# Infinite Products of Random Matrices & Repeated Interaction Dynamics

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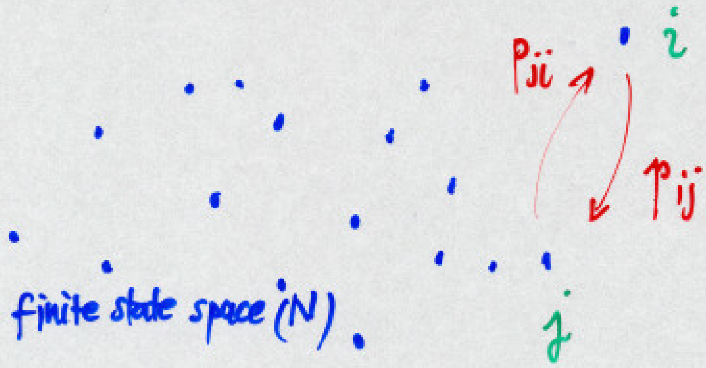
# Random Reduced Dynamics Operators (RRDO)

Motivated by :

## 1. Markov chains




jump probabilities :

$$0 \leq p_{ij} \leq 1$$

transition matrix :

$$(M)_{ij} = p_{ij}$$

Random variable : $X_n$ (position at step $n$)

Initial proba vector
$$\mu = \begin{bmatrix} \mathbb{P}(X_0 = 1) \\ \vdots \\ \mathbb{P}(X_0 = N) \end{bmatrix}^T$$

$$\mathbb{P}(X_n = j \mid X_{n-1} = i) = p_{ij} = p_{ij}(\underline{n}) \qquad \text{(inhomogeneous)}$$

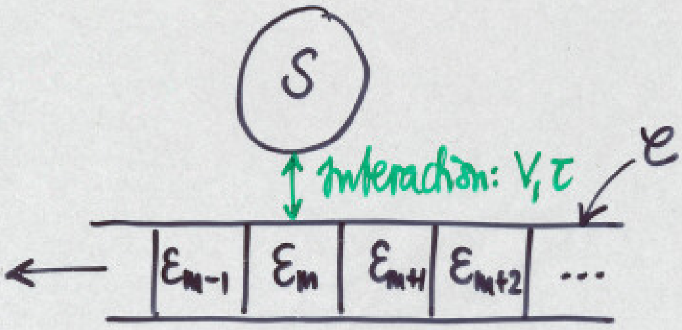
$$\mathbb{P}(X_n = j) = \left( \mu\, M(1)\, M(2) \cdots M(n) \right)_j$$

- $M(k)$ stochastic matrix $\left( M_{ij} \geq 0, \ \sum_j M_{ij} = 1, \forall i \right)$

  $\Rightarrow$ $M(k)$ contraction for norm
  $$|||\psi||| = \max_{1 \leq j \leq N} |\psi_j| \qquad (\psi \in \mathbb{C}^N)$$

- $\psi_S := \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T$ is invariant : $M(k)\psi_S = \psi_S, \ \forall k.$

## 2. Repeated Interaction Quantum systems




$S$: syst. of interest

$\mathcal{C}$: chain of identical, independent elements $\mathcal{E}$

$V$: interaction operator

$\tau$: interaction time

**Hilbert space**: $\mathcal{H}_S \otimes \mathcal{H}_\mathcal{E} \otimes \mathcal{H}_\mathcal{E} \otimes \cdots$

**Reference vector**: $\psi = \psi_S \otimes \psi_\mathcal{E} \otimes \psi_\mathcal{E} \otimes \cdots \quad \in \mathcal{H}_S$

**Interaction operator**: $V$ acts on $\mathcal{H}_S \otimes \mathcal{H}_\mathcal{E}$

**Repeated interaction dynamics**: $\psi \mapsto e^{-i\tau H_n} \cdots\cdots e^{-i\tau H_2} e^{-i\tau H_1} \psi$

where $$H_k = H_S + \sum_{j=1}^{\infty} H_{\mathcal{E},j} + \lambda V_k$$

coupling constant

**Observables**: Operators on $\mathcal{H}_S \otimes \mathcal{H}_\mathcal{E} \otimes \mathcal{H}_\mathcal{E} \otimes \cdots$

$A_S \in \mathcal{B}(\mathcal{H}_S)$ "observable of $S$"

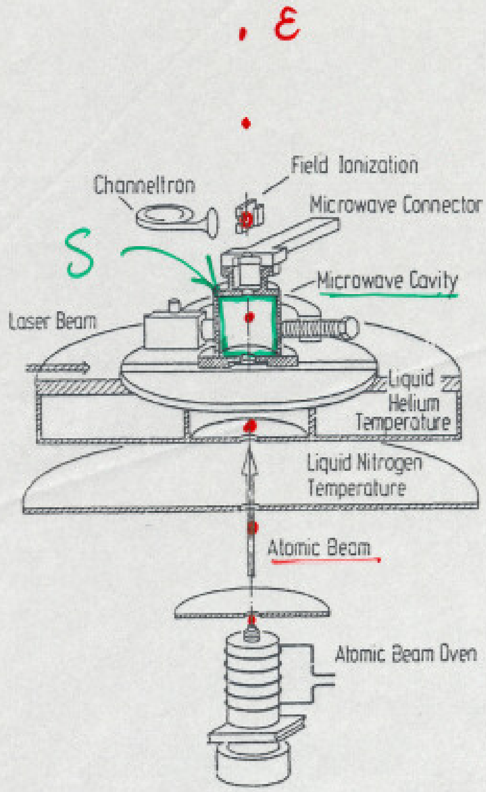



FIG. 1. Vacuum chamber with the atomic-beam arrangement and the microwave cavity. The upper part is cooled to liquid-helium temperature.

Meschede et al. "One-Atom Maser"

Phys. Rev. Lett. 54, 551 (1985)

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**Reduced dynamics of $S$:** $\quad A_S, \psi$ :

$$\langle \psi, e^{i\tau H_1} \cdots e^{i\tau H_n} A_S e^{-i\tau H_n} \cdots e^{-i\tau H_1} \psi \rangle$$

$$= \langle \psi, e^{i\tau K_1} \cdots e^{i\tau K_n} A_S \psi \rangle \qquad (*)$$

$$K_j \text{ s.t. } \begin{cases} e^{itK_j} A_S e^{-itK_j} = e^{itH_j} A_S e^{-itH_j} & (\forall A_S) \\ e^{-itK_j} \psi = \psi \end{cases}$$

$$P = \mathbb{1}_{\mathcal{H}_S} \otimes P_{\Omega_\mathcal{E}} \otimes P_{\Omega_\mathcal{E}} \otimes \cdots \quad \text{projects onto } \mathcal{H}_S \qquad (P_{\Omega_\mathcal{E}} = |\Omega_\mathcal{E}\rangle\langle\Omega_\mathcal{E}|)$$

$$\rightarrow \quad A_S \psi = A_S P \psi = P A_S \psi, \text{ so}$$

$$(*) = \langle \psi, P e^{i\tau K_1} \cdots e^{i\tau K_n} P A_S \psi \rangle$$

$$= \langle \psi, P e^{i\tau K_1} P \cdots P e^{i\tau K_n} P A_S \psi \rangle$$

(by independence of the $\mathcal{E}_j$)

$$= \langle \psi_S, M_1 \cdots M_n A_S \psi_S \rangle$$

($M_j = P e^{i\tau K_j} P$: operator on $\mathcal{H}_S$)

- $\| M_j A_S \psi_S \| \leq \| A_S \| := ||| A_S \psi_S |||$ (defines norm on $\mathcal{H}_S$)

  $\Rightarrow$ $M_j$ contraction for $||| \cdot |||$

- $e^{-itK_j} \psi = \psi \Rightarrow M_j \psi_S = \psi_S, \ \forall j$ (invariant vector)

We consider complex systems described by random charact.

$\Rightarrow$ $M_1, \dots, M_n$ iid random matrices.

Definition (RRDO) Let $M(\omega)$ be a random matrix on $\mathbb{C}^N$, with probability space $(\Omega, \mathcal{F}, p)$. $M(\omega)$ is a random reduced dynamics operator (RRDO) if

(1) $\exists$ norm $|||\cdot|||$ on $\mathbb{C}^N$ with respect to which $M(\omega)$ is a contraction, $\forall \omega$.

(2) $\exists\ \psi_S$, constant in $\omega$, s.t. $M(\omega)\psi_S = \psi_S$, $\forall \omega$.

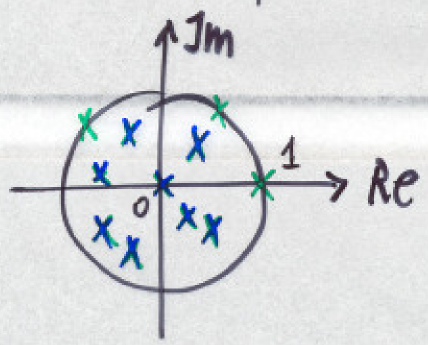
A RRDO generates the dynamical process

$$\Psi_n(\bar{\omega}) = M(\omega_1) \cdots M(\omega_n) \quad , \quad \bar{\omega} \in \Omega^{\mathbb{N}}.$$

Goal: determine the ergodic properties of the process $\Psi_n$

(1) $\Rightarrow$ $\mathrm{spec}(M(\omega)) \subset \{|z| \leq 1\}$

(2) $\Rightarrow$ $1 \in \mathrm{spec}(M(\omega))$




x : fluctuations

x : decay

Decompose $\Psi_n$ into decaying part and fluctuating part.

$P_1(\omega)$ := projection of $M(\omega)$ corr. to eigenval. 1

$$\begin{cases} P(\omega) := |\psi_s\rangle\langle\psi_s|\, P_1(\omega) & \text{a projection} \\ Q(\omega) := \mathbb{1} - P(\omega) \end{cases}$$

$$M(\omega) = P(\omega) + Q(\omega)\, M(\omega)\, Q(\omega) =: P(\omega) + M_Q(\omega)$$

$$\Rightarrow \begin{cases} \Psi_n(\bar{\omega}) = M(\omega_1)\cdots M(\omega_n) = |\psi_s\rangle\langle\theta_n(\bar{\omega})| \\ \qquad\qquad\qquad + M_Q(\omega_1)\cdots M_Q(\omega_n) \\ \theta_n(\bar{\omega}) = M^*(\omega_n)\cdots M^*(\omega_2)\, P_1(\omega_1)^*\, \psi_s \end{cases}$$

(Markov process)

__Def.__ $\mathcal{M}_{(E)}$ = those RRDO whose spectrum on the complex unit circle consists only of a simple eigenvalue $\{1\}$.

Natural probability on $\Omega^{\mathbb{N}}$: $d\mathbb{P} = \prod_{j\geq 1} dp_j$, $dp_j = dp$, $\forall j$.

__Theorem 1.__ (Decaying process)

$M(\omega)$: a RRDO. Suppose that $p\left(M(\omega) \in \mathcal{M}_{(E)}\right) \neq 0$. Then there is a set $\Omega_1 \subset \Omega^{\mathbb{N}}$ and constants $C, \alpha > 0$, s.t. $\mathbb{P}(\Omega_1) = 1$ and s.t. $\forall \bar{\omega} \in \Omega_1$ and any $n \in \mathbb{N}$:

$$\| M_Q(\omega_1) \cdots M_Q(\omega_n) \| \leq C e^{-\alpha n}.$$

$$\mathbb{E}[f] := \int_\Omega f(\omega)\, dp(\omega)$$

$P_{1,\mathbb{E}[M]}$ : spectral projection of $\mathbb{E}[M]$ onto eigenvalue $\{1\}$.

**Theorem 2** (Fluctuating process)

$M(\omega)$: a RRDO. Suppose that $p(M(\omega) \in \mathcal{M}_{(E)}) \neq 0$. Then $\mathbb{E}[M] \in \mathcal{M}_{(E)}$. Moreover, $\exists\, \Omega_2 \subset \Omega^{\mathbb{N}}$ s.t. $\mathbb{P}(\Omega_2) = 1$, and $\forall \bar{\omega} \in \Omega_2$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \theta_n(\bar{\omega}) = \theta,$$

where $\theta = (\mathbb{1} - \mathbb{E}[M_Q]^*)^{-1}\, \mathbb{E}[P_1^*(\omega)\psi_S] = P^*_{1,\mathbb{E}[M]}\psi_S$.

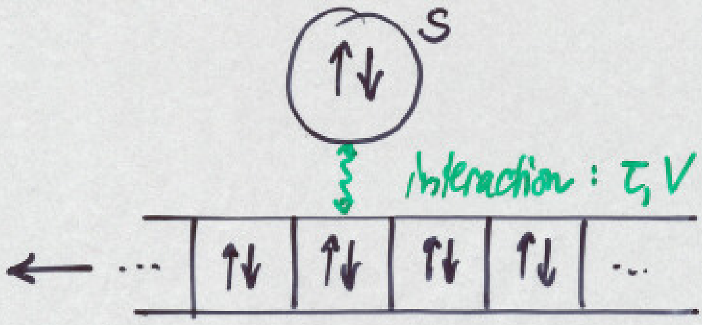
The combination of Theorems 1 & 2 yields:

**Theorem 3** ( Ergodic theorem for RRDP)

$M(\omega)$: a RRDO. Suppose that $p(M(\omega) \in \mathcal{M}_{(E)}) \neq 0$. Then $\exists\, \Omega_3 \subset \Omega^{\mathbb{N}}$ s.t. $\mathbb{P}(\Omega_3) = 1$, and $\forall \bar{\omega} \in \Omega_3$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} M(\omega_1)\cdots M(\omega_n) = |\psi_S\rangle\langle\theta| = P_{1,\mathbb{E}[M]}.$$

# Spin systems: an explicit example




Only 2 levels of $S, \mathcal{E}$ are involved in physical process.

**Hilbert spaces:** $\mathcal{H}_S = \mathbb{C}^2 = \mathcal{H}_\mathcal{E}$

**Reference vectors:** $\begin{cases} \psi_\mathcal{E} : \text{Gibbs equilibrium state at temp } T = 1/\beta \\ \psi_S : \text{anything} \end{cases}$

**Non-interacting Hamiltonians:**

$$H_S = \begin{bmatrix} 0 & 0 \\ 0 & E_S \end{bmatrix}, \quad H_\mathcal{E} = \begin{bmatrix} 0 & 0 \\ 0 & E_\mathcal{E} \end{bmatrix}$$

**Interaction operator:** $V = I \otimes a^* + I^* \otimes a \quad (\text{on } \mathcal{H}_S \otimes \mathcal{H}_\mathcal{E})$

$$I = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \begin{cases} a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \text{annihilation op.} \\ a^* = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \text{creation op.} \end{cases}$$

$$\Rightarrow \quad H_k = H_S + \sum_{n \geq 1} H_{\mathcal{E},n} + \lambda V_k \qquad (k : \text{time-step})$$

Repeated interaction dynamics leads to reduced dynamics operator

$$M = P e^{i\tau K_\lambda} P \qquad (2 \times 2 \text{ matrix})$$

## Deterministic Result

Assume a non-resonance condition: $\tau E_\mathcal{E} \notin \pi\mathbb{Z}$ &

either (S1) $b \neq 0$ & $\tau(E_\mathcal{E} - E_S) \notin 2\pi\mathbb{Z}$

or (S2) $c \neq 0$ & $\tau(E_\mathcal{E} + E_S) \notin 2\pi\mathbb{Z}$

**Theorem** There is a $\lambda_0 > 0$ s.t. if $0 < |\lambda| < \lambda_0$, the following statements are true. Let $S$ be initially in an arbitrary state, and let $A$ be an arbitrary observable of $S$. Under the repeated interaction dynamics, we have

$$\langle A \rangle_n \xrightarrow{n\to\infty} \rho_+(A) \qquad (*)$$

where

$$\rho_+(A) = \frac{\alpha_1}{\alpha_1+\alpha_2} \left\langle \begin{bmatrix}1\\0\end{bmatrix}, A\begin{bmatrix}1\\0\end{bmatrix}\right\rangle + \frac{\alpha_2}{\alpha_1+\alpha_2} \left\langle \begin{bmatrix}0\\1\end{bmatrix}, A\begin{bmatrix}0\\1\end{bmatrix}\right\rangle + O(\lambda^2)$$

and

$$\alpha_1 = |b|^2 \operatorname{sinc}^2\left[\frac{\tau(E_\mathcal{E}-E_S)}{2}\right] + e^{-\beta E_\mathcal{E}} |c|^2 \operatorname{sinc}^2\left[\frac{\tau(E_\mathcal{E}+E_S)}{2}\right]$$

$\left(\operatorname{sinc} x = \frac{\sin x}{x}\right)$; $\alpha_2$ has similar expression.

Moreover, the convergence in (*) is exponentially fast, with rate $1/\gamma_0$, with

$$\gamma_0 = \lambda^2\tau^2 \min\left[\frac{\alpha_1+\alpha_2}{1+e^{-\beta E_\mathcal{E}}},\ \frac{1}{2}\frac{\alpha_1+\alpha_2}{1+e^{-\beta E_\mathcal{E}}} + \frac{|a-d|^2}{2}\operatorname{sinc}^2\left(\frac{\tau E_\mathcal{E}}{2}\right)\right]$$

Applications:
- decoherence
- control of asymptotic state
- monitoring of $S$ (inv. scattering)

## Probabilistic Result

Assume $\tau = \tau(\omega)$ random interaction time $\Rightarrow$ $M = M(\omega)$ RRDO.

(other possibilities: $E_\mathcal{E}$, $E_S$, temperature, $I$ random)

(S3) $b \neq 0$ & there are $\eta_- > 0$, $\tau_{max} > 0$ s.t.

$$p\left(\tau E_\mathcal{E} \notin \pi\mathbb{Z},\ \left|\tau(E_\mathcal{E}-E_S)/2 - \pi\mathbb{Z}\right| \geq \eta_-,\ \tau \leq \tau_{max}\right) \neq 0$$

or (S4) $c \neq 0$ & similar

**Theorem** There is a $\lambda_0 > 0$ (depending explicitly on $E_\mathcal{E}, E_S, \eta_-, \tau_{max}$) s.t. if $0 < |\lambda| < \lambda_0$, then

$$p\left(M(\omega) \in \mathcal{M}_{(E)}\right) \neq 0.$$

Thus the theorems about the process $\Psi_n = M(\omega_1)\cdots M(\omega_n)$ apply:

<u>Corollary</u> Let $0<|\lambda|<\lambda_0$. Let $\langle A\rangle_n$ be the average of an observable $A$ of $S$ at time-step $n$ (random quantity). Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle A\rangle_n = \langle \theta, A\,\psi_S\rangle . \quad \text{almost surely}$$

Here, $\psi_S$ is the ref. state of $S$, and

$$\theta = P^*_{1,\mathbb{E}[M]}\,\psi_S$$

($\theta$ can be expanded in coupling parameter $\lambda$)

# Rough outlines of proofs

• $\| M_Q(\omega_1) \cdots M_Q(\omega_n) \| \le C e^{-\alpha n}$ a.s. $\bar{\omega} \in \Omega^{\mathbb{N}}$

Provided $p\left( M(\omega) \in \mathcal{M}_{(E)} \right) \neq 0$ (*)

1. $\exists M_0 \in \mathcal{M}_{(E)}$ s.t. $\forall \varepsilon > 0$, $p\left( \| M(\omega) - M_0 \| < \varepsilon \right) > 0$

(Use $\mathcal{M}_{(E)} = \bigcup_{n \geq 1} \mathcal{M}_{(n)}$, $\mathcal{M}_{(n)} = \{ \text{RRDO} : \operatorname{spec}(M_Q) \subset \{ |z| \le 1 - \frac{1}{n} \} \}$

& compactness of $\mathcal{M}_{(n)}$ )

(On a set of non-vanishing measure, the $M(\omega)$ can be approximated by a constant $M_0$ (indep. of $\omega$))

2. $\exists \Omega_\varepsilon \subset \Omega$, $p(\Omega_\varepsilon) > 0$, s.t. $\forall \omega \in \Omega_\varepsilon$,

$$M_Q(\omega) = M_{Q_0} + \Delta(\omega) , \quad \| \Delta(\omega) \| < \varepsilon$$

Then use combinatorial argument & estimates to get

$$\| M_Q(\omega_1) \cdots M_Q(\omega_k) \| \le C e^{-\delta k}$$

$\forall k$, $\forall \omega_1, \dots, \omega_k \in \Omega_1$.

3. Bootstrapping argument to upgrade estimate to $\bar{\omega} \in \Omega_1 \subset \Omega^{\mathbb{N}}$
$\mathbb{P}(\Omega_1) = 1$

- <u>Fluctuating process</u>

$$M(\omega_1)\cdots M(\omega_n) = |\psi_S\rangle\langle\theta_n(\bar\omega)| + M_Q(\omega_1)\cdots M_Q(\omega_n)$$

Want: $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \theta_n(\bar\omega) = \theta$ a.s.

Can write

$$\sum_{n=1}^{N}\theta_n(\bar\omega) = \sum_{k=1}^{N}\sum_{j=0}^{N-k} \theta^{(k)}(T^j\bar\omega), \qquad (*)$$

where

$$\theta^{(k)}(\bar\omega) = \theta^{(k)}(\omega_1,\dots,\omega_k) = M_Q^*(T^{k-1}\omega)\, M_Q^*(T^{k-2}\omega)\cdots M_Q(T\omega)\, P_1^*(\omega)\,\psi_S$$

with $T: \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ shift, $(T\bar\omega)_j = \omega_{j+1}$

$$(*) \Rightarrow \quad \frac{1}{N}\sum_{n=1}^{N}\theta_n(\bar\omega) = \sum_{k=1}^{\infty} \chi_{(k\le N)} \underbrace{\sum_{j=0}^{N-k}\theta^{(k)}(T^j\bar\omega)\frac{1}{N}}_{g(k,N,\bar\omega)}$$

by ergodicity:

$$\lim_{N\to\infty} g(k,N,\bar\omega) = \mathbb{E}[\theta^{(k)}] \quad \text{a.s.}$$

$$= \left(\mathbb{E}[M_Q^*]\right)^{k-1}\mathbb{E}[P_1^*(\omega)\psi_S]$$

$$\Rightarrow \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\theta_n(\bar\omega) = \left(\mathbb{1} - \mathbb{E}[M_Q]^*\right)^{-1}\mathbb{E}[P_1^*(\omega)\psi_S] =: \theta$$

a.s.

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