x+\epsilon)^{N-1}+x\big[(x+\epsilon)^{N-1}-x^{N-1}\big],$$

from which it follows by iteration that

$$(x+\epsilon)^N-x^N=\epsilon\sum_{s=1}^N x^{s-1}(x+\epsilon)^{N-s}.$$

Hence we have

$$\begin{aligned}
\frac{1}{h}\big[(\zeta-z)^{k+1}-(\zeta-z-h)^{k+1}\big] &= \frac{1}{-h}\big[(\zeta-z+(-h))^{k+1}-(\zeta-z)^{k+1}\big] \\
&= \sum_{s=1}^{k+1}(\zeta-z)^{s-1}(\zeta-z-h)^{k+1-s}.
\end{aligned}$$

We estimate the difference of I_1 and I_2 as follows.

$$\begin{aligned}
|I_1-I_2| &= \left|\int_{\mathcal{C}} f(\zeta)\left[\frac{k+1}{(\zeta-z)^{k+2}}-\frac{\sum_{s=1}^{k+1}(\zeta-z)^{s-1}(\zeta-z-h)^{k+1-s}}{(\zeta-z-h)^{k+1}(\zeta-z)^{k+1}}\right]\mathrm{d}\zeta\right| \\
&\le L(\mathcal{C})\max_{\zeta\in\mathcal{C}}|f(\zeta)|\max_{\zeta\in\mathcal{C}}\frac{1}{|\zeta-z|^{k+1}}\left|\frac{k+1}{\zeta-z}-\sum_{s=1}^{k+1}(\zeta-z)^{s-1}(\zeta-z-h)^{-s}\right| \\
&\le \frac{L(\mathcal{C})\max_{\zeta\in\mathcal{C}}|f(\zeta)|}{d^{k+2}}\max_{\zeta\in\mathcal{C}}\left|k+1-\sum_{s=1}^{k+1}\left[\frac{\zeta-z}{\zeta-z-h}\right]^s\right|.
\end{aligned}$$

Note that for each $\zeta\in\mathcal{C}$ fixed, the last sum converges to $k+1$ as $h\to 0$. However, we need to show that this convergence is *uniform* in $\zeta\in\mathcal{C}$.

$$\begin{aligned}
k+1-\sum_{s=1}^{k+1}\left[\frac{\zeta-z}{\zeta-z-h}\right]^s &= \sum_{s=1}^{k+1}\left\{1-\left[\frac{\zeta-z}{\zeta-z-h}\right]^s\right\} \\
&= \sum_{s=1}^{k+1}\left\{1-\left[1+\frac{h}{\zeta-z-h}\right]^s\right\} \\
&= -\sum_{s=1}^{k+1}\frac{h}{\zeta-z-h}\sum_{s'=1}^{s}\left[1+\frac{h}{\zeta-z-h}\right]^{s-s'}.
\end{aligned}$$

In the last step, we have again used the above formula for $(x+\epsilon)^N-x^N$. We arrive thus at the upper bound

$$\left|k+1-\sum_{s=1}^{k+1}\left[\frac{\zeta-z}{\zeta-z-h}\right]^s\right|\le\frac{2|h|}{d}\sum_{s=1}^{k+1}\sum_{s'=1}^{s}\left|1+\frac{h}{\zeta-z-h}\right|^{s-s'}\le\frac{2^{k+2}|h|(k+1)^2}{d},$$

provided $|h|\le d/2$, because then $\left|1+\frac{h}{\zeta-z-h}\right|\le 1+\frac{2|h|}{d}\le 2$.

This shows that $\lim_{h\to 0}|I_1-I_2|=0$, and the proof of Theorem 7 is complete. □

Example 18. Let $\mathcal{C}$ be a simple closed contour enclosing the origin. Then

$$\int_{\mathcal{C}}\frac{\sinh(z)}{z^3}\mathrm{d}z=\frac{2\pi\mathrm{i}}{2!}\sinh''(0)=0.$$

Theorem 9 (Liouville Theorem). *If f is entire and $|f(z)|\le M$ for some $M<\infty$ and all $z\in\mathbb{C}$, then f is constant.*

Proof. From Cauchy's formula, we have

$$f'(z)=\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}}\frac{f(\zeta)}{(\zeta-z)^2}\mathrm{d}\zeta,$$

where $\mathcal{C}$ is the circle of radius r, centered at z. Thus $|f'(z)|\le\frac{1}{2\pi}\frac{M}{r^2}2\pi r=\frac{M}{r}$. We can choose r as large as we wish, so $f'(z)=0$, for all $z\in\mathbb{C}$. This implies that f is constant, as can be seen e.g. by the fundamental theorem of calculus:

$$\int_{z_0}^{z}f'(\zeta)\mathrm{d}\zeta=f(z)-f(z_0)=0,$$

for all $z,z_0\in\mathbb{C}$. □

The next result says that if f is analytic (and not constant) in a domain D, then the maximum of the function $|f|$ can only be attained on the boundary of D.

Theorem 10 (Maximum modulus principle). *Suppose that f is analytic and non-constant in a domain D. Then $|f(z)|$ has no maximum value in D, that is, there is no point $z_0 \in D$ such that $|f(z)| \le |f(z_0)|$, for all $z \in D$.*

We can rephrase this result as a positive statement. Suppose that f is analytic on a domain D whose closure K is compact, and suppose that $|f|$ is continuous on K. Then $|f|$ has a maximum on K. Thus we have the following result.

Corollary 4. *Suppose that f is analytic in a domain D whose closure K is compact, and suppose that $|f|$ is continuous on K. Then the maximum value of $|f(z)|$, for $z \in K$, occurs on the boundary $K\backslash D$.*

A proof of Theorem 10 is quite easy once we have the following result of independent interest.

Lemma 5. *Suppose that f is analytic in $B(z_0, r)$ and that $|f(z)| \le |f(z_0)|$ for all $z \in B(z_0, r)$. Then $f(z) = f(z_0)$ throughout $B(z_0, r)$.*

Proof of Theorem 10. Suppose that $|f(z)|$ attains its maximum at $z_0 \in D$, and let the maximum of $|f|$ be M, $|f(z_0)| = M$. Let $\zeta \in D$ be arbitrary. We show that $f(z_0) = f(\zeta)$. To do so, we choose a polygonal line $\mathcal{L}$ linking z_0 to ζ, lying entirely in D. The distance from $\mathcal{L}$ to the boundary of D is strictly positive, $d = \inf\{|z - z'| : \ z \in \mathcal{L}, z' \in \mathbb{C}\backslash D\} > 0$. We can cover the polygonal line $\mathcal{L}$ with balls of radius d and centres $c_1, \ldots, c_N \in \mathcal{L}$, with $|c_j - c_{j-1}| < d$, s.t. $c_1 = z_0$ and $c_N = \zeta$, and such that $\mathcal{L} \subset \cup_{j=1}^N B(c_j, d) \subset D$. Lemma 5 tells us that $f(z) = f(z_0)$ throughout $B(c_1, d)$. Now take $z \in B(c_2, d)$. Then, since $f(z_0)$ is the maximum througout D, and since $f(z_1) = f(z_0)$, we have $|f(z)| \le |f(z_1)|$, for all $z \in B(c_2, d)$. So by Lemma 5, $f(z) = f(z_0)$ for all $z \in B(c_1, d) \cap B(c_2, d)$. We continue this argument and see that f is constant on the union of all the balls. In particular, we have $f(z_0) = f(\zeta)$. □

Proof of Lemma 5. Take $z_1 \in B(z_0, r)$, s.t. $\rho = |z_0 - z_1| > 0$. We want to show that $f(z_1) = f(z_0)$. By the Cauchy formula, we have

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}_\rho} \frac{f(z)}{z - z_0} \mathrm{d}z = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho \mathrm{e}^{\mathrm{i}\phi})\, \mathrm{d}\phi,$$

where $\mathcal{C}_\rho$ is the circle around z_0 with radius ρ. It follows that

$$|f(z_0)| \le \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho \mathrm{e}^{\mathrm{i}\phi})|\, \mathrm{d}\phi.$$

On the other hand, since z_0 is the point where $|f|$ is maximal, we have

$$|f(z_0)| \ge \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \rho \mathrm{e}^{\mathrm{i}\phi})|\, \mathrm{d}\phi,$$

and consequently, $|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+\rho e^{i\phi})|\, d\phi$, which in turn is equivalent to

$$\int_0^{2\pi} \big[|f(z_0)| - |f(z_0+\rho e^{i\phi})|\big]\, d\phi = 0.$$

Since the integrand is a non-negative continuous function of ϕ, it follows that it must vanish at all points $\phi \in [0, 2\pi]$. Consequently we have $|f(z_0)| = |f(z)|$, for all z s.t. $|z-z_0| = \rho$. In particular, $|f(z_0)| = |f(z_1)|$. Since z_1 was chosen arbitrary ($z_1 \neq z_0$) we see that $|f|$ must be constant througout $B(z_0, r)$.

Finally, we need to show that since $|f|$ is constant, f itself is constant. If $|f(z)| = 0$ then $f(z) = 0$, so we may assume that $|f(z)|$ is constant and nonzero. Then $|f(z)|^2 = c > 0$, so $\overline{f}(z) = c/f(z)$, and in particular, both f and $\overline{f}$ are analytic in $B(z_0, r)$. Let $f = u + iv$ and $\overline{f} = U + iV$. The Cauchy-Riemann equations are satisfied for both f and $\overline{f}$, and they give

$$\partial_x u = \partial_y v,\ \partial_y u = -\partial_x v, \quad \text{and} \quad \partial_x U = \partial_y V,\ \partial_y U = -\partial_x V.$$

But since $U = u$ and $V = -v$, we can add up the equations to obtain $\partial_x u = 0 = \partial_y u$. This implies that u is constant, and hence so is v. □

3 Power Series

Let $f_n : A \to \mathbb{C}$, $n = 1, 2, \ldots$, be a function sequence defined on a set $A \subset \mathbb{C}$. Suppose that for each $z \in A$, $\lim_{n\to\infty} f_n(z)$ exists, and denote this limit by $f(z)$. The function f on A is called the *pointwise limit* of the sequence f_n. We say the sequence $\{f_n\}$ *converges uniformly* to its pointwise limit f if and only if for all $\epsilon > 0$ there exists an N such that $|f_n(z) - f(z)| < \epsilon$ whenever $n > N$, for all $z \in A$. (We write also "$f_n \to f$ uniformly".) Here, N depends on ϵ (in general), but it does not depend on z. It is clear that f_n converges to f uniformly on A if and only if

$$\lim_{n\to\infty} \sup_{z\in A} |f_n(z) - f(z)| = 0.$$

The following result is easy to prove, just like in the case for real valued functions.

Theorem 11. *Suppose f_n converges uniformly to f on a set $A \subset \mathbb{C}$. If each f_n is continuous on A, then so is f.*

Uniform convergence plays a central role in deciding whether one can interchange limits and integrals, or limits and infinite sums, or infinite sums and integrals, etc.

Theorem 12. *Let $\mathcal{C}$ be a contour. Suppose that $f_n \to f$ uniformly on $\mathcal{C}$, and that each f_n is continuous on $\mathcal{C}$. Then*

$$\lim_{n\to\infty} \int_{\mathcal{C}} f_n(z) dz = \int_{\mathcal{C}} f(z) dz.$$

Proof. We know from the previous theorem that f is continuous on $\mathcal{C}$. Therefore $\int_{\mathcal{C}} f(z)dz$ is well defined. Fix $\epsilon > 0$. There exists an $N = N(\epsilon)$ such that $|f_n(z) - f(z)| < \epsilon$, whenever $n > N$, for all $z \in \mathcal{C}$. Thus,

$$\left|\int_{\mathcal{C}} f_n(z)dz - \int_{\mathcal{C}} f(z)dz\right| = \left|\int_{\mathcal{C}} \big[f_n(z) - f(z)\big]dz\right| \leq \epsilon L(\mathcal{C}),$$

whenever $n > N$. □

The next result shows that analyticity is preserved under uniform limits.

Theorem 13. *Let D be a simply connected domain, and suppose that f_n is a function sequence s.t. f_n is analytic in D for each n, and s.t. $f_n \to f$ uniformly on each compact subset of D. Then f is analytic on D, and $f_n' \to f'$ uniformly on each compact subset of D.*

Proof. We first show that f is analytic in D. Let $\mathcal{C}$ be any closed contour in D. From Theorem 12 we have

$$\int_{\mathcal{C}} f(z)\mathrm{d}z = \lim_{n\to\infty} \int_{\mathcal{C}} f_n(z)\mathrm{d}z.$$

But each f_n is analytic in the simply connected domain D and thus, by Corollary 2 the integral of f_n along the closed contour $\mathcal{C}$ vanishes for each n. Consequently $\int_{\mathcal{C}} f(z)\mathrm{d}z = 0$ for any closed contour $\mathcal{C}$ in D. Since f is continuous on D (it being a uniform limit of continuous functions), Morera's theorem yields that f is analytic on D.

Next, we show that $f_n' \to f'$ uniformly on compact sets in D. We first show this on disks and then for compact sets. Let $z_0 \in D$ and let $r > 0$ be such that $B(z_0, 2r) \subset D$. Let $\mathcal{C}$ be a circle of radius $\rho < r$, centered at z_0. By Cauchy's integral formula for derivatives, we have for all $z \in B(z_0, \rho/2)$,

$$|f'(z) - f_n'(z)| = \frac{1}{2\pi}\left|\int_{\mathcal{C}} \frac{f(\zeta) - f_n(\zeta)}{(\zeta - z)^2}\mathrm{d}\zeta\right| \le \frac{4}{\rho}\sup_{\zeta\in\mathcal{C}} |f(\zeta) - f_n(\zeta)|.$$

Let now $K \subset D$ be compact. Choose for each $z \in K$ a $\rho_z > 0$ such that $B(z, \rho_z) \subset D$, and cover the set K as $K \subset \bigcup_{z\in K} B(z, \rho_z/2) \subset D$. By compactness, this cover has a finite refinement: $K \subset \bigcup_{j=1}^J B(z_j, \rho_{z_j}/2) \subset D$. We have shown above that for every $\epsilon > 0$ fixed, and for each $j = 1, 2, \ldots, J$, there exists an $N_j(\epsilon)$ such that if $n > N_j(\epsilon)$, then $|f'(z) - f_n'(z)| < \epsilon$ for all $z \in B(z_j, \rho_{z_j}/2)$. Let $N = \max_{1\le j\le J} N_j(\epsilon)$. For $n \ge N$ we have $|f'(z) - f_n'(z)| < \epsilon$ for all $z \in \bigcup_{j=1}^J B(z_j, \rho_{z_j}/2)$. But the latter union contains K, so $f_n' \to f'$ uniformly on K. □

Let z_n be a sequence in $\mathbb{C}$ and set $S_k = \sum_{n=1}^k z_n$. Just as for real series, we call S_k the k-th partial sum of the infinite series $z_1 + z_2 + \cdots$. We say that the infinite series *converges* if the complex sequence S_k has a limit S, as $k \to \infty$. In this case we write $S = \sum_{n=1}^\infty z_n$. We say the series *diverges* if S_k diverges. The following theorem collects some basic facts which are very easy to prove.

Theorem 14. *Suppose that $\sum_{n=1}^\infty z_n$ and $\sum_{n=1}^\infty \zeta_n$ converge. Then*

1. *The series $\sum_{n=1}^\infty (\lambda z_n)$ converges and has the value $\lambda \sum_{n=1}^\infty z_n$, for all $\lambda \in \mathbb{C}$.*
2. *The series $\sum_{n=1}^\infty (z_n + \zeta_n)$ converges and has the value $\sum_{n=1}^\infty z_n + \sum_{n=1}^\infty \zeta_n$.*
3. *$z_n \to 0$ as $n \to \infty$.*

Note that an easy proof of 3. is obtained by observing that $z_n = S_n - S_{n-1}$. It is also very easy to see that $\sum_{n=1}^\infty z_n$ converges to S if and only if $\sum_{n=1}^\infty \mathrm{Re}(z_n)$ converges to $\mathrm{Re}(S)$ and $\sum_{n=1}^\infty \mathrm{Im}(z_n)$ converges to $\mathrm{Im}(S)$. We say that the series $\sum_{n=1}^\infty z_n$ *converges absolutely* if and only if $\sum_{n=1}^\infty |z_n|$ converges.

A *series of functions* (or function series) $\sum_{n=1}^{\infty} f_n(z)$ is said to *converge pointwise* to the function $F(z)$ on a set $A \subset \mathbb{C}$ if the sequence of partial sums $S_k(z) = \sum_{n=1}^{k} f_n(z)$ converges pointwise, that is, if for all $z \in A$ fixed, $\lim_{k\to\infty} S_k(z) = F(z)$. The function series $\sum_{k=1}^{\infty} f_n(z)$ is said to *converge uniformly* to $F(z)$ on a set $A \subset \mathbb{C}$ if the sequence of partial sum functions converges uniformly to F, that is, if $\lim_{k\to\infty} \sup_{z\in A} |S_k(z) - F(z)| = 0$.

Examples 19. 1. If $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly to $F(z)$, for $z \in A \subset \mathbb{C}$, and if each f_n is continuous on A, then F is continuous on A. This follows simply from Theorem 11.

2. If $\sum_{n=1}^{\infty} f_n(z)$ converges to $F(z)$ uniformly on a contour $\mathcal{C}$, and if each f_n is continuous on $\mathcal{C}$, then by Theorem 12,

$$\int_{\mathcal{C}} F(z)\mathrm{d}z = \int_{\mathcal{C}} \sum_{n=1}^{\infty} f_n(z)\mathrm{d}z = \sum_{n=1}^{\infty} \int_{\mathcal{C}} f_n(z)\mathrm{d}z.$$

3. If $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly to $F(z)$ in any compact subset of a simply connected domain D, and if each f_n is analytic in D, then F is analytic in D, and $F'(z) = \sum_{n=1}^{\infty} f_n'(z)$, the convergence being uniform on compacta in D. This follows from Theorem 13.

The following result gives a very useful criterion for absolute convergence of a function series.

Theorem 15 (Weierstrass M-test)**.** *Suppose that $f_n : A \to \mathbb{C}$ satisfies $|f_n(z)| \le M_n$, for all $n = 1, 2, \ldots$, and for all $z \in A$, and suppose that $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely and uniformly on A.*

Proof. Set $S_k = \sum_{n=1}^{k} |f_n(z)|$. Then we have, for $k' > k$,

$$|S_{k'} - S_k| = \sum_{n=k+1}^{k'} |f_n(z)| \le \sum_{n=k+1}^{k'} M_n.$$

It follows that S_k is a Cauchy sequence since the tail of the convergent series on the r.h.s. tends to zero as $k \to \infty$. Therefore $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely. Call the (pointwise) limit function $F(z)$. We have

$$F(z) - \sum_{n=1}^{k} f_n(z) = \sum_{n=k+1}^{\infty} f_n(z),$$

from which it follows that

$$\left| F(z) - \sum_{n=1}^{k} f_n(z) \right| \le \sum_{n=k+1}^{\infty} |f_n(z)| \le \sum_{n=k+1}^{\infty} M_n.$$

Once again, the r.h.s. converges to zero as $k \to \infty$, and it does so uniformly in z. □

Example 20. Evaluate the integral

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{e}^{-\mathrm{i}\varphi}/R}}{R^n \mathrm{e}^{\mathrm{i}n\varphi}} \mathrm{d}\varphi,$$

where $R > 0$ is fixed, and $n \in \mathbb{Z}$. To do this, we expand the double exponential as $\mathrm{e}^{\mathrm{e}^{-\mathrm{i}\varphi}/R} = \sum_{k\geq 0} \frac{\mathrm{e}^{-\mathrm{i}k\varphi}}{k!R^k}$. By the Weierstrass M-test, this series converges uniformly in $\varphi \in \mathbb{R}$. Therefore we have

$$a_n = \sum_{k\geq 0} \frac{1}{k!R^{n+k}} \, \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}(k+n)\varphi} \mathrm{d}\varphi = \sum_{k\geq 0} \frac{\delta_{k,-n}}{k!},$$

where $\delta_{k,-n}$ is the Kronecker symbol. Hence $a_n = 0$ for $n \geq 1$ and $a_n = 1/(-n)!$ for $n \leq 0$.

A *power series* is a series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, where $a_n, z_0 \in \mathbb{C}$, and where z belongs to some subset of $\mathbb{C}$. By a shift of the variable, $w = z - z_0$, the power series takes the form $\sum_{n=0}^{\infty} a_n w^n$.

Recall the definition of the limit superior for a sequence of real numbers,

$$\limsup_{n\to\infty} x_n \equiv \overline{\lim}_{n\to\infty} x_n := \lim_{n\to\infty} \sup_{k\geq n} x_k.$$

It is well known from basic real analysis that $\limsup_{n\to\infty} x_n$ can be characterized equivalently as follows. Let S be the set of all accumulation points of the set $\{x_1, x_2, \ldots\}$, i.e., S is the set of all limit points of all convergent subsequences of $\{x_n\}$. Then $\limsup_{n\to\infty} x_n = \sup S$ (the least upper bound on S). From this relation it also immediately follows that if $\lim_{n\to\infty} x_n = x$ exists, then $\limsup_{n\to\infty} x_n = x$.

The following elementary properties are often useful. Suppose that $\overline{\lim}_{n\to\infty} x_n = L$ for some $-\infty < L < \infty$. Then we have:

1. For all $\epsilon > 0$ there exists N such that $x_k \leq L + \epsilon$, for all $k > N$.

2. For all N, for all $\epsilon > 0$, there exists $k > N$ such that $x_k \geq L - \epsilon$.

3. $\overline{\lim}_{n\to\infty} c x_n = cL$, for all $c \geq 0$.

Theorem 16. *Let a_n be a sequence of complex numbers, and set $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = L$. Then*

1. *If $L = 0$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all z.*

2. *If $L = \infty$, then $\sum_{n=0}^{\infty} a_n z^n$ converges for $z = 0$ only.*

3. *If $0 < L < \infty$ then set $R := 1/L$. The series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$ and diverges for $|z| > R$.*

The number R is called the *radius of convergence* of the series $\sum_{n=0}^{\infty} a_n z^n$. Our convention is to allow the values $R = 0, \infty$ (situations 2. and 1. of the theorem).

Proof. We just write $\overline{\lim}$ instead of $\overline{\lim}_{n\to\infty}$ in this proof.

1. Since $\overline{\lim}\, |a_n|^{1/n} = 0$ we have $\overline{\lim}\, |a_n|^{1/n} |z| = 0$ for all z. Thus $\overline{\lim}\, |a_n z^n|^{1/n} = 0$, so there is an N such that $|a_n z^n|^{\frac{1}{n}} < 1/2$ whenever $n > N$. Hence $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely since $\sum_{n=0}^{\infty} 2^{-n}$ converges.

2. If $z \neq 0$, then $|a_n|^{1/n} \geq 1/|z|$ for infinitely many values of n. Thus we have $|a_n z^n|^{1/n} \geq 1$ for infinitely many n, so $|a_n z^n|$ cannot converge to zero. It follows that the series cannot converge.

3. Assume first that $|z| < R$. Then $|z| = R(1-\epsilon)$ for some $\epsilon > 0$. It follows that $\overline{\lim}\,|a_n|^{1/n}|z| = \frac{1}{R}R(1-\epsilon) = 1-\epsilon$, so we have $|a_n|^{1/n}|z| = |a_n z^n|^{1/n} < 1-\epsilon/2$, for n sufficiently large. Thus $|a_n z^n| < (1-\epsilon/2)^n$ for n sufficiently large, and thus $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely (compare to the geometric series).

   Now assume that $|z| > R$. Then $\overline{\lim}\,|a_n|^{1/n}|z| > 1$, so that for infinitely many values of n, we have $|a_n z^n|^{1/n} > 1$. Hence the series does not converge.

□

The last assertion of Theorem 16 shows that a power series converges for all z such that $|z| < R$ and diverges for all z such that $|z| > R$, hence the name "radius of convergence" for R. For points z such that $|z| = R$, the series may or may not converge.

Example 21. Suppose that $\sum_{n\geq 0} a_n z^n$ converges for $\alpha < |z| < \beta$, some $\alpha > 0$. Then the series converges absolutely for all $|z| < \beta$. Indeed, we have $|a_n z^n| \to 0$ for all $\alpha < |z| < \beta$. In particular, for $|z| = \beta - \epsilon$ we get $|a_n| \leq (\beta-\epsilon)^{-n}$, provided n is large enough. It follows that $|a_n z^n| < [|z|/(\beta-\epsilon)]^n$ for n large enough. Given $|z| < \beta$ we can find ϵ s.t. $|z|/(\beta-\epsilon) < 1$. It follows that for all $|z| < \beta$, the series $\sum_{n\geq 0} a_n z^n$ converges absolutely.

Examples 22. 1. Consider $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$. Here, $L = \overline{\lim}\,|a_n|^{1/n} = \overline{\lim}\, n^{-2/n} = \overline{\lim}\, \mathrm{e}^{-2\log(n)/n} = 1$. So the radius of convergence is $R = 1/L = 1$. For $|z| = 1$, the series converges absolutely as well. (Compare it to the series with general term n^{-2}.)

2. For $\sum_{n=1}^{\infty} \frac{z^n}{n}$ we have $L = \overline{\lim}\, n^{-1/n} = 1$, so the radius of convergence is again $R = 1$. However, for $z = 1$, the series diverges. Observe that this is the derivative of the previous series, and the latter did converge for $z = 1$. One can show that for $z = \mathrm{e}^{\mathrm{i}\varphi}$, $\varphi \notin 2\pi\mathbb{Z}$, the series converges, by looking at real and imaginary parts separately. To do this, we first recall Dirichlet's Test: *Let $\sum_{n=1}^{\infty} a_n$ be a series with bounded partial sums, and let $\{b_n\}$ be a monotone decreasing sequence such that $b_n \to 0$. Then $\sum_{n=1}^{\infty} a_n b_n$ converges.* The trigonometric identity

$$\sum_{k=1}^{n} \sin(kx) = \frac{\cos(\frac{x}{2}) - \cos[(n+\frac{1}{2})x]}{2\sin(\frac{x}{2})},$$

valid for $x \neq 0, \pm 2\pi, \pm 4\pi, \ldots$ implies that $|\sum_{k=1}^{n} \sin(kx)| \leq |\sin(\frac{x}{2})|^{-1}$. Moreover obviously $\sin(kx) = 0$ for $x = 0, \pm 2\pi, \pm 4\pi, \ldots$ Thus the series $\sum_{n=1}^{\infty} \sin(nx)$ has bounded partial sums for all x and consequently, $\sum_{n=1}^{\infty} \frac{1}{n}\sin(n\varphi)$ converges for all φ by Dirichlet's test. Similarly, one can use the identity

$$\sum_{k=1}^{n} \cos(kx) = \frac{\sin[(n+\frac{1}{2})x] - \sin(\frac{x}{2})}{2\sin(\frac{x}{2})},$$

valid for $x \neq 0, \pm 2\pi, \pm 4\pi, \ldots$ to see that $\sum_{n=1}^{\infty} \frac{1}{n}\cos(n\varphi)$ converges for all φ. Hence we have shown that $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for all $|z| = 1$ except for $z = 1$.

Example 23. Consider the geometric series $\sum_{n\geq 0} z^n$. The radius of convergence is $R = 1$, and for any $|z| = 1$, the series diverges, because if $|z| = 1$, then z^n cannot converge to zero, as $n \to \infty$. Also, just as in the case of the real geometric series, one easily finds the partial sums explicitly, $S_k = \frac{1-z^{k+1}}{1-z}$. Hence we get $\lim_{k\to\infty} S_k = \sum_{n\geq 0} z^n = (1-z)^{-1}$, for z s.t. $|z| < 1$. Do the partial sum functions $S_k(z)$ converge to the limit function $(1-z)^{-1}$ uniformly in z? We estimate the difference

$$\left| S_k(z) - \frac{1}{1-z} \right| = \frac{1}{|1-z|}|z|^{k+1}.$$

For every $k \in \mathbb{N}$ we can find a z with modulus sufficiently close to, but strictly less than 1, such that $\frac{1}{|1-z|}|z|^{k+1} > 1$. This means that S_k does not converge to its limit uniformly in $\{z \in \mathbb{C} : \ |z| < 1\}$. However, let $0 < \rho < 1$ be fixed, then for $z \in B(0, \rho)$, we have

$$\left| S_k(z) - \frac{1}{1-z} \right| \leq \frac{\rho^{k+1}}{1-\rho},$$

and the r.h.s. converges to zero as $k \to \infty$, uniformly in $z \in B(0, \rho)$. Hence the geometric series converges uniformly on any $B(0, \rho)$, with $0 < \rho < R = 1$.

The next result shows that the situation illustrated in the previous example is generic.

Theorem 17. *Let $R > 0$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$. Then the series converges absolutely and uniformly on the set $|z| \leq \rho$, for any $0 < \rho < R$.*

Proof. Let $f_n(z) = a_n z^n$, then $|f_n(z)| \leq M_n := |a_n|\rho^n$ for all z with $|z| \leq \rho$. We have $\overline{\lim}\, M_n^{1/n} = \rho\, \overline{\lim} |a_n|^{1/n} = \rho/R < 1$, so $\sum_{n=0}^{\infty} M_n$ converges (by the root test for real numerical series). The Weierstrass M-test implies that $\sum_{n=0}^{\infty} f_n(z) = \sum_{n=0}^{\infty} a_n z^n$ converges absolutely and uniformly on $|z| \leq \rho < R$. □

It follows now directly from Theorem 12 that summation and integration can be interchanged.

Theorem 18. *Suppose that $\sum_{n=0}^{\infty} a_n z^n$ converges to $f(z)$ for $|z| < R$. Let $\mathcal{C}$ be any contour lying in the interior of the circle $|z| = R$. Then*

$$\int_{\mathcal{C}} f(z)\mathrm{d}z = \int_{\mathcal{C}} \sum_{n=0}^{\infty} a_n z^n \mathrm{d}z = \sum_{n=0}^{\infty} a_n \int_{\mathcal{C}} z^n \mathrm{d}z.$$

Example 24. Suppose the series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ has radius of convergence $R > 0$, and let $\mathcal{C}$ be a simple contour, $\mathcal{C} \subset B(z_0, R)$. Then we have $\int_{\mathcal{C}} \sum_{n=0}^{\infty} a_n (z-z_0)^n \mathrm{d}z = \sum_{n=0}^{\infty} a_n \int_{\mathcal{C}} (z-z_0)^n \mathrm{d}z = 0$. Since the series is continuous in z in the domain $|z - z_0| < R$, Morera's theorem implies that the series defines an analytic function in this domain. We now give an alternative proof of the analyticity of power series.

Theorem 19. *Suppose that $\sum_{n=0}^{\infty} a_n z^n$ converges to $f(z)$, with radius of convergence R. Then f is analytic inside $|z| < R$, and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges to $f'(z)$, and has the same radius of convergence R.*

Proof. Fix z_0 with $|z_0| < 1$. There is an $\epsilon > 0$ s.t. $|z_0| = R - \epsilon$. Let $\rho := R - \epsilon/2$. Since each $F_N(z) = \sum_{n=0}^N a_n z^n$ is analytic in $|z| \le \rho$ and, by Theorem 17, F_N to converges uniformly in $|z| \le \rho$ to f, the latter function is analytic in $|z| \le \rho$ (Theorem 13). This shows that f is analytic at any point $z_0 \in B(0, R)$.

Next, we know from Theorem 13 that $F_N'(z)$ converges to $F'(z)$ uniformly on compact sets inside $|z| < R$, and that on each such compact set,

$$\sum_{n=1}^{\infty} n a_n z^{n-1} = \frac{\mathrm{d}}{\mathrm{d}z}\left(\sum_{n=0}^{\infty} a_n z^n\right).$$

Finally we show that the radii of convergence of $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1} z^n$ are the same. We have

$$\begin{aligned}
\overline{\lim}\,|(n+1)a_{n+1}|^{1/n} &= \overline{\lim}\,|a_{n+1}|^{1/n} \\
&= \overline{\lim}\,|a_n|^{\frac{1}{n}\cdot\frac{n}{n-1}} \\
&= \overline{\lim}\,|a_n|^{1/n}.
\end{aligned}$$

In the first step, we use that $\lim_{n\to\infty}(n+1)^{1/n} = 1$, and in the last step we use the following two facts. Let $L = \overline{\lim}\,|a_n|^{1/n}$, then

1. For all $\epsilon > 0$ there exists N such that $|a_n|^{1/n} < L + \epsilon$ for all $n > N$. If follows that $\overline{\lim}\,|a_n|^{\frac{1}{n}\cdot\frac{n}{n-1}} \le \overline{\lim}\,(L+\epsilon)^{\frac{n}{n-1}} = L+\epsilon$. Since $\epsilon > 0$ is arbitrary, we get $\overline{\lim}\,|a_n|^{\frac{1}{n}\cdot\frac{n}{n-1}} \le L$.

2. For all $\epsilon > 0$ and all N, there exists $n > N$ such that $|a_n|^{1/n} > L - \epsilon$. Consequently, in view of the characterization of the limit superior in terms of subsequential limit points, $\overline{\lim}\,|a_n|^{\frac{1}{n}\cdot\frac{n}{n-1}} \ge \overline{\lim}\,(L-\epsilon)^{\frac{n}{n-1}} = L - \epsilon$. It follows that $\overline{\lim}\,|a_n|^{\frac{1}{n}\cdot\frac{n}{n-1}} \ge L$.

This completes the proof of the Theorem. □

Corollary 5. *Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has a radius of convergence $R > 0$. Then we have $a_n = f^{(n)}(0)/n!$.*

Proof. The function f is analytic on $|z| < R$, hence all derivatives of f exist on $|z| < R$. Moreover,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n z^{n-k},$$

from which we see that $f^{(k)}(0) = k!a_k$. □

How can we state the corollary for a power series centered at a general z_0, $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$? Let R be the radius of convergence of the last series. For $|z| < R$, define $g(z) = f(z+z_0) = \sum_{n\ge 0} a_n z^n$, a series with radius of convergence R. Thus we have $n!a_n = g^{(n)}(0) = f^{(n)}(z_0)$. It follows that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

The converse to the corollary is true as well, as shows the following result.

Theorem 20. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is analytic in $B(z_0, r)$, for some $r > 0$. Then we have*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

for every $z \in B(z_0, r)$. The series converges absolutely for $|z - z_0| < r$, and it converges absolutely and uniformly in any $B(z_0, \rho)$, with $\rho < r$.

Proof. We first consider $z_0 = 0$. Take $z \in B(0, r)$, then $|z| = \rho r$ for some $0 \leq \rho < 1$. Set $\mathcal{C} = \{z \in \mathbb{C} : |z| = \rho' r\}$, for a ρ' satisfying $\rho < \rho' < 1$. The circle $\mathcal{C}$ contains the point z and is contained inside $B(0, r)$, so f is analytic inside and on $\mathcal{C}$. By the Cauchy integral formula

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{f(\zeta)}{\zeta} \frac{1}{1 - z/\zeta} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{f(\zeta)}{\zeta} \sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n \mathrm{d}\zeta,$$

where we have inserted the geometric series (see also Example 23). Since z is in the interior of $\mathcal{C}$ and ζ is on $\mathcal{C}$ we have $|z/\zeta| = \rho/\rho' < 1$. Hence the geometric series converges uniformly in $\zeta \in \mathcal{C}$ (for fixed z; use e.g. the Weierstrass M-test). By Theorem 12 we can interchange the sum and the integral and we obtain

$$f(z) = \frac{1}{2\pi \mathrm{i}} \sum_{n=0}^{\infty} z^n \int_{\mathcal{C}} \frac{f(\zeta)}{\zeta^{n+1}} \mathrm{d}\zeta = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n,$$

where we use the Cauchy integral formula for derivatives in the last step.

This shows the result for $z_0 = 0$. A simple translation gives the result as stated: Assume that f analytic in $B(z_0, r)$ and set $g(z) = f(z + z_0)$, which is analytic in $B(0, r)$. Thus we have

$$f(z + z_0) = g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n, \quad \text{so} \quad f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

□

Example 25. Let $f(z) = 1/z$, $z \neq 0$. Then $f^{(n)}(z) = (-1)^n n! z^{-n-1}$, $n = 0, 1, 2, \ldots$ and in particular, $f^{(n)}(1) = (-1)^n n!$. The Taylor series of $f(z)$ around $z_0 = 1$ is therefore given by

$$1/z = \sum_{n=0}^{\infty} (-1)^n (z - 1)^n = \sum_{n=0}^{\infty} (1 - z)^n.$$

The radius of convergence is $R = 1$ and for $|z - 1| \geq 1$ the series diverges. Of course, $f(z)$ is analytic everywhere except at the origin, so according to Theorem 20, it has a power series expansion around any point $z_0 \neq 0$. It is also clear from that Theorem that the radius of convergence of the Taylor expansion around z_0 is $R = |z_0|$, since $1/z$ is analytic in $B(z_0, |z_0|)$, but not in any ball centered at z_0 with radius larger than $|z_0|$.

Example 26. Let $f(z) = (z - \mathrm{i})^{-2} \log(2z)$. What is the radius of convergence of the Taylor series representing f at the point $z_0 = 1 + \mathrm{i}$? The function is analytic except on the set $\mathcal{S} = (-\infty, 0] \cup \{\mathrm{i}\}$. The largest radius r s.t. f is analytic in $B(z_0, r)$ is thus $r = 1$. This is the radius of convergence of the Taylor series: indeed, the radius of convergence has to be at least r by Theorem 20, and on the other hand, it cannot be larger than r, since if it was, the function f would be analytic at some points in $\mathcal{S}$.

Example 27. Find a neighbourhood of $z_0 = 0$ such that $\sinh(z)$ is approximated by its lowest order approximation z, to a precision $1/100$. The Taylor series of sinh around the origin is

$$\sinh(z) = \sum_{n\geq 0} \frac{z^{2n+1}}{(2n+1)!},$$

having infinite radius of convergence (this is so because $\sinh(z)$ is analytic in $B(0, r)$ for any $r > 0$). Consequently,

$$|\sinh(z) - z| = \left|\sum_{n\geq 1} \frac{z^{2n+1}}{(2n+1)!}\right| \leq \sum_{n\geq 1} \frac{|z|^{2n+1}}{(2n+1)!} \leq |z|^3 \sum_{n\geq 1} \frac{|z|^{2n-2}}{(2n+1)!}.$$

For $|z| \leq 1$ we can bound the r.h.s. from above by $|z|^3 \sinh(1) < \frac{e}{2}|z|^3 < \frac{3}{2}|z|^3$. Therefore for $|z| < 150^{-1/3} \approx 0.19$ we have $|\sinh(z) - z| < 1/100$.

The following result gives the Taylor series of a product of analytic functions.

Theorem 21. *Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ have radii of convergence R_1 and R_2, respectively. Then we have $f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n$, where $c_n = \sum_{r=0}^{n} a_r b_{n-r}$, and the latter series has radius of convergence $R \geq \min\{R_1, R_2\}$.*

Proof. Since f and g are analytic in $B(0, \min\{R_1, R_2\})$, so is the product fg. Therefore we have $f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n$ with

$$\begin{aligned}
c_n &= \frac{1}{n!}(fg)^{(n)}(0) \\
&= \frac{1}{n!}\sum_{r=0}^{n} \binom{n}{r} f^{(r)}(0) g^{(n-r)}(0) \\
&= \frac{1}{n!}\sum_{r=0}^{n} \frac{n!}{(n-r)!r!}\, r! a_r\, (n-r)! b_{n-r} \\
&= \sum_{r=0}^{n} a_r b_{n-r}.
\end{aligned}$$

□

3.1 Laurent Series

In the previous chapter we have shown that a function which is analytic in a disk has a power series expansion there. We want to try to do the same for a function $f(z)$ which is analytic in an annulus $A(z_0; R_1, R_2) = \{z : R_1 < |z - z_0| < R_2\}$, where $0 < R_1 < R_2 < \infty$ are called the *inner and outer radius* of the annulus centered at z_0.

The main difference is that the functions now considered may have singularities in the inner circle $|z| < R_1$. For instance, consider $f(z) = z^{-n}$ for some $n \geq 1$, which is analytic in any $A(0; r, R)$, $0 < r < R < \infty$. If we were to try to find a power series expansion of f in $A(0; r, R)$, then in order to take into account the unboundedness of f at zero we

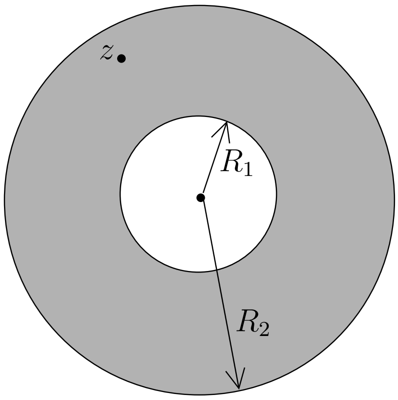


Figure 17: Annulus $A(z_0; R_1, R_2)$

should not only take positive powers of z in the series, but also negative ones. Then indeed, the power series of f is trivially given by the single term z^{-n}. It may happen that z_0 is a "stronger" singularity in the sense that a series expansion would involve all negative powers of z: consider e.g. $f(z) = \mathrm{e}^{1/z}$. From the power series expansion of the exponential we see that $f(z) = \sum_{n\geq 0} \frac{1}{n!} z^{-n}$, for all $z \neq 0$. On the other hand, if a function is analytic in a disk $B(z_0, R)$ then it is obviously analytic in the annulus $A(z_0; r, R)$, for any $0 < r < R$. In this case we know already that f does have a Taylor series expansion for all $z \in B(z_0, R)$, and hence for all $z \in A(z_0; r, R)$, any $0 < r < R$, and no negative powers show up in the series.

Let z_n, $n \in \mathbb{Z}$ be a collection of complex numbers. We say that a (two-sided) series $\sum_{n=-\infty}^{\infty} z_n$ converges if and only if both the series $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_{-n}$ converge. If these two series converge, then the value of the series $\sum_{n=-\infty}^{\infty} z_n$ is the sum $\sum_{n=0}^{\infty} z_n + \sum_{n=1}^{\infty} z_{-n}$. If either of the two (one-sided) series diverges, then we say that the two-sided series diverges. Absolute convergence and uniform convergence of a two-sided series (of complex numbers or of functions) are defined in the analogous way.

Theorem 22 (Laurent series expansion)**.** *Let f be analytic inside an annulus $A = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$, where $0 < R_1 < R_2 < \infty$. Then we have for all $z \in A$*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

and the series converges absolutely and uniformly in any closed annulus $\{z \in \mathbb{C} : \; r_1 \leq |z - z_0| \leq r_2\}$, where $R_1 < r_1 < r_2 < R_2$.

Moreover, if $\mathcal{C}$ is any simple closed contour inside the annulus encircling z_0, then the coefficients a_n, $n = 0, \pm 1, \pm 2 \ldots$, can be expressed as

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta.$$

In the Laurent expansion above, we call $\sum_{n\leq -1} a_n (z - z_0)^n$ the *principal part of the Laurent series*. The coefficient a_{-1} is called the *residue of f at z_0*, denoted by $\mathrm{Res}(f; z_0)$.

Remarks. 1. In general, the coefficients a_n in the Laurent series expansion are not given by the derivatives of f at z_0. Indeed, f may not be analytic (or even defined!) at z_0. The function f is supposed to be analytic only at points in the annulus.

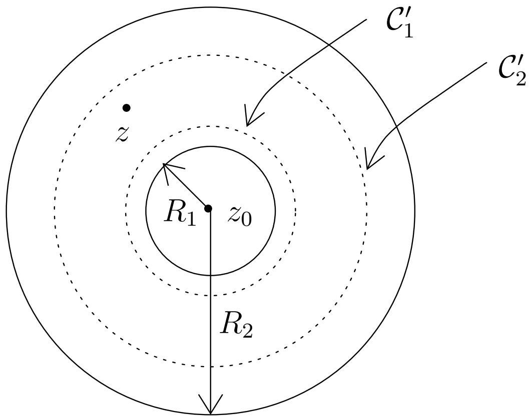
2. If f is analytic also inside the hole of the annulus, i.e., inside all of $B(z_0, R_2)$, then so is $\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}$, for $n = -1, -2, \ldots$ It follows from the Cauchy-Goursat theorem that $a_n = 0$ for all $n \le -1$, and hence the Laurent series reduces to the Taylor series.

3. The Laurent series is unique. Suppose that $f(z) = \sum_{n=-\infty}^{\infty} \alpha_n (z - z_0)^n$ converges in the interior of the annulus $\{R_1 < |z - z_0| < R_2\}$. Then it converges uniformly on all closed sets in the interior of the annulus. Let $\mathcal{C}$ be the circle around z_0 with radius $(R_2 - R_1)/2$. For all $n \in \mathbb{Z}$ we have

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \sum_{k=-\infty}^{\infty} \alpha_k \int_{\mathcal{C}} \frac{(\zeta - z_0)^k}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta.$$

Now $\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} (\zeta - z_0)^{k-n-1} \mathrm{d}\zeta = 1$ if $k = n$ and this integral vanishes if $k \neq n$. Thus it follows that $a_n = \alpha_n$, for all $n \in \mathbb{Z}$.

Proof of Theorem 22. Take $z \in A$ and let $\mathcal{C}_1'$ and $\mathcal{C}_2'$ be concentric cirlces with centre z_0 and radii r_1 and r_2, chosen so that z lies in the annulus defined by $\mathcal{C}_1'$ and $\mathcal{C}_2'$, i.e., $R_1 < r_1 < |z - z_0| < r_2 < R_2$.



The Cauchy integral formula gives

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}_2' - \mathcal{C}_1'} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}_2'} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}_1'} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta.$$

We are going to rewrite the term $(\zeta - z)^{-1}$ in the integrands in a suitable manner. For $\zeta \in \mathcal{C}_2'$ we have $|z - z_0| < |\zeta - z_0| = r_2$, and so we get

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

The series converges uniformly in ζ, for $\zeta \in \mathcal{C}_2'$. Therefore, we can integrate termwise to obtain (Theorem 12)

$$\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}_2'} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \sum_{n=0}^{\infty} (z - z_0)^n \int_{\mathcal{C}_2'} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n := \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}_2'} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta, \quad n = 0, 1, 2, \ldots$$

Similarly, for $\zeta \in \mathcal{C}_1'$, we have $r_1 = |\zeta - z_0| < |z - z_0|$, so

$$\frac{1}{\zeta - z} = -\frac{1}{z - z_0} \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = -\sum_{n=1}^{\infty} \frac{(\zeta - z_0)^{n-1}}{(z - z_0)^n}.$$

Again, the convergence is uniform in $\zeta \in \mathcal{C}_1'$ and thus

$$-\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}_1'} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta = \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n},$$

where

$$a_{-n} := \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}_1'} \frac{f(\zeta)}{(\zeta - z_0)^{-n+1}} \mathrm{d}\zeta, \quad n = 1, 2, \ldots$$

Let us consider the expressions for a_n, $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, the function $\zeta \mapsto \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}$ is analytic in the annulus defined by $\mathcal{C}_1'$ and $\mathcal{C}_2'$, and on these curves. So by the deformation lemma,

$$\int_{\mathcal{C}_{1,2}'} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta = \int_{\mathcal{C}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta,$$

where $\mathcal{C}$ is any simple closed curve around z_0 inside the annulus A. This gives the formula for a_n, as stated in the theorem.

The series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$ converge for all $z \in \{R_1 < |z| < R_2\}$, so they converge uniformly in any closed annulus $\{r_1 \le |z| \le r_2\}$, with $R_1 < r_1 < r_2 < R_2$ (see also Example 21). $\square$

Example 28. Find the Laurent series of $\mathrm{e}^{1/z}$. This function is analytic in any annulus $0 < |z| < R_2 < \infty$. The coefficients are given by

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{e}^{-\mathrm{i}\varphi}/R}}{R^{n+1} \mathrm{e}^{\mathrm{i}(n+1)\varphi}} \mathrm{i} R \mathrm{e}^{\mathrm{i}\varphi} \mathrm{d}\varphi.$$

Since the Taylor series of e^w is $\sum_{n=0}^{\infty} \frac{w^n}{n!}$ and has radius of convergence $R = \infty$, we have

$$\mathrm{e}^{1/z} = \sum_{n=0}^{\infty} \frac{1}{z^n n!},$$

for all $0 < |z| < \infty$. We conclude that

$$a_n = \begin{cases} 0 & , \quad n \ge 1 \\ 1/(-n)! & , \quad n = 0, -1, -2, \ldots \end{cases}.$$

This gives an explicit formula for the above integral expressing a_n. Compare this with Example 20.

Example 29. The function $f(z) = (1+z^2)^{-1}$ is analytic everywhere except at $z = \pm\mathrm{i}$. What is the Laurent series expansion of f around $z_0 = \mathrm{i}$? We know that $R_1 = 0$ and $R_2 = 2$, since f is analytic in the annulus $\{z \in \mathbb{C} : \ 0 < |z - \mathrm{i}| < 2\}$.

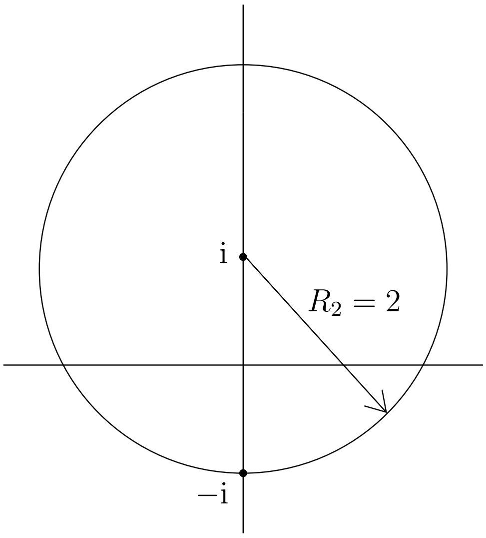


Figure 18: Radii for Laurent series of f.

Setting $w = z - \mathrm{i}$ we get

$$f(z) = \frac{1}{w(w+2\mathrm{i})} = \frac{1}{2\mathrm{i}w}\frac{1}{1+\frac{w}{2\mathrm{i}}} = \frac{1}{2\mathrm{i}w}\sum_{n=0}^{\infty}\left(-\frac{w}{2\mathrm{i}}\right)^n,$$

provided that $|w| < 2$, i.e., $|z-\mathrm{i}| < 2$. So

$$f(z) = \frac{1}{2\mathrm{i}(z-\mathrm{i})}\sum_{n=0}^{\infty}\left(\frac{\mathrm{i}}{2}\right)^n (z-\mathrm{i})^n$$

is the Laurent series expansion of f around $z_0 = \mathrm{i}$. Note that all a_n with $n \le -2$ vanish.

Example 30. $f(z) = \frac{1}{z(z+1)(z+2)}$ is not analytic at $z = 0, -1, -2$. So there are Laurent series expansions of f in the annuli $A_1 = \{0 < |z| < 1\}$, $A_2 = \{1 < |z| < 2\}$, and $A_3 = \{2 < |z| < \infty\}$. We decompose f into partial fractions, $f(z) = \frac{1}{2z} + \frac{1}{2(z+2)} - \frac{1}{z+1}$ and note that

$$\frac{1}{2(z+2)} = \frac{1}{4}\frac{1}{1+z/2} = \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n, \qquad |z| < 2$$

and

$$-\frac{1}{z+1} = -\sum_{n=0}^{\infty}(-z)^n, \qquad |z| < 1.$$

Hence for $z \in A_1$ we obtain

$$f(z) = \frac{1}{2z} - \sum_{n=0}^{\infty}(-z)^n + \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n = \frac{1}{2z} + \sum_{n=0}^{\infty}\left[(-1)^{n+1} + (-2)^{-n-2}\right]z^n.$$

For $|z|>1$ we have
$$-\frac{1}{1+z}=-\frac{1}{z}\frac{1}{1+1/z}=-\frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n.$$
Therefore the Laurent series of f in the annulus A_2 is
$$f(z)=\frac{1}{2z}-\frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n+\frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n=\sum_{n=0}^{\infty}(-2)^{-n-2}z^n-\frac{1}{2z}+\sum_{n\geq 2}(-1)^n z^{-n}.$$
Finally, for $z\in A_3$, we have
$$\frac{1}{2(z+2)}=\frac{1}{2z}\frac{1}{1+2/z}=\frac{1}{2z}\sum_{n=0}^{\infty}\left(-\frac{2}{z}\right)^n.$$
Therefore, the Laurent expansion of f in A_3 is
$$f(z)=\frac{1}{2z}+\frac{1}{2z}\sum_{n=0}^{\infty}(-2)^n z^{-n}-\frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n=-\sum_{n\geq 1}(-1)^n(2^{n-2}-1)z^{-n}.$$

3.2 Isolated singularities

Recall that we say that $z_0\in\mathbb{C}$ is a singular point, or a singularity, of a function f if f is not analytic at z_0, but every neighbourhood of z_0 contains a point at which f is analytic. We call z_0 an *isolated singularity of f* if f is analytic in a punctured neighbourhood $0<|z-z_0|<R$ of z_0, for some $R>0$, but f is not analytic at z_0. In particular, if z_0 is an isolated singularity of f, then f has a Laurent series expansion centered at z_0.

Example 31. The origin is an isolated singularity of $f(z)=1/z$. No other point is a singularity of f. Since $\sin(1/z)=0$ for $z=\frac{1}{n\pi}$, $n\in\mathbb{Z}$, the origin is a singular point of $g(z)=\frac{1}{\sin(1/z)}$, but it is not an isolated singularity.

Let z_0 be an isolated singularity of f. We say that z_0 is a *removable singularity of f* if and only if $\lim_{z\to z_0}f(z)$ exists. The point z_0 is a *pole of order* $n\in\mathbb{N}$ if and only if $\lim_{z\to z_0}(z-z_0)^n f(z)$ exists and is nonzero. A pole of order one is called a *simple pole.* A singular point of a function which is not a removable singularity, nor a pole, nor associated with a branch point or cut of a multi-valued function, is called an *essential singularity.* A function f which is analytic in a domain, except possible for having poles, is called a *meromorphic function* in that domain.

The nature of an isolated singularity can be read off the Laurent series expansion, as shows the following result.

Theorem 23 (Classification of isolated singularities)**.** *Let z_0 be an isolated singularity of f, and let $f(z)=\sum_{n\in\mathbb{Z}}a_n(z-z_0)^n$, $0<|z-z_0|<R$, be the Laurent series expansion of f around z_0. Then*

1. *z_0 is a removable singularity if and only if $a_n=0$, for $n\leq -1$.*

2. z_0 is a pole of order m if and only if $a_{-m} \neq 0$ and $a_{-k} = 0$ for all $k > m$.

3. z_0 is an essential singularity if and only if infinitely many a_n, with $n \leq -1$, are nonzero.

Proof. 1. If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for $0 < |z - z_0| < R$, then $\lim_{z \to z_0} f(z) = a_0$, so z_0 is a removable singularity. Conversely, if z_0 is a removable singularity, then f is analytic in the annuli $r < |z - z_0| < R$, for all $0 < r < R$, and $\lim_{z \to z_0} f(z) = \alpha$ exists. We notice that f is bounded on $\{|z| < R\}$. Indeed for all $\epsilon > 0$, there exists an r such that if $|z - z_0| < r$, then $|f(z) - \alpha| < \epsilon$, so $|f(z)| < |\alpha| + \epsilon$ whenever $|z - z_0| < r$. Next, by analyticity, f is continuous in the closed annulus $r/2 \leq |z - z_0| \leq R/2$, so it is bounded there. Thus there exists C_0 such that $|f(z)| < C_0$ whenever $|z - z_0| \leq R/2$. Now let $f(z) = \sum_{n \in \mathbb{Z}} a_n(z - z_0)^n$, $0 < |z - z_0| < R$, be the Laurent expansion of f around z_0. Then we have

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{f(z_0 + \rho \mathrm{e}^{\mathrm{i}\varphi})}{\rho^{n+1} \mathrm{e}^{\mathrm{i}(n+1)\varphi}} \rho \mathrm{i} \mathrm{e}^{\mathrm{i}\varphi} \mathrm{d}\varphi,$$

where $\rho > 0$ is arbitrary ($\rho < R$). Thus we have $|a_n| \leq C_0 \rho^{-n}$, for all ρ such that $0 < \rho < R/2$. Consequently, we obtain $a_n = 0$ for all $n \leq -1$, by taking $\rho \to 0$.

2. If $a_{-m} \neq 0$ and $a_{-k} = 0$ for all $k > m$, then

$$\lim_{z \to z_0} (z - z_0)^m f(z) = \lim_{z \to 0} (z - z_0)^m \sum_{n=-m}^{\infty} a_n(z - z_0)^n = a_{-m}.$$

Thus, z_0 is a pole of order m. Conversely, suppose that z_0 is a pole of order m. The function $g(z) = (z - z_0)^m f(z)$ has the Laurent series expansion

$$g(z) = \sum_{n \in \mathbb{Z}} a_n(z - z_0)^{n+m} = \sum_{n \in \mathbb{Z}} a_{n-m}(z - z_0)^n.$$

Furthermore, g has a removable singularity at z_0. Thus by 1., $a_{n-m} = 0$ for all $n < 0$, that is, $a_n = 0$ for all $n \leq -m - 1$. Therefore,

$$f(z) = \sum_{n=-m}^{\infty} a_n(z - z_0)^n.$$

Moreover $\lim_{z \to z_0} (z - z_0)^m f(z) = a_{-m} \neq 0$.

3. Suppose $f(z) = \sum_{n \in \mathbb{Z}} a_n(z - z_0)^n$, $0 < |z - z_0| < R$ with only finitely many nonzero a_{-n}, $n \geq 1$. Then, by 1. and 2., z_0 is neither a pole nor a removable singularity, so it is an essential singularity. Conversely, if z is an essential singularity of f, then infinitely many a_{-n}, $n \geq 0$, have to be nonzero (for otherwise we are in situation 1. or 2.).

□

Example 32. $f(z) = \frac{\mathrm{e}^z - 1}{z}$ is analytic everywhere, except at $z_0 = 0$. We have

$$\mathrm{e}^z - 1 = \sum_{n=0}^{\infty} \frac{z^n}{n!} - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!},$$

and this series has radius of convergence $R = \infty$. It follows that

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!},$$

thus $z_0 = 0$ is a removable singularity.

Example 33. $f(z) = \frac{2}{z(z-3)^2}$ has isolated singular points at $z_0 = 0$ and $z_0 = 3$. Since $\lim_{z\to 0} zf(z) = \frac{2}{9} \neq 0$ we know that $z_0 = 0$ is a simple pole. Since $\lim_{z\to 3}(z-3)^2 f(z) = \frac{2}{3} \neq 0$ we know that $z_0 = 3$ is a pole of order 2. The Laurent series of f around 0 is obtained as follows:

$$\frac{2}{z(z-3)^2} = \frac{2}{z}\left(\frac{1}{z-3}\right)^2 = \frac{2}{z}\left(\frac{-1}{3}\right)^2\left(\frac{1}{1-z/3}\right)^2 = \frac{2}{9z}\left[\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right]^2 \quad \text{valid for } |z| < 3.$$

The right hand side can be expanded as

$$\frac{2}{z(z-3)^2} = \frac{2}{9z}\left[1 + \frac{z}{3} + \frac{z^2}{9} + \cdots\right]^2 = \frac{2}{9}\left[\frac{1}{z} + \frac{2}{3} + \frac{z}{3} + \cdots\right].$$

In particular, $\operatorname{Res}(f;0) = \frac{2}{9}$.

To calculate the Laurent series of f around $z_0 = 3$ we expand

$$\frac{2}{z}\frac{1}{(z-3)^2} = \frac{2}{3+(z-3)}\frac{1}{(z-3)^2} = \frac{2}{3}\frac{1}{1+(z/3-1)}\frac{1}{(z-3)^2} = \frac{2}{3}\frac{1}{(z-3)^2}\sum_{n=0}^{\infty}\left(1-\frac{z}{3}\right)^n,$$

for all z with $|z-3| < 3$. Consequently, for $z \in B(3,3)$, we have

$$f(z) = \frac{2}{3}\frac{1}{(z-3)^2}\sum_{n=0}^{\infty} 3^{-n}(3-z)^n = \frac{2}{3}\left[\frac{1}{(z-3)^2} - \frac{1}{3}\frac{1}{z-3} + \frac{1}{9} - \frac{1}{27}(z-3) + \cdots\right].$$

In particular, $\operatorname{Res}(f;3) = -\frac{2}{9}$.

There is another characterization of an isolated singularity z_0 of f in terms of the limit $\lim_{z\to z_0} f(z)$. By definition, z_0 is a removable singularity if and only if $\lim_{z\to z_0} f(z) = \alpha \in \mathbb{C}$ exists. Next suppose that z_0 is a pole of order $m \geq 1$ of f. Then we have (Theorem 23)

$$f(z) = \sum_{n=-m}^{\infty} a_n(z-z_0)^n = (z-z_0)^{-m}\left[a_{-m} + \sum_{n\geq 1} a_{-m+n}(z-z_0)^n\right],$$

for $|z-z_0|$ sufficiently small, and where $a_{-m} \neq 0$. The series on the right side defines an analytic function which vanishes at $z = z_0$, so there exists a δ s.t. if $|z-z_0| < \delta$, then $|\sum_{n\geq 1} a_{-m+n}(z-z_0)^n| < |a_{-m}|/2$. Therefore, an application of the inverse triangle inequality yields $|f(z)| \geq |z-z_0|^{-m}|a_{-m}|/2$ for z sufficiently close to z_0. This implies that if z_0 is a pole of f, then $\lim_{z\to z_0}|f(z)| = \infty$. That the converse of this statement is correct follows from the following result.

Theorem 24 (Casorati-Weierstrass)**.** *Suppose z_0 is an essential singularity of f, and let R be radius of the deleted neighbourhood $A(z_0; 0, R)$ of z_0 in which f has a Laurent expansion. Fix any $0 < r < R$, $\epsilon > 0$ and $w \in \mathbb{C}$. Then there exists a $z \in A(z_0; 0, r)$ s.t. $|f(z) - w| < \epsilon$.*

This result implies that in any deleted neighbourhood of an essential singularity one can find a sequence of points converging to any pre-established limit w.

Proof. The argument is by contradiction. Suppose there were $r > 0$, $\epsilon > 0$ and $w \in \mathbb{C}$ s.t. $|f(z) - w| \geq \epsilon$ for all $z \in A(z_0; 0, r)$. Then the function

$$g(z) = \frac{1}{f(z) - w}$$

is analytic in $z \in A(z_0; 0, r)$, and satisfies the bound $|g(z)| \leq 1/\epsilon$ for these z. Let a_n be the coefficients of the Laurent series of g in $A(z_0; 0, r)$. We have $|a_n| \leq \rho^{-n}/\epsilon$ for all $0 < \rho < r$ (see also the proof of Theorem 23). Thus, taking $\rho \to 0$, one obtains $a_n = 0$ for all $n < 0$. Hence g has a removable singularity at z_0, $g(z) = \sum_{n\geq 0} a_n(z - z_0)^n$ for $0 < |z - z_0| < r$. Let n_0 be the smallest integer satisfying $a_n \neq 0$. Expressing f in terms of g, we have

$$f(z) - w = \frac{1}{g(z)} = (z - z_0)^{-n_0} \frac{1}{a_{n_0} + \sum_{n\geq 1} a_{n+n_0}(z - z_0)^n}.$$

The last fraction is analytic in z for z sufficiently close to z_0, since the denominator is analytic and does not vanish for such z. It follows that $f(z) - w$ has a pole of order n_0 at z_0. This is in contradiction to the assumption that z_0 is an essential singularity of f. □

If $f(z_0) = 0$ then we say z_0 is a *zero of f*. $z_0 \in \mathbb{C}$ is called an *isolated zero of f* if and only if there is a deleted neighbourhood of z_0 in which f does not vanish. If f is analytic at z_0 and if z_0 is a zero of f, and if there is a neighbourhood of z_0 in which we have $f(z) = (z - z_0)^n g(z)$ for some $n \geq 1$ and some function g such that $\lim_{z\to z_0} g(z)$ exists and is nonzero, then we call z_0 a *zero of f of order n*.

Suppose that f is analytic at z_0. Then it has a Taylor expansion $f(z) = \sum_{m=0}^{\infty} a_m(z - z_0)^m$, for $|z - z_0| < R$. If f has a zero of order n at z_0, then by the above definition, there is a function g defined in a neighbourhood of z_0, such that $g(z) = (z - z_0)^{-n} f(z)$, for $z \neq z_0$, and such that $\lim_{z\to z_0} g(z)$ exists and takes a nonzero value ζ. It follows that $a_0 = a_1 = \cdots = a_{n-1} = 0$, and that $a_n = \zeta \neq 0$. Thus $g(z) = \sum_{m\geq 0} a_{m+n}(z - z_0)^m$ is analytic at z_0 as well, and $a_n = \zeta \neq 0$. We thus see that:

Suppose that f is analytic at z_0. Then z_0 is a zero of order n of f if and only if $f^{(k)}(z_0) = 0$ for $k = 0, 1, 2, \ldots, n - 1$, and $f^{(n)}(z_0) \neq 0$.

Theorem 25. *Let f be analytic at z_0, and let z_0 be a zero of order n of f. Then z_0 is an isolated zero of f.*

Proof. We have $f(z) = (z - z_0)^n g(z)$, where g is analytic at z_0, and $g(z_0) \neq 0$. Since g is continuous at z_0, there is a $\delta > 0$ such that if $|z - z_0| < \delta$, then $|g(z)| > |g(z_0)|/2 > 0$. Therefore, for $|z - z_0| < \delta$ and $z \neq z_0$, we have $|f(z)| = |z - z_0|^n |g(z)| > 0$. □

Theorem 26. *Suppose that f is analytic at z_0, and that there is a sequence $\{z_n\}_{n\geq 1}$ with infinitely many different z_n such that $z_n \to z_0$, and such that $f(z_n) = 0$, for all n. Then f is identically zero in a neighbourhood of z_0.*

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, $|z-z_0| < R$, be the Taylor expansion of f around z_0. Suppose that f is not identically zero in $\{|z-z_0| < R\}$, then there is a smallest, finite $n_0 \geq 0$ such that $a_{n_0} \neq 0$. Hence

$$f(z) = (z-z_0)^{n_0}[a_{n_0} + \sum_{n=1}^{\infty} a_{n_0+n}(z-z_0)^{n_0+n}].$$

But $f(z_k) = 0$ and $(z_k - z_0)^{n_0} \neq 0$, so

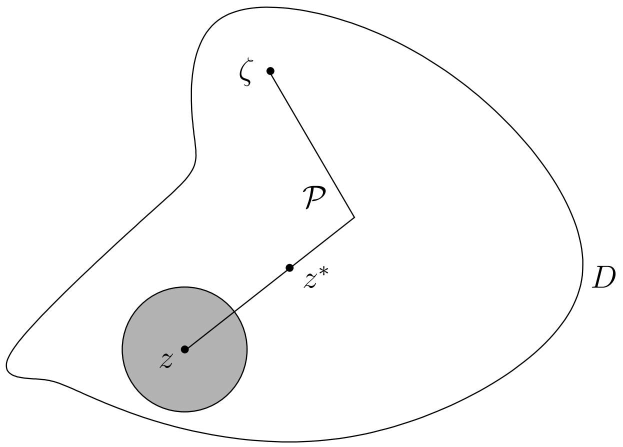
$$a_{n_0} + \sum_{n=1}^{\infty} a_{n_0+n}(z_k - z_0)^{n_0+n} = 0,$$

for all k. By taking $k \to \infty$, we get $a_{n_0} = 0$, a contradiction. □

Theorem 27 (Identity Theorem)**.** *Suppose that f and g are analytic in a domain D. Suppose that there is a sequence $\{z_n\}$, with infinitely many different $z_n \in D$, such that $z_n \to z$, for some $z \in D$. If $f(z_n) = g(z_n)$ for all n, then $f = g$ on D.*

Note that for the result to hold it is important that the limit point z is in D. Indeed, the function $\sin(1/z)$ has zeroes $z_n = (\pi n)^{-1}$, $n \in \mathbb{Z}$, so $\sin(1/z)$ is analytic and nonzero in $D = B(1,1)$, but has a sequence of zeroes accumulating at the boundary of D.

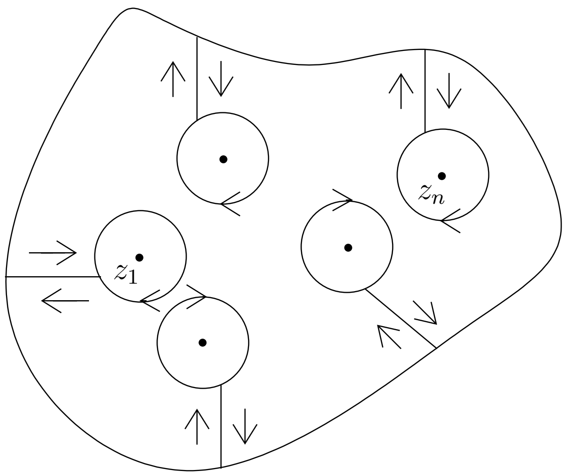
Proof. Consider $h := f - g$, which is analytic on D. From the above theorem, we know that $h = 0$ in a neighbourhood of z. Take now any $\zeta \in D$, and link it to z by a polygonal path $\mathcal{P}$ lying inside D.



Suppose that h does not vanish on all points of $\mathcal{P}$. Then, starting from z and moving towards ζ on $\mathcal{P}$, there will be a last point $z^* \in \mathcal{P}$ where $h(z) = 0$. This is so since the zeros of h form a closed set. By construction, there is a sequence $z_n \to z^*$, $z_n \in \mathcal{P}$, such that $h(z_n) = 0$. By the previous theorem, h has to vanish in a neighbourhood of z^*. In particular, there are points after z^* on $\mathcal{P}$ where h vanishes, a contradiction. Thus z^* does not exist. This means that $h = 0$ on all of $\mathcal{P}$. Therefore we have $h(\zeta) = 0$, and thus $f(\zeta) = g(\zeta)$. Since $\zeta \in D$ was arbitrary, the result is shown. □

3.3 The Residue Theorem

Let f be a function analytic inside and on a simple closed contour $\mathcal{C}$, except possibly at finitely many points $z_1, \ldots, z_n$ inside $\mathcal{C}$.



Let $\mathcal{C}_k$ be a small circle around z_k, lying inside $\mathcal{C}$, and such that all other points z_l, $l \neq k$, lie outside $\mathcal{C}_k$. By the deformation lemma we have

$$\int_{\mathcal{C}} f(z)\mathrm{d}z = \sum_{k=1}^{n} \int_{\mathcal{C}_k} f(z)\mathrm{d}z.$$

Let R_k be the outer radius of convergence of the Laurent series of f around z_k. We may without loss of generality choose the radius of $\mathcal{C}_k$ smaller than R_k. So we have

$$f(z) = \sum_{m=-\infty}^{\infty} a_m^{(k)} (z - z_k)^m,$$

for $0 < |z - z_k| < R_k$, and $\mathcal{C}_k \subset B(z_k, R_k)$. Therefore we have

$$\int_{\mathcal{C}_k} f(z)\mathrm{d}z = \sum_{m=-\infty}^{\infty} a_m^{(k)} \int_{\mathcal{C}_k} (z - z_k)^m \mathrm{d}z.$$

The last integral vanishes if $m \neq -1$ and equals $2\pi\mathrm{i}$ if $m = -1$. Consequently,

$$\int_{\mathcal{C}_k} f(z)\mathrm{d}z = 2\pi\mathrm{i} a_{-1}^{(k)} = 2\pi\mathrm{i}\mathrm{Res}(f; z_k)$$

and we have shown the following result.

Theorem 28 (Residue Theorem). *Let $\mathcal{C}$ be a simple closed contour within and on which f is analytic, except at a finite number of singular points $z_1, \ldots, z_n$ inside $\mathcal{C}$. Then we have*

$$\int_{\mathcal{C}} f(z)\mathrm{d}z = 2\pi\mathrm{i} \sum_{k=1}^{n} \mathrm{Res}(f; z_k).$$

The residue theorem is useful in the calculation of contour integrals.

Example 34. We want to calculate $\int_{\mathcal{C}} z^2 \mathrm{e}^{1/z} \mathrm{d}z$, where $\mathcal{C}$ is a simple closed contour enclosing the origin. Since

$$z^2 \mathrm{e}^{1/z} = z^2 \sum_{n=0}^{\infty} \frac{1}{n! z^n} = \sum_{n=0}^{\infty} \frac{z^{-n+2}}{n!},$$

for $0 < |z| < \infty$, the origin is an essential singularity of $z^2 \mathrm{e}^{1/z}$. Since $\mathrm{Res}(z^2 \mathrm{e}^{1/z}; 0) = 1/3!$, we get $\int_{\mathcal{C}} z^2 \mathrm{e}^{1/z} \mathrm{d}z = \mathrm{i}\pi/3$.

If z_0 is a pole of order $m \geq 1$ of f, then, in a deleted neighbourhood of z_0,

$$f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0)^n = (z - z_0)^{-m} \sum_{n=0}^{\infty} a_{n-m} (z - z_0)^n$$

and so the function

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_{n-m} (z - z_0)^n$$

is analytic in a neighbourhood of z_0 (has a removable singularity at z_0). It follows from Taylor's theorem that

$$n! a_{n-m} = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\bigg|_{z=z_0} (z - z_0)^m f(z).$$

In particular, we have

$$\mathrm{Res}(f; z_0) = a_{-1} = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\bigg|_{z=z_0} (z - z_0)^m f(z).$$

This shows the following little result.

Lemma 6 (Residue at a pole)**.** *Let z_0 be a pole of order $m \geq 1$ of f. Then*

$$\mathrm{Res}(f; z) = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\bigg|_{z=z_0} (z - z_0)^m f(z).$$

Example 35. Let $f(z) = (z-1)^{-1}(z+\mathrm{i})^{-1}(z-\mathrm{i})^{-1}$. The singularities of f are the three simple poles $z = 1, \pm\mathrm{i}$. It follows from Lemma 6 that

$$\mathrm{Res}(f; 1) = (z-1)f(z)\big|_{z=1} = 1/2 \quad \text{and} \quad \mathrm{Res}(f; \pm\mathrm{i}) = (z \mp \mathrm{i}) f(z)\big|_{z=\pm\mathrm{i}} = (\pm\mathrm{i} - 1)^{-1}(\pm\mathrm{i} \mp \mathrm{i})^{-1}.$$

Example 36. Calculate $\int_{\mathcal{C}} \frac{\mathrm{Log}(z)}{(z^2+1)^2} \mathrm{d}z$, where $\mathcal{C}$ is the contour given in Figure 19 below.

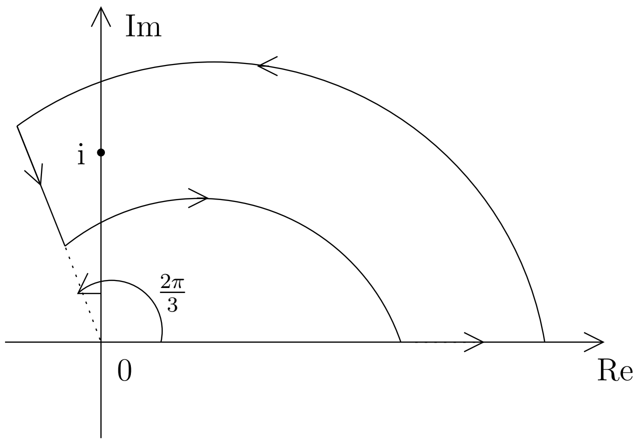


Figure 19: Contour of integration

The only singularity of $f(z) = \frac{\mathrm{Log}(z)}{(z^2+1)^2}$ inside $\mathcal{C}$ is a pole of order two at $z = \mathrm{i}$. (The order is easily seen to be two since $(z^2+1)^2 = (z+\mathrm{i})^2(z-\mathrm{i})^2$.) In order to calculate the integral with the help of the residue theorem, we need to find the residue of f at $z = \mathrm{i}$. By Lemma 6, we find

$$\mathrm{Res}\,(f; \mathrm{i}) = \frac{\mathrm{d}}{\mathrm{d}z}\Big|_{z=\mathrm{i}}(z-\mathrm{i})^2 \frac{\mathrm{Log}(z)}{(z+\mathrm{i})^2(z-\mathrm{i})^2} = \frac{\pi}{8} + \frac{\mathrm{i}}{4}.$$

We conclude that

$$\int_{\mathcal{C}} \frac{\mathrm{Log}(z)}{(z^2+1)^2}\mathrm{d}z = -\frac{\pi}{2} + \mathrm{i}\frac{\pi^2}{4}.$$

The next result is useful in many applications. It is based on the following very simple observation. If f has a zero of order n at z_0 then the residue of $f'(z)/f(z)$ at z_0 is simply n. Similarly, if z_0 is a pole of order n of f, then the residue of $f'(z)/f(z)$ is $-n$.

Theorem 29. *Let $\mathcal{C}$ be a simple closed contour and suppose that f is nonzero on $\mathcal{C}$ and analytic inside and on $\mathcal{C}$, except possibly for having finitely many poles inside D. Then we have*

$$\frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}} \frac{f'(z)}{f(z)}\mathrm{d}z = N - P,$$

where N and P are the numbers of zeroes and of poles of f inside D, counted according to their multiplicity.

Proof. Denote by D the interior of $\mathcal{C}$ and suppose that f has a zero of order n at $z_0 \in D$. Then for z in a neighbourhood of z_0, we have

$$f(z) = (z-z_0)^n g(z),$$

for some function g analytic at z_0 satisfying $g(z_0) \neq 0$, and hence not vanishing in a neighbourhood of z_0. It follows that in a neighbourhood of z_0,

$$\frac{f'(z)}{f(z)} = \frac{n}{z-z_0} + \frac{g'(z)}{g(z)}.$$

The residue of this quotient at z_0 is therefore n. Next suppose that f has a pole of order n at $z_0 \in D$, so that in a deleted neighbourhood of z_0, we have

$$f(z) = (z - z_0)^{-n} g(z),$$

for some function g analytic at z_0 satisfying $g(z_0) \neq 0$, and hence not vanishing in a neighbourhood of z_0. It follows that in a deleted neighbourhood of z_0,

$$\frac{f'(z)}{f(z)} = \frac{-n}{z - z_0} + \frac{g'(z)}{g(z)},$$

which has residue $-n$ at z_0. Finally, if $z_0 \in D$ is neither a pole nor a zero of f, then $\frac{f'(z)}{f(z)}$ is analytic at z_0. Therefore, by the residue theorem, we have

$$\int_{\mathcal{C}} \frac{f'(z)}{f(z)} \mathrm{d}z = 2\pi\mathrm{i} \sum_{j=1}^{J} n_j - 2\pi\mathrm{i} \sum_{k=1}^{K} p_k,$$

where n_j is the multiplicity of the j-th zero of f inside D, and p_k is the order of the k-th pole of f inside D. $\square$

Example 37. The number of eigenvalues of the matrix

$$\begin{bmatrix} \mathrm{i} & 0 & 3\mathrm{i}+1 \\ 0 & 5\mathrm{i} & \mathrm{i} \\ 10\mathrm{i} & 2 & 0 \end{bmatrix}$$

having modulus smaller than a fixed $r > 0$ (for r not being the modulus of any root) is

$$N(r) = \frac{1}{2\pi\mathrm{i}} \int_0^{2\pi} \frac{\chi'(r\mathrm{e}^{\mathrm{i}\varphi})}{\chi(r\mathrm{e}^{\mathrm{i}\varphi})} \mathrm{i}r\mathrm{e}^{\mathrm{i}\varphi} \mathrm{d}\varphi,$$

where

$$\chi(z) = -z^3 + 6\mathrm{i}z^2 - 2(5 - 6\mathrm{i})z + 50(3\mathrm{i} + 1)$$

is the characteristic polynomial. Note that explicitly evaluating this integral seems to be a task of the same order of difficulty as directly calculating the eigenvalues of the matrix. Of course we see that $N(r) \to 0$ as $r \to 0$ and $N(r) \to 3$ as $r \to \infty$. These two facts tell us (rather trivially) that zero is not an eigenvalue of the matrix, and that the total number of eigenvalues of the matrix is 3.

A useful application of the residue theorem is the following. Let $P(x)$ and $Q(x)$ be two polynomials in the real variable $x \in \mathbb{R}$. (We allow the coefficients of P, Q to be complex.) Assume that the degree of Q is by at least two bigger than that of P. Then there is a constant $C < \infty$ s.t. we have

$$\left| \frac{P(z)}{Q(z)} \right| \leq C|z|^{-2},$$

for all $|z|$ sufficiently large. (Here, $P(z)$ is the polynomial of the complex variable z, obtained by simply replacing x by z in the expression of $P(x)$.) Let us in addition assume that Q has no roots on the real line. Then the integral $\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\mathrm{d}x$ converges. Consider

$$I_R = \int_{-R}^{R} \frac{P(x)}{Q(x)}\mathrm{d}x + \int_{\mathcal{C}_R} \frac{P(z)}{Q(z)}\mathrm{d}z,$$

where $\mathcal{C}_R$ is the semi-circle $z(\varphi) = R\mathrm{e}^{\mathrm{i}\varphi}$, $\varphi \in [0,\pi]$. Since the modulus of the integrand of the last integral is bounded above by CR^{-2} but the length of $\mathcal{C}_R$ is only πR, we have $\lim_{R\to\infty} \int_{\mathcal{C}_R} \frac{P(z)}{Q(z)}\mathrm{d}z = 0$, and so

$$\lim_{R\to\infty} I_R = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\mathrm{d}x.$$

On the other hand, if R is larger than the modulus of the largest root of $P(z)$ in the upper half plane ($\mathrm{Im}z > 0$), then the residue theorem implies that $\frac{1}{2\pi\mathrm{i}} I_R$ equals the sum over all residues of $P(z)/Q(z)$ in the upper half plane. We conclude that

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\mathrm{d}x = 2\pi\mathrm{i} \sum_{j:\mathrm{Im}z_j>0} \mathrm{Res}\left(\frac{P(z)}{Q(z)}; z_j\right),$$

where the sum is over all residues of $P(z)/Q(z)$ in the upper half plane.

Example 38. Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{x^4+\alpha x^2+1}\mathrm{d}x$, where $-2 < \alpha < 2$. Here $P(z) = z^2$, and the roots of $Q(z) = z^4 + \alpha z^2 + 1$ satisfy $z^2 = \frac{1}{2}(-\alpha \pm \mathrm{i}\sqrt{4-\alpha^2})$, and none of them are real since $|\alpha| < 2$. Note that since the coefficients of Q are real, z is a root if and only if $\overline{z}$ is a root. Let $z = x + \mathrm{i}y$. We are only interested in the roots with $y > 0$. We have $x^2 - y^2 = -\alpha/2$ and $2xy = \pm\frac{1}{2}\sqrt{4-\alpha^2}$. By squaring the latter relation and using the former one, one obtains the two roots in the upper half plane, $z_\pm = \frac{1}{2}[\pm\sqrt{2-\alpha} + \mathrm{i}\sqrt{2+\alpha}]$. Therefore the other two roots are $\overline{z}_\pm$ and so

$$Q(z) = (z - z_+)(z - \overline{z}_+)(z - z_-)(z - \overline{z}_-).$$

It follows that

$$\mathrm{Res}\left(\frac{P}{Q}; z_+\right) = \frac{P(z)}{(z-\overline{z}_+)(z-z_-)(z-\overline{z}_-)}\Big|_{z=z_+} = \frac{1}{4}\left(\frac{1}{\sqrt{2-\alpha}} - \mathrm{i}\frac{1}{\sqrt{2+\alpha}}\right).$$

and similarly,

$$\mathrm{Res}\left(\frac{P}{Q}; z_-\right) = \frac{P(z)}{(z-z_+)(z-\overline{z}_+)(z-\overline{z}_-)}\Big|_{z=z_-} = \frac{1}{4}\left(-\frac{1}{\sqrt{2-\alpha}} - \mathrm{i}\frac{1}{\sqrt{2+\alpha}}\right).$$

We conclude that

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+\alpha x^2+1}\mathrm{d}x = \frac{\pi}{\sqrt{2+\alpha}}.$$

An interesting consequence of Theorem 29 is Rouché's theorem given below. This theorem states that if an analytic function has n roots inside a contour $\mathcal{C}$, and if we "perturb" f by adding another analytic function g (which is smaller than f on $\mathcal{C}$), then the number of zeroes of $f+g$ inside $\mathcal{C}$ stays the same, n. Of course, the zeroes of f will "move" under the perturbation g, but their total number is conserved.

Theorem 30 (Rouché)**.** *Let $\mathcal{C}$ be a simple closed contour within and on which f and g are analytic. If*

$$|g(z)| < |f(z)| \qquad \text{for all } z \in \mathcal{C},$$

then the functions f and $f+g$ have the same number of zeroes in the interior of $\mathcal{C}$ (counting multiplicity).

Proof. For $0 \le a \le 1$ the function $f_a(z) = f(z) + ag(z)$ does not vanish on $\mathcal{C}$ and is analytic inside and on $\mathcal{C}$. Note that f_a interpolates between $f_0 = f$ and $f_1 = f+g$. The number of zeroes of f_a inside $\mathcal{C}$ is given by

$$N_a = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{f_a'(z)}{f_a(z)} \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{f'(z) + ag'(z)}{f(z) + ag(z)} \mathrm{d}z.$$

We show below that $a \mapsto N_a$ is continuous in $0 \le a \le 1$. But then, since N_a takes integer values, N_a must be constant in $0 \le a \le 1$. Now N_0 is the number of zeroes of f inside $\mathcal{C}$ while N_1 is the number of zeroes of $f+g$ inside $\mathcal{C}$.

It remains to show the continuity of N_a. Let $z \in \mathcal{C}$ be fixed and take $a \in [0,1]$ and $h \in \mathbb{R}$ s.t. $a+h \in [0,1]$. Then

$$\frac{f_{a+h}'(z)}{f_{a+h}(z)} - \frac{f_a'(z)}{f_a(z)} = h \frac{f(z)g'(z) - f'(z)g(z)}{[f(z)+ag(z)]^2 + hg(z)[f(z)+ag(z)]}.$$

Choose h so small that

$$|h| \max_{z \in \mathcal{C}} |g(z)| < \frac{1}{2} \min_{z \in \mathcal{C}} |f(z) + ag(z)|,$$

where we point out again that the r.h.s. is strictly positive. We obtain then

$$\left| \frac{f_{a+h}'(z)}{f_{a+h}(z)} - \frac{f_a'(z)}{f_a(z)} \right| \le 2|h| \frac{|f(z)g'(z) - f'(z)g(z)|}{|f(z)+ag(z)|^2}.$$

By integrating over $z \in \mathcal{C}$, it follows from this bound that N_a is continuous (in fact, differentiable) in $a \in [0,1]$. □

Example 39. To estimate the location of the roots of the equation $p(z) = z^6 + 7z + 1 = 0$ we write the polynomial on the left side as $f(z) + g(z)$, where $f(z) = z^6$, $g(z) = 7z + 1$. The condition $|g(z)| < |f(z)|$ is implied by $7|z| + 1 < |z|^6$. The latter inequality is satisfied if $|z| = 2$. Consequently, since f has six roots inside the circle of radius 2 centered at the origin, all six roots of p have modulus less than 2. On the other hand, for $|z| = 1$, we have the reverse inequality $|g(z)| > |f(z)|$. Since g has one root inside the cirlce of radius 1 around the origin, p has one root with modulus smaller than 1 (and hence by the previous argument p has five roots with moduli in the annulus $1 \le |z| < 2$).

Example 40. Let $n \in \mathbb{N}$. The equation $\mathrm{e}^{2\mathrm{i}z} = 12z^n$ has n solutions inside the circle $|z| = 1$. To see this we let $f(z) = 12z^n$ and $g(z) = \mathrm{e}^{2\mathrm{i}z}$. Then $|g(z)| = \mathrm{e}^{-2|z|\sin(\arg(z))} \le \mathrm{e}^{2|z|}$, while $|f(z)| = 12|z|^n$. For $|z| = 1$ we have thus $|g(z)| \le \mathrm{e}^2 < 12 = |f(z)|$, from which the result follows by Rouché's theorem.

Let $\mathcal{C}$ be a simple closed contour and suppose that f_n, f are analytic inside and on $\mathcal{C}$, that $f(z) \neq 0$ for all $z \in \mathcal{C}$ and that f_n converges to f uniformly on $\mathcal{C}$. Let $m = \min_{z\in\mathcal{C}} |f(z)|$. Then for n large enough, we have

$$|f_n(z) - f(z)| < m \le |f(z)| \qquad \text{for all } z \in \mathcal{C}.$$

Rouché's theorem implies that the number of roots of f and of $f + f_n - f = f_n$ inside $\mathcal{C}$ is the same. This shows the following result.

Theorem 31 (Hurwitz)**.** *Let $\mathcal{C}$ be a simple closed contour. Let f_n, f be functions analytic inside and on $\mathcal{C}$, such that $f(z) \neq 0$ for all $z \in \mathcal{C}$ and s.t. $f_n \to f$ uniformly on $\mathcal{C}$. Then there is an n_0 s.t. for all $n \ge n_0$ the functions f_n and f have the same number of zeroes in the interior of $\mathcal{C}$ (counted including multiplicity).*

Example 41. INTERSTING EXAMPLE: $f(z) = \mathrm{e}^z$, $P_N(z) = \sum_{n=0}^N z^n/n!$. P_N has N zeroes in $\mathbb{C}$, while f has none. On $|z| = R$, R fixed, $P_N \to f$ uniformly, so all zeroes of P_N escape to infinity as $N \to \infty$. Task: given R, find N s.t. all roots of $P_n(z)$ satisfy $|z| \ge R$. Answer: $N > eR - 1$ (if $R > 1$). Method: $|e^z| \ge e^{-R}$ on $|z| = R$, so need to find N s.t. $|e^z - P_N(z)| < e^{-R}$ unif. on $|z| = R$. L.h.s. $\le \frac{R^{N+1}}{(N+1)!}\mathrm{e}^R$, then take ln and convert $\ln(N+1)! = \sum_{k=1}^N \ln(k+1) \ge \int_1^{N+1} \ln(x)\mathrm{d}x$ and estimate.

3.4 Using contour integrals to evaluate and estimate sums

A method for evaluating sums of the form $\sum_{n=-\infty}^{\infty} f(n)$, where f is analytic at $n \in \mathbb{Z}$, is based on finding a function g whose residues are $\{f(n)\}_{n\in\mathbb{Z}}$. Let us first see how to construct such a function. Suppose we can find a function $\varphi(z)$ which has simple poles with residue one at every $n \in \mathbb{Z}$. Then the function $g(z) = f(z)\varphi(z)$ has residue $f(n)$ at each $n \in \mathbb{Z}$. Indeed,

$$\mathrm{Res}(g; n) = \lim_{z\to n}(z - n)f(z)\varphi(z) = f(n)\mathrm{Res}(\varphi; n) = f(n)$$

The function

$$\varphi(z) = \pi\frac{\cos(\pi z)}{\sin(\pi z)} = \pi\cot(\pi z)$$

does the job. Indeed, the singularities of φ are at z satisfying $\sin(\pi z) = \frac{1}{2\pi\mathrm{i}}[\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}] = 0$, i.e. $z = n$ with $n \in \mathbb{Z}$. The function $\sin(\pi z)$ has thus zeroes of order one at $z \in \mathbb{Z}$, and hence $[\sin(\pi z)]^{-1}$ has simple poles at $z \in \mathbb{Z}$. Moreover, $\mathrm{Res}(\varphi; n) = 1$.

Let us assume that f has finitely many poles $\{z_k\}_{k=1}^K$. Take a simple closed contour $\mathcal{C}_N$ enclosing the integers $n = 0, \pm1, \pm2, \ldots, \pm N$, as well as all the poles of f, as illustrated in Figure 20 below.

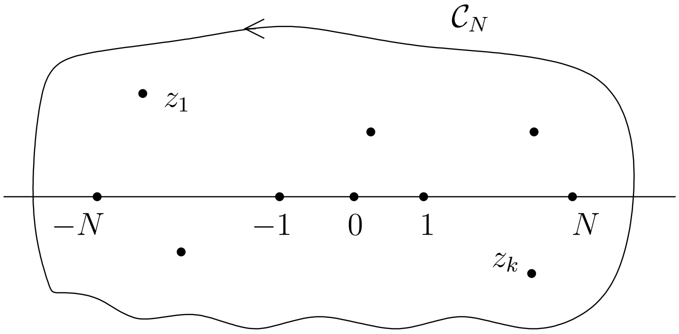


Figure 20: $\mathcal{C}_N$ encloses the integers $-N, ..., N$ and all singularities of f

The residue theorem gives

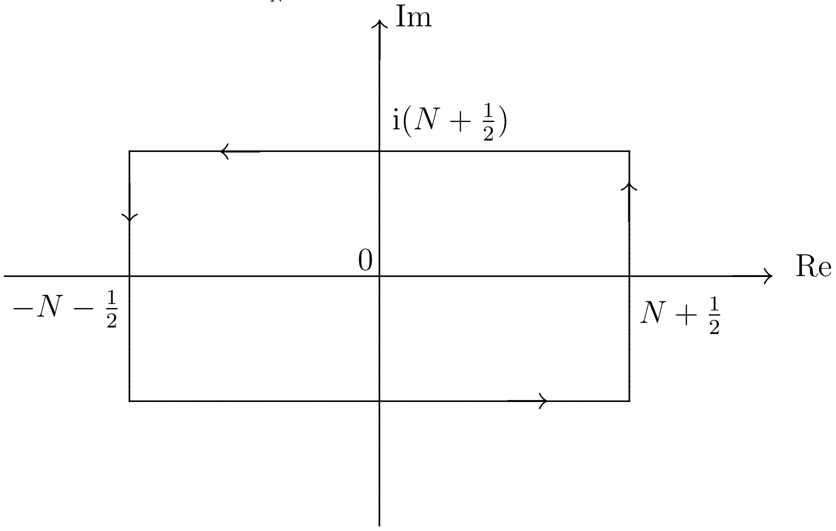
$$\pi \int_{\mathcal{C}_N} f(z)\cot(\pi z)\mathrm{d}z = 2\pi\mathrm{i} \sum_{\substack{n=-N,\dots,N,\\ n\notin\{z_k\}}} f(n) + 2\pi\mathrm{i}\sum_{k=1}^{K} \mathrm{Res}(\pi f(z)\cot(\pi z); z_k).$$

The first sum is due to the residues of $f(z)\cot(\pi z)$ at the integers $-N, \dots, N$ and the second sum comes from the singularities of f at z_k. In particular, if f is analytic then the second sum is not present. The next little result shows that the l.h.s. of the above equality actually vanishes in the limit $N \to \infty$ and we conclude that

$$\sum_{n\in\mathbb{N},\ n\notin\{z_k\}} f(n) = -\sum_{k=1}^{K} \mathrm{Res}(\pi f(z)\cot(\pi z); z_k).$$

Lemma 7. *Suppose that $|f(z)| \le A|z|^{-\alpha}$ for some constants $A > 0$ and $\alpha > 1$. Let $\mathcal{C}_N$ be the square with vertices $\pm(N + 1/2) \pm \mathrm{i}(N + 1/2)$. Then*

$$\lim_{N\to\infty} \int_{\mathcal{C}_N} f(z)\cot(\pi z)\mathrm{d}z = 0.$$



Proof. Writing $z = x + \mathrm{i}y$ we have

$$\cot(\pi z) = \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} = \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\pi x}\mathrm{e}^{-\pi y} + \mathrm{e}^{-\mathrm{i}\pi x}\mathrm{e}^{\pi y}}{\mathrm{e}^{\mathrm{i}\pi x}\mathrm{e}^{-\pi y} - \mathrm{e}^{-\mathrm{i}\pi x}\mathrm{e}^{\pi y}} = \mathrm{i}\frac{\mathrm{e}^{2\pi\mathrm{i}x}\mathrm{e}^{-2\pi y} + 1}{\mathrm{e}^{2\pi\mathrm{i}x}\mathrm{e}^{-2\pi y} - 1}.$$

On the right vertical side of $\mathcal{C}_N$, we have $x = N + 1/2$ and $y \in [-N - 1/2, N + 1/2]$ and so

$$\cot(\pi z) = -\mathrm{i}\frac{1 - \mathrm{e}^{-2\pi y}}{1 + \mathrm{e}^{-2\pi y}} \quad \text{and} \quad |\cot(\pi z)| = \frac{1 - \mathrm{e}^{-2\pi y}}{1 + \mathrm{e}^{-2\pi y}} < 1.$$

Similarly, on the lower horizontal part of $\mathcal{C}_N$ we have $x \in [-N - 1/2, N + 1/2]$ and $y = -N - 1/2$ and thus

$$\cot(\pi z) = \mathrm{i}\frac{\mathrm{e}^{2\pi \mathrm{i} x}\mathrm{e}^{2\pi(N+1/2)} + 1}{\mathrm{e}^{2\pi \mathrm{i} x}\mathrm{e}^{2\pi(N+1/2)} - 1} \quad \text{and} \quad |\cot(\pi z)| \le \frac{\mathrm{e}^{2\pi(N+1/2)} + 1}{\mathrm{e}^{2\pi(N+1/2)} - 1} < 2,$$

where the last inequality holds for N large enough. (Note that we have used the triangle inequality and the inverse triangle inequality in the numerator and the denominator to arrive at the first upper bound on $|\cot(\pi z)|$.) It is easy to obtain similar bounds on the other sides of $\mathcal{C}_N$, and we thus have $\sup_{z\in\mathcal{C}_N} |\cot(\pi z)| \le 2$. Consequently

$$\left|\int_{\mathcal{C}_N} f(z)\cot(\pi z)\mathrm{d}z\right| \le 4(2N+1)\sup_{z\in\mathcal{C}_N} |f(z)\cot(\pi z)| \le 8(2N+1)\sup_{z\in\mathcal{C}_N} |f(z)|.$$

However, for $z \in \mathcal{C}_N$ we have $|z| \ge N$, and by the decay condition on f, $|f(z)| \le AN^{-\alpha}$, for some $A > 0$ and $\alpha > 1$. Therefore,

$$\left|\int_{\mathcal{C}_N} f(z)\cot(\pi z)\mathrm{d}z\right| \le 8A\frac{2N+1}{N^\alpha} \longrightarrow 0,$$

as $N \to \infty$. This completes the proof. □

Example 42. We want to calculate

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

According to the above formula, we have

$$2\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{0\ne n\in\mathbb{Z}} \frac{1}{n^2} = -\mathrm{Res}\left(\pi\frac{1}{z^2}\cot(\pi z); 0\right).$$

To find the residue, we find the Laurent expansion of $\pi\frac{1}{z^2}\cot(\pi z)$ around the origin. First, using the Taylor expansions of $\sin(z)$ and $\cos(z)$ around the origin, we get

$$\cot(z) = \frac{\cos(z)}{\sin(z)} = \frac{1 - \frac{z^2}{2} + \frac{z^4}{4!} - \cdots}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots} = \frac{1}{z}\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right].$$

In order to expand the fraction $\frac{1}{1-a}$, $a = \frac{z^2}{3!} - \frac{z^4}{5!} + \cdots$ we use the geometric series $\frac{1}{1-a} = \sum_{n\ge 0} a^n$. Thus

$$\begin{aligned}
\cot(z) &= \frac{1}{z}\left[1 + \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^4}{(3!)^2} + \cdots\right]\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right] \\
&= \frac{1}{z}\left[1 - \frac{1}{3}z^2 + z^4\left(\frac{1}{4!} - \frac{1}{2!3!} - \frac{1}{5!} + \frac{1}{(3!)^2}\right) + \cdots\right] \\
&= \frac{1}{z} - \frac{z}{3} - \frac{1}{45}z^3 + \cdots
\end{aligned}$$

We can now read off the residue, $\mathrm{Res}\left(\frac{\pi\cot(\pi z)}{z^2};0\right) = -\frac{\pi^2}{3}$. So we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Next, consider the function $\varphi(z) = \pi\csc(\pi z) = \frac{\pi}{\sin(\pi z)}$. φ has simple poles at $n\in\mathbb{Z}$ (as before), and

$$\mathrm{Res}(\varphi;n) = (z-n)\varphi(n)|_{z=n} = \lim_{z\to n}\pi\frac{z-n}{\sin(\pi z)} = [\cos(n\pi)]^{-1} = (-1)^n.$$

Moreover, $\csc^2(z) = \sin^{-2}(z) = \frac{\sin^2(z)+\cos^2(z)}{\sin^2(z)} = 1+\cot^2(z)$, so $\csc^2(z)$ is bounded for $z\in\mathcal{C}_N$, the square of the previous lemma, and we obtain the following result. If f has finitely many singularities $\{z_k\}_{k=1}^K$, then

$$\sum_{n\in\mathbb{N},\ n\notin\{z_k\}}(-1)^n f(z) = -\sum_{k=1}^{K}\mathrm{Res}(\pi f(z)\csc(z);z_k).$$

Example 43. We have

$$2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \sum_{n\in\mathbb{N},\ n\neq 0}\frac{(-1)^n}{n^2} = -\frac{1}{2}\mathrm{Res}\left(\pi\frac{\csc(\pi z)}{z^2};0\right).$$

$$\begin{aligned}\csc(\pi z) &= \frac{1}{\sin(\pi z)}\\ &= \frac{1}{0+\pi z-\frac{1}{3!}(\pi z)^3+\cdots}\\ &= \frac{1}{\pi z}\left[1+\frac{1}{3!}(\pi z)^2+\cdots\right]\\ &= \frac{1}{\pi z}+\frac{\pi^2}{6\pi}z+\cdots.\end{aligned}$$

It follows that $\mathrm{Res}\left(\pi\frac{\csc(\pi z)}{z^2};0\right) = \frac{\pi^3}{6\pi} = \frac{\pi^2}{6}$, and hence

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

We can evaluate sums involving binomial coefficients in the following way. Since $C(n,k) = \binom{n}{k}$ is just the coefficient of z^k in $(1+z)^n$, we have

$$C(n,k) = a_k = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{C}}\frac{(1+z)^n}{z^{k+1}}\mathrm{d}z,$$

where $\mathcal{C}$ is a simple closed curve enclosing the origin. This relation may be used in estimates. For example, taking $\mathcal{C} = \{|z| = 1\}$,

$$C(2n, n) = \frac{1}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{(1 + \mathrm{e}^{\mathrm{i}\varphi})^{2n}}{\mathrm{e}^{\mathrm{i}(n+1)\varphi}} \mathrm{e}^{\mathrm{i}\varphi} \mathrm{i} \mathrm{d}\varphi$$

so

$$C(2n, n) \leq 2^{2n} = 4^n.$$

Example 44. To evaluate $\sum\limits_{n=0}^{\infty} \frac{C(2n,n)}{5^n}$, which converges since $C(2n, n) \leq 4^n$, we write

$$\sum_{n=0}^{\infty} \frac{C(2n, n)}{5^n} \quad = \quad \frac{1}{2\pi \mathrm{i}} \sum_{n=0}^{\infty} \int_{\mathcal{C}} \frac{(1 + z)^{2n}}{(5z)^n} \frac{\mathrm{d}z}{z},$$

where $\mathcal{C} = \{|z| = 1\}$. Since for $|z| = 1$ we have $\left|\frac{(1+z)^{2n}}{(5z)^n} \frac{1}{z}\right| \leq \left(\frac{4}{5}\right)^n$, the series converges uniformly on $\mathcal{C}$ by the Weierstrass M-test. Therefore,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{C(2n, n)}{5^n} &= \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \sum_{n=0}^{\infty} \left[\frac{(1 + z)^2}{5z}\right]^n \frac{\mathrm{d}z}{z} \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{1}{1 - \frac{(1+z)^2}{5z}} \frac{\mathrm{d}z}{z} \\
&= -\frac{5}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{\mathrm{d}z}{\left(z - \frac{3+\sqrt{5}}{2}\right)\left(z - \frac{3-\sqrt{5}}{2}\right)} \\
&= \sqrt{5}
\end{aligned}$$

We have used the Cauchy formula to evaluate the last integral: the contour contains the singularity $z_0 = (3-\sqrt{5})/2$ and has thus the form $\frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{f(z)}{z-z_0} \mathrm{d}z$, with $f(z) = -5(z - \frac{3+\sqrt{5}}{2})^{-1}$.

Example 45. To evaluate $\sum_{k=0}^{n} C(n, k)^2$, we note that $C(n, k)$ is the coefficient of z^k in $(1 + z)^n$ and also the coefficient of z^{-k} in $(1 + \frac{1}{z})^n$. So it follows that $\sum_{k=0}^{n} C(n, k)^2$ must be the constant term in $(1 + z)^n(1 + \frac{1}{z})^n$. Consequently, we have

$$\sum_{k=0}^{n} C(n, k)^2 = a_0 = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} (1 + z)^n \left(1 + \frac{1}{z}\right)^n \frac{1}{z} \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{C}} \frac{(1 + z)^{2n}}{z^{n+1}} \mathrm{d}z,$$

where $\mathcal{C}$ is a simple closed contour around the origin. The latter integral is also the coefficient of the power z^n in the Taylor expansion of $(1+z)^{2n}$ around the origin, hence it equals $C(2n, n)$. So we have shown that

$$\sum_{k=0}^{n} C(n, k)^2 = C(2n, n).$$

4 Analytic Continuation

The identity theorem for analytic functions shows that an analytic function f on a domain D is entirely determined by its values on any subdomain. How can we calculate the values in the larger domain, starting from the knowledge of f on the smaller domain?

Example 46. The series

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

converges for $|z| < 1$. It defines an analytic function $f(z)$ in the open unit disk. For $|z| \geq 1$, the series does not converge. Nevertheless, $g(z) := \frac{1}{1-z}$ is analytic everywhere except at $z = 1$ (simple pole). We call g an analytic continuation of f.

Theorem 32. *Let D_1, D_2 be two domains such that $D_1 \cap D_2 \neq \emptyset$. Suppose that f_1 is analytic in D_1, f_2 is analytic in D_2, and that $f_1(z) = f_2(z)$ for all $z \in D_1 \cap D_2$. Then there exists a unique function $F(z)$ that is analytic in $D_1 \cup D_2$, and that coincides with $f_1(z)$ on D_1.*

Proof. Set $F(z) = f_1(z)$ for $z \in D_1$, and $F(z) = f_2(z)$ for $z \in D_2$. Since $f_1(z) = f_2(z)$ for $z \in D_1 \cap D_2$, F is well defined. Moreover, F is analytic or $D_1 \cup D_2$. By the identity theorem, F is the only analytic function in $D_1 \cup D_2$ with $F(z) = f_1(z)$ for $z \in D_1$. $\square$

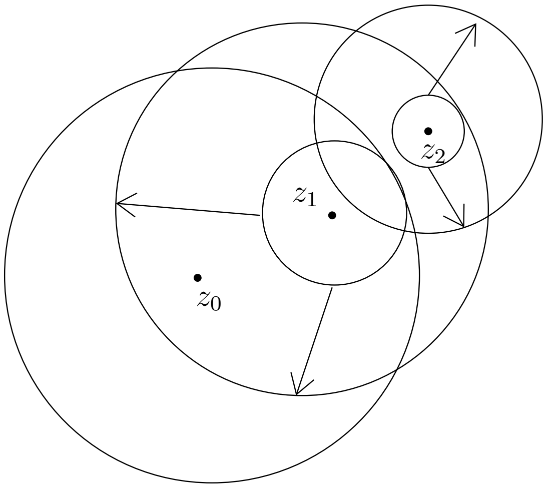
The function F is called an *analytic continuation* of f_1 (into the domain D_2). How can we obtain construct the values of an analytic extension? Since the function to be extended is determined by its Taylor series, and since the analytic extension is unique, this extension must be determined by the Taylor coefficient of the function to be extended. Let

$$f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be the Taylor expansion of f, convergent for $|z - z_0| < R$. Let z_1 be such that $|z_1 - z_0| < R$. Then f is analytic at z_1, and can thus be expanded as

$$f_1(z) = \sum_{n=0}^{\infty} b_n (z - z_1)^n,$$

at least for z such that $|z - z_1| < R - |z_1 - z_0|$.

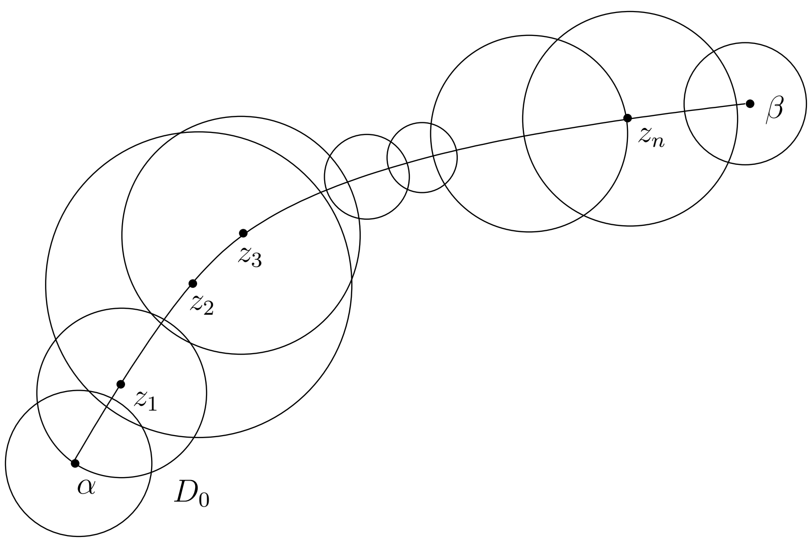


The coefficients in the latter Taylor series are given by

$$\begin{aligned} b_n &= \frac{1}{n!} f_1^{(n)}(z_1) \\ &= \frac{1}{n!} \sum_{m=n}^{\infty} a_m m(m-1)\cdots(m-n+1)(z_1-z_0)^{m-n} \\ &= \sum_{m=n}^{\infty} C(m,n) a_m (z_1-z_0)^{m-n} \end{aligned}$$

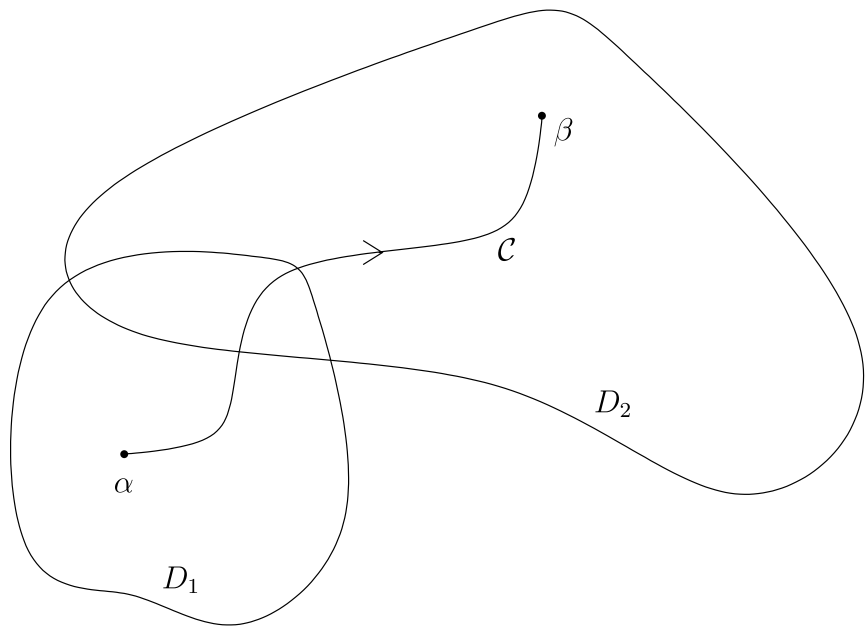
We know that the above series with coefficients b_n converges for $|z-z_1| < R - |z_1-z_0|$, but *it may well be that it converges in a bigger disk $B(z_1, R_1)$, with $R_1 > R - |z_1 - z_0|$*. If this happens, then we have an extension f_2 of f_1 into the region $B(z_0, R) \cup B(z_1, R_1)$. Now we can continue this procedure, pick a point z_2 in $B(z_1, R_1)$ and expand around z_2, etc. In this way we may obtain a chain of circles, each center lying inside the previous circle in the chain.

One may extend a function along a given curve. Let $\mathcal{C}$ be a curve linking $\alpha \in \mathbb{C}$ to $\beta \in \mathbb{C}$.



The function f is given by its Taylor series, in a disk B_0 centered at α. Going along the curve $\mathcal{C}$, starting at α, we may be able to find a point $z_1 \in \mathcal{C}$ so that the piece (α,z_1) on $\mathcal{C}$ lies entirely in B_0, and so that the Taylor series of f at z_1 converges in a disk B_1 that goes beyond B_0. Then we have an analytic continuation of f into B_1. We continue the process: we may be able to find a $z_2 \in B_1 \cap \mathcal{C}$, such that the piece (z_1,z_2) on $\mathcal{C}$ lies entirely inside B_1, and such that the Taylor expansion of f at z_2 converges in some disk B_2 going beyond B_1; then f is extended to B_2. If we can continue the procedure until we reach a point $z_n \in \mathcal{C}$ such that the Taylor series of f at z_n converges in a disk containing the point β, then we have extended f from α to β, along the curve $\mathcal{C}$. This extension is independent of the choice of intermediate points $z_1, \ldots, z_n$. Indeed, let F and G be two such extensions, defined and analytic on domains D and D' (consisting of unions of circles), respectively. Then, since F and G coincide in a neighbourhood of α, they coincide on all points $D \cap D'$. In particular, $F(z) = G(z)$ for all $z \in \mathcal{C}$. Thus $F(\beta) = G(\beta)$.

Theorem 33. *Let D_1, D_2 be domains with $D_1 \cap D_2 \neq \emptyset$. Suppose that f_1 is analytic in D_1, and that f_2 is an analytic continuation of f_1 into D_2. Let $\alpha \in D_1$ and $\beta \in D_2$, and let $\mathcal{C}$ be a contour from α to β, so that $\mathcal{C} \subset D_1 \cup D_2$. Then $f_2(\beta)$ can be calculated via an extension by a circle chain along $\mathcal{C}$.*

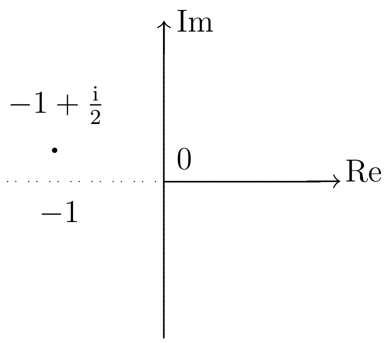


Proof. We need to find intermediate points $z_1, \ldots, z_n$ and disks $B_1, \ldots, B_n$, having the properties of the corresponding quantities in the circle chain extension. First assume that $\mathcal{C}$ is a curve given by $z(t)$, $0 \leq t \leq 1$, which is an arc equivalent, continuous and simple with $z(0) = \alpha$, $z(1) = \beta$. Since $\mathcal{C}$ is a compact set in the open set $D_1 \cup D_2$, we can find a radius $r > 0$ such that all balls with centers on $\mathcal{C}$ and radius r still belong to $D_1 \cup D_2$ (e.g., take $r = \frac{1}{2} \min\{|z - \zeta| : z \in \mathcal{C}, \zeta \in \mathbb{C} \backslash (D_1 \cup D_2)\}$). Since $z(t)$ is continuous on the compact interval $[0, 1]$, it is uniformly continuous. This means that there is a $\delta > 0$ such that if $|t - t'| < \delta$, then $|z(t) - z(t')| < r$.

Let us now divide the interval $0 \le t \le 1$ into n equal parts of length $\frac{1}{n}$, with n so large that $\frac{1}{n} < \delta$, and call the corresponding points in the partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$. Let $z_k = z(t_k)$ be the corresponding points on $\mathcal{C}$, $z_0 = \alpha$, $z_n = \beta$. Now we have $|z_{k+1} - z_k| = |z(t_{k+1}) - z(t_k)| < r$, since $|t_{k+1} - t_k| < \delta$. Therefore, $z_k \in B(z_{k-1}, r)$ and the $B(z_k, r)$, $k = 0, \ldots, n$, form a circle chain lying inside $D_1 \cup D_2$ and covering $\mathcal{C}$. Moreover, the piece (z_{k-1}, z_k) on $\mathcal{C}$ lies inside $B(z_{k-1}, r)$ and so we have found a circle chain linking α to β, along $\mathcal{C}$.

Finally, if $\mathcal{C}$ is an arbitrary contour, then it is a finite composition of arcs, and one proceeds on each arc as above. $\square$

The last result shows that if f_1 has an analytic continuation from a domain D_1 into a domain D_2, then the continuation can be obtained using the circle chain procedure along an arbitrary curve $\mathcal{C}$ inside $D_1 \cup D_2$. Is the converse true as well? Suppose that f_1 is analytic in a domain D_1, and that there is a domain D_2, s.t. $D_1 \cap D_2 \neq \emptyset$, with the property that we can extend f_1 along any arbitrary curve $\mathcal{C} \subset D_1 \cup D_2$ into $D_1 \cup D_2$. Do we thus get an analytic extension of f_1 into $D_1 \cup D_2$? The answer is no, in general, since the value of an extension along one curve may not coincide with that one along another curve.



Example 47. The principal value square root, $f_1(z) = \text{P.V.}\, z^{\frac{1}{2}} = \mathrm{e}^{\frac{1}{2}\mathrm{Log}(z)}$, is analytic on $D_1 = \mathbb{C}\backslash(-\infty, 0]$. Let $z_+ = -1 + \frac{\mathrm{i}}{2}$. The function f_1 has the Taylor series expansion $f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_+)^n$ around z_+, with $a_n = \frac{f_1^{(n)}(z_+)}{n!}$. For $n \ge 2$, we have

$$f_1^{(n)}(z) = z^{\frac{1}{2}-n} \frac{1}{2} \left(-\frac{1}{2}\right) \left(-\frac{3}{2}\right) \cdots \left(\frac{1}{2} - n + 1\right) = (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} z^{\frac{1}{2}-n},$$

where the root is the principal one. Let φ_+ be the principal argument of z_+. Then

$$z_+^{\frac{1}{2}-n} = \left(\frac{\sqrt{5}}{2}\right)^{\frac{1}{2}-n} \mathrm{e}^{\mathrm{i}(\frac{1}{2}-n)\varphi_+}.$$

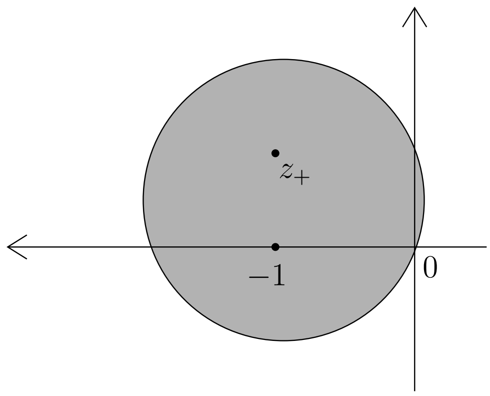
We thus have

$$f_1(z) = \left(\frac{\sqrt{5}}{2}\right)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\frac{\varphi_+}{2}} \left[1 + \frac{z - z_+}{\sqrt{5}} \mathrm{e}^{-\mathrm{i}\varphi_+} - \sum_{n=2}^{\infty} (-1)^n \frac{1 \cdot 3 \cdots (2n-3)}{n!} \left(\frac{z - z_+}{\sqrt{5}}\right)^n \mathrm{e}^{-\mathrm{i}n\varphi_+}\right].$$

We now calculate the radius of convergence of the last series. Since

$$\frac{1 \cdot 3 \cdot 5 \cdots (2(n+1)-3)}{(n+1)!} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-3)} = \frac{2(n+1)-3}{n+1} \xrightarrow{n \to \infty} 2,$$

we see from the ratio test that the series converges for $|z - z_+| < \sqrt{5}/2$. It is clear that the radius of convergence cannot exceed $\sqrt{5}/2$, since this is the distance from the centre z_+ of the series to the origin, which is a singularity of f_1. However, note that points of non-analyticity of P.V.$z^{\frac{1}{2}}$ on the negative axis $(-\infty, 0)$ lie inside the disk of convergence of the power series for f_1.



We have therefore constructed f_+, an analytic continuation of f_1 into $D_1 \cup B(z_+, \sqrt{5}/2)$. The extension f_+ is given by $f_+ = f_1$ on D_1, and it is defined by the power series above for $z \in B(z_+, \sqrt{5}/2)$. In particular, the point $z = -1$ is in the domain of analyticity of f_+. By the identity theorem, the continuation of f_1 along any contour $\mathcal{C} \subset D_1 \cup B(z_+, \frac{\sqrt{5}}{2})$ from z_+ to -1 will coincide with f_+.

Next we consider $z_- = -1 - \frac{\mathrm{i}}{2}$, and we expand f_1 around z_-. It is by now easy to see that f_1 is given by a Taylor series centered at z_-, and that this series has radius of convergence $\sqrt{5}/2$. In particular, this gives an analytic extension f_- of f_1 into $D_1 \cup B(z_-, \sqrt{5}/2)$. The point $z = -1$ is in the domain of analyticity of f_-.

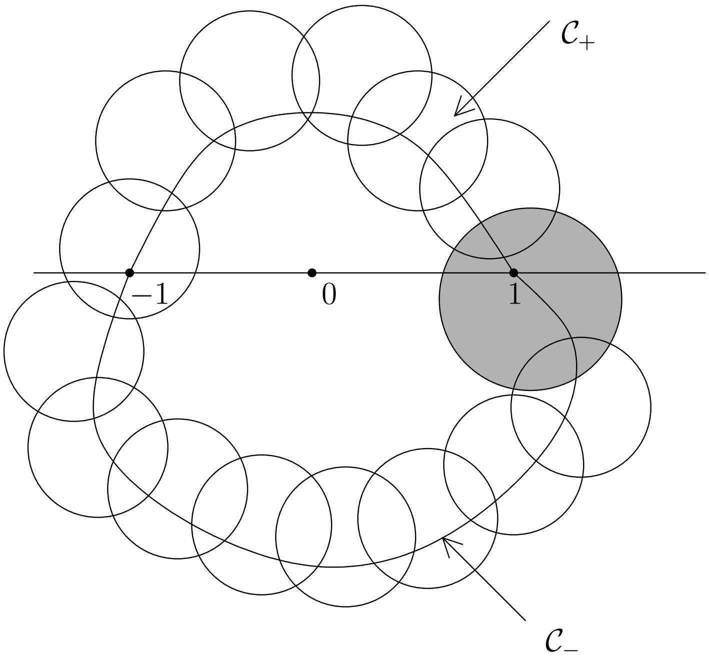
We have thus *two* analytic continuations of f_1, denoted f_+ and f_-, both defined and analytic at the point $z = -1$. Since $f_\pm$ are analytic in the disks $B(z_\pm, \frac{\sqrt{5}}{2})$, they are in particular continuous, so

$$f_+(-1) = \lim_{\alpha \uparrow \pi} f(\mathrm{e}^{\mathrm{i}\alpha}) = \lim_{\alpha \uparrow \pi} \mathrm{e}^{\frac{1}{2}\mathrm{Log}(\mathrm{e}^{\mathrm{i}\alpha})} = \lim_{\alpha \uparrow \pi} \mathrm{e}^{\frac{\mathrm{i}}{2}\mathrm{Arg}(\alpha)} = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}} = \mathrm{i},$$

and

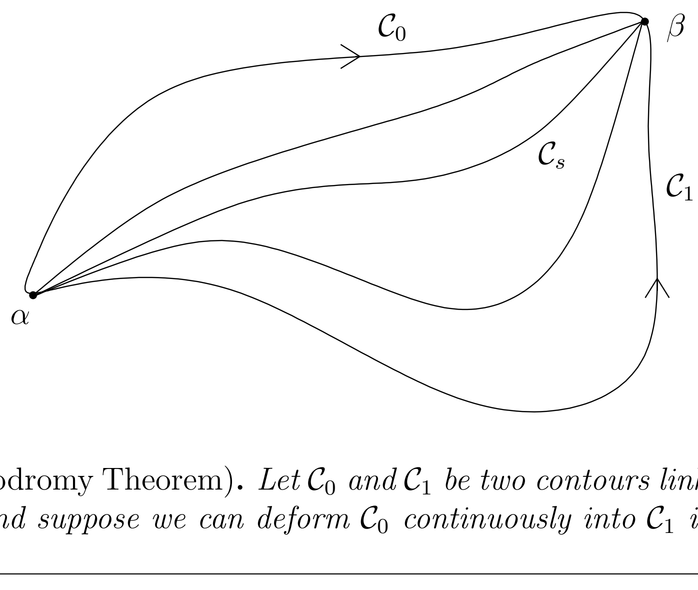
$$f_-(-1) = \lim_{\alpha \downarrow -\pi} f(\mathrm{e}^{\mathrm{i}\alpha}) = \mathrm{e}^{\frac{-\mathrm{i}\pi}{2}} = -\mathrm{i}.$$

Therefore, *the two extensions do not coincide* !



The above example shows that we can start with f_1 defined on $B(0,1)$ and extend f_1 in two ways, along $\mathcal{C}_+$ and $\mathcal{C}_-$, into domains containing -1. However, the two continuations do not coincide at the endpoint -1. Under what conditions can one guarantee uniqueness (single-valuedness) of analytic extensions? Note that in the above example, $\mathcal{C} = \mathcal{C}_+ \cup (-\mathcal{C}_-)$ encloses a singularity of the function $f_1(z) = z^{\frac{1}{2}}$. As we shall see, this is the origin of non-uniqueness of analytic continuations.

Let $\alpha, \beta \in \mathbb{C}$ be fixed and consider two contours $\mathcal{C}_0$, $\mathcal{C}_1$ linking α to β. We say that $\mathcal{C}_0$ can be deformed continuously into $\mathcal{C}_1$ if there is a continuous map $z : [0,1] \times [0,1] \to \mathbb{C}$, $(s,t) \mapsto z(s,t)$ such that $z(0,t)$ is a parametrization of $\mathcal{C}_0$, $z(1,t)$ is a parametrization of $\mathcal{C}_1$ (in particular, $z(0,0) = z(1,0) = \alpha$, $z(0,1) = z(1,1) = \beta$), and such that $t \mapsto z(s,t)$ defines a contour $\mathcal{C}_s$ linking α to β, for each $0 \le s \le 1$. The curves $\mathcal{C}_0$ and $\mathcal{C}_1$ are then also called *homotopically equivalent*.

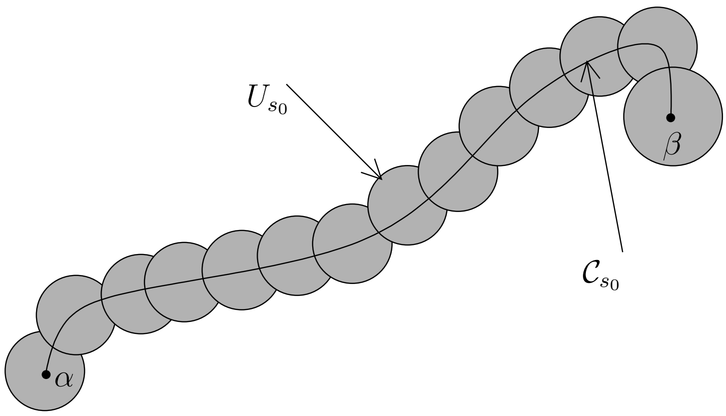


Theorem 34 (Monodromy Theorem)**.** *Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be two contours linking α to β. Suppose f is analytic at α and suppose we can deform $\mathcal{C}_0$ continuously into $\mathcal{C}_1$ in such a way that f*

can be continued along any intermediate contour $\mathcal{C}_s$*. Then the continuation along* $\mathcal{C}_0$ *and* $\mathcal{C}_1$ *coincide at* β*.*

In the example above, the hypothesis of the theorem is not satisfied, since one has to sweep the curve $\mathcal{C}_+$ continuously onto $\mathcal{C}_-$. So for some value of s, the intermediate curve $\mathcal{C}_s$ has to pass through the origin, which is a singularity of f_1. For this value of s, one cannot, of course, perform an analytic continuation from $\alpha = +1$ to $\alpha = -1$ along $\mathcal{C}_s$.

Proof. Consider $0 \le s_0 \le 1$. We can extend f from α to β along $\mathcal{C}_{s_0}$, via a chain of circles. The union of those circles defines an open set U_{s_0} containing $\mathcal{C}_{s_0}$, and we have an analytic function $F_{s_0}(z)$ on U_{s_0}.



There exists a $\delta > 0$ such that if $|s - s_0| < \delta$, then $\mathcal{C}_s \subset U_{s_0}$. This follows from the continuity of the map $z : [0,1] \times [0,1] \to \mathbb{C}$. Indeed, by the uniform continuity of this map, we have that for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|(s_0, t_0) - (s,t)\| < \delta \Rightarrow |z(s_0,t_0) - z(s,t)| < \epsilon$. (Here, $\|(x,y)\| = \sqrt{x^2+y^2}$ denotes the Euclidean norm of $\mathbb{R}^2$.) In particular, if $\|(s_0,t) - (s,t)\| = |s_0 - s| < \delta$, then $|z(s_0,t) - z(s,t)| < \epsilon$ (for all t). Thus $|s_0 - s| < \delta$ implies that $\max_{t\in[0,1]} |z(s_0,t) - z(s,t)| < \epsilon$. Thus $\mathcal{C}_s$ lies in an ϵ-neighbourhood of $\mathcal{C}_{s_0}$, if $|s_0 - s| < \delta$. It suffices to take $\epsilon = \frac{1}{2}\mathrm{dist}(\mathcal{C}_{s_0}, \mathbb{C}\backslash U_{s_0})$, then $\mathcal{C}_s \subset U_{s_0}$ for all s such that $|s_0 - s| < \delta$.

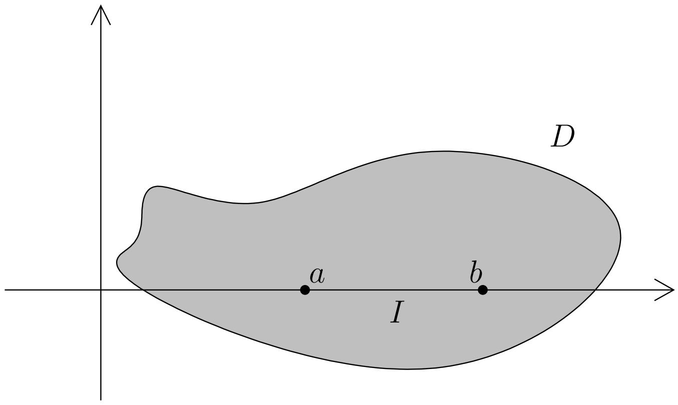
Now for each s such that $|s_0 - s| < \delta$, we get an analytic function $z \mapsto F_s(z)$ in some neighbourhood U_s of $\mathcal{C}_s$, again by the circle chain procedure. The identity theorem gives $F_s = F_{s_0}$. Hence, for any $s_0 \in [0,1]$, we find a $\delta_{s_0} > 0$ such that the extension along all curves $\mathcal{C}_s$, $|s - s_0| < \delta_{s_0}$, yield the same analytic function at β. Since $[0,1]$ is compact, we can find finitely many such $s_1, \dots, s_n$ and $\delta_1, \dots, \delta_n$ such that $\cup_{j=1}^n B(s_j, \delta_j)$ covers $[0,1]$. Therefore, the extensions along $\mathcal{C}_0$ and $\mathcal{C}_1$ yield the same value at β.

Actually, this proof shows that if a point $\zeta \in \mathbb{C}$ belongs to the domain of any two analytic functions F_1, F_2, obtained in this procedure, then $F_1(\zeta) = F_2(\zeta)$, so the extension yields a single-valued analytic function between $\mathcal{C}_0$ and $\mathcal{C}_1$. □

4.1 Extension of real functions to analytic ones

Suppose $f : I = [a,b] \to \mathbb{R}$ is a real function. When can we find a complex domain D containing I, an a function $F : D \to \mathbb{C}$, analytic on D, such that $F(x) = f(x)$, for all $x \in I$?

If such an extension exists, it must be unique: Suppose F_1, F_2 and D_1, D_2 do the job. Then $D_1 \cap D_2$ contains I, and F_1 and F_2 must coincide on I. By the identity theorem, F_1 coincides with F_2 on all of $D_1 \cap D_2$.



Of course, if f is not infinitely many times differentiable, then one cannot find a complex analytic extension (since if it existed, the function F, restricted to I, would have to be infinitely many times differentiable). However, $f \in C^\infty(I)$ does not guarantee the existence of a complex analytic extension.

Theorem 35. *Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$. Then there exists a complex domain D and an extension F of f, analytic on D, if any only if $f(x)$ can be expanded in a real power series around any point $x \in I$.*

Proof. ($\Leftarrow$) Suppose we have, for any $x_0 \in I$ fixed, an expansion

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

for some $a_n \in \mathbb{R}$, with radius of convergence $r_0 > 0$. Then define

$$F(z) := \sum_{n=0}^{\infty} a_n (z - x_0)^n.$$

This series converges for $|z - x_0| < r_0$. Indeed, the radius convergence is $R_0 = 1/L_0$, where $L_0 = \limsup |a_n|^{\frac{1}{n}} = 1/r_0$. Now set $D = \cup_{x_0 \in I} B(x_0, r_0) \subset \mathbb{C}$. D is a domain. By the identity theorem, F is well defined and analytic on D, and by construction, $F(x) = f(x)$ for all $x \in I$.

($\Rightarrow$) Suppose we have a complex domain D containing I, and a function F analytic on D, such that $F(x) = f(x)$ for all $x \in I$. Let $x_0 \in I$ be fixed. Since F is analytic at x_0, if has a Taylor expansion

$$F(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

convergent for $|x - x_0| < r_0$. Thus f has a power series expansion around any $x_0 \in I$. To complete the proof we just have to show that the coefficients a_n are real. We know that

$a_n = \frac{1}{n!} F^{(n)}(x_0)$. Now

$$F^{(n)}(x_0) = \frac{\mathrm{d}^n F}{\mathrm{d}z^n}(x_0) = \frac{\partial^n F}{\partial x^n}(x_0) = \frac{\partial^n f}{\partial x^n}(x_0) = f^{(n)}(x_0),$$

since F and f coincide on I. Thus $a_n = \frac{1}{n!} f^{(n)}(x_0) \in \mathbb{R}$. □

Example 48. The trigonometric functions

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

have unique complex (entire) extensions. (Just replace x by z in the Taylor series.)