Some Applications of Resonance Theory to Open Spin Systems

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I Open Quantum Systems

- **Total system**: { system S } + { reservoir(s) R} + { interactions }
- **S**: *few degrees of freedom*, N-level system (finitely many spins)
- **R**: *many degrees of freedom*, spatially extended free (bosonic) quantum field in thermal equilibrium at temperature $T > 0$
- **Dynamics of total density matrix**:

$$\rho_{\rm SR}(t) = {\rm e}^{-{\rm i}tH/\hbar}\rho_{\rm SR}(0)\,{\rm e}^{{\rm i}tH/\hbar}$$

$H = H_{\rm S} + H_{\rm R} + H_{\rm I}$: total Hamiltonian

- **Reduced density matrix**: $\rho(t) = {\rm Tr}_{\rm R}\ \rho_{\rm SR}(t)$ (partial trace over R)
- **Dynamics of reduced density matrix**: $\rho(t) = V(t)\rho(0)$, $V(t)$ *dynamical map* (not (semi-)group)
- **Time-scales**:

$$\left\{\begin{array}{ll} \tau_{\rm S} & \text{isolated S} \quad (\leftrightarrow \omega_{\rm S} = (E-E')/\hbar) \\ \tau_{\rm relax} & \text{relaxation time of S} \quad (\leftrightarrow H_{\rm I}) \\ \tau_{\rm R} = \frac{\hbar}{k_{\rm B}T} & \text{thermal reservoir correlation time} \end{array}\right.$$

Quantum Optical Master Equation

[Legget et al. '81, Palma et. al. '96, Gardiner-Zoller '04, Weiss '99]

- **Finite system coupled to bosonic reservoir**

$$H = H_{\rm S} + \sum_k \hbar\omega_k a_k^\dagger a_k + \lambda G \sum_k g_k(a_k^\dagger + a_k)$$

$H_{\rm S}, G$: $N \times N$ matrices, g_k: coupling function; reduced evolution

$$\frac{\rm d}{{\rm d}t}\rho(t) = -\frac{1}{\hbar^2}\int_0^t {\rm Tr}_{\rm R}\big[H_{\rm I}(t), [H_{\rm I}(s), \rho_{\rm SR}(s)]\big]{\rm d}s$$

- **Born-Markov approximation**: system relaxation much slower than decay of reservoir correlations (memory effects weak) + **Rotating wave approximation**: syst. relax. much slower than free system dynamics

$=$ **Quantum Optical Regime:** $\max\{\tau_{\rm R}, \tau_{\rm S}\} \ll \tau_{\rm relax}$

Master equation, van Hove limit, resonance representation

- **Markovian master equation** ($\Leftarrow$ Born-Markov + rotating wave approx.)

$$\rho(t) = \mathrm{e}^{t\mathcal{L}_\lambda}\rho(0),$$

  Lindblad generator $\mathcal{L}_\lambda = \mathcal{L}_0 + \lambda^2 K^\#$.

- **Weak coupling (van Hove) limit**: $\forall a > 0$

$$\lim_{\lambda\to 0} \sup_{\lambda^2 t\in(0,a)} ||V_\lambda(t) - \mathrm{e}^{t\mathcal{L}_\lambda}|| = 0.$$

- **Resonance representation**

$$\sup_{t\geq 0} ||V_\lambda(t) - \mathrm{e}^{tM_\lambda}|| \leq C\lambda^2$$

  Valid for small λ. M_λ contains all orders in λ, $M_\lambda = \mathcal{L}_\lambda + O(\lambda^4)$. Necessitates regularity of interaction and positive temperature.

II Resonance representation of reduced dynamics

S: N-level system,

$$H_{\rm S} = {\rm diag}(E_1, \ldots, E_N), \qquad H_{\rm S}\Phi_n = E_n\Phi_n, \quad n = 1, \ldots, N$$

$\rm R = R_1 + \cdots + R_K$: collection of reservoirs,

$$H_{{\rm R}_j} = \int_{{\rm R}^3} |k| a_j^*(k) a_j(k) {\rm d}^3 k, \quad j = 1, \ldots, K$$

Interactions $\rm S \leftrightarrow R_j$:

$$H_{{\rm I},j} = \alpha_j G_j \otimes \varphi(g_j), \quad j = 1, \ldots, K$$

α_j: coupling constant, G_j: matrix on $\rm S$, $g_j(k) \in L^2({\rm R}^3, {\rm d}^3 k)$ form factor,

$$\varphi(g_j) = \frac{1}{\sqrt{2}} \int_{{\rm R}^3} \big\{ g_j(k) a^*(k) + g_j(k)^* a(k) \big\} {\rm d}^3 k$$

Evolution of reduced density matrix elements:

$$[\rho_t]_{mn} = \langle \Phi_m, \rho_t \Phi_n \rangle = \mathrm{Tr}_{\mathrm{SR}}\ \mathrm{e}^{-\mathrm{i}tH} \rho_{\mathrm{SR}}(0) \mathrm{e}^{\mathrm{i}tH} |\Phi_n\rangle\langle\Phi_m|$$

$$H = H_{\mathrm{S}} + \sum_{j=1}^{K} H_{\mathrm{R}_j} + \sum_{j=1}^{K} H_{\mathrm{I},j}$$

Uncoupled dynamics, $\alpha := \max|\alpha_j| = 0$

$$[\rho_t]_{mn} = \mathrm{e}^{\mathrm{i}t(E_n - E_m)} [\rho_0]_{mn}$$

Effects of coupling:

- *Irreversibility*
  $E_n - E_m \to \varepsilon^{(s)}_{E_n - E_m} = E_n - E_m + \delta^{(s)}_{E_n - E_m} + O(\alpha^4) \in \mathbf{C}$
  with $\delta^{(s)}_{E_n - E_m} = O(\alpha^2)$ and $\mathrm{Im}\,\delta^{(s)}_{E_n - E_m} \geq 0$

- *Joint evolution of elements*
  $[\rho_t]_{mn} = F_t\big([\rho_0]_{kl}\ :\ E_k - E_l = E_m - E_n\big) + O(\alpha^2)$
  define cluster $\mathcal{C}(e) = \{(k, l)\ :\ E_k - E_l = e\}$

Assumptions

1. Regularity of form factors: translation analyticity; IR behaviour $f, g \sim |k|^p$, $p = -\frac{1}{2} + \mathbf{N}$ and UV cutoff (e.g. $\sim \mathrm{e}^{-|k|/|k_0|}$)

2. Fermi golden rule condition: resonance energies $\varepsilon_e^{(s)}$ are distinct at 2nd order in α.

3. System and reservoirs not entangled initially,

$$\rho_{\mathrm{SR}}(0) = \rho_{\mathrm{S}}(0) \otimes \rho_{\mathrm{R}_1}(0) \cdots \otimes \rho_{\mathrm{R}_K}(0)$$

   and reservoirs in thermal state at temperature $T = 1/\beta > 0$.

Remark:

$$\tau_{\mathrm{S}} = \max_{E \neq E'} \frac{\hbar}{E - E'}, \quad \tau_{\mathrm{R}} = \frac{\hbar}{k_{\mathrm{B}} T}, \quad \tau_{\mathrm{relax}} \propto \lambda^{-2}$$

Assumptions imply $\max\{\tau_{\mathrm{S}}, \tau_{\mathrm{R}}\} \ll \tau_{\mathrm{relax}}$, quantum optical regime.

Theorem *There is an* $\alpha_0 > 0$ *s.t. if* $\alpha < \alpha_0$ *then we have for all* $t \geq 0$

$$[\rho_t]_{mn} = \sum_{(k,l)\in\mathcal{C}(E_m-E_n)} A_t(m,n;k,l)[\rho_0]_{kl} + O(\alpha^2),$$

where the remainder is uniform in t. *The* A_t *satisfy the Chapman-Kolmogorov relation*

$$A_{t+s}(m,n;k,l) = \sum_{(p,q)\in\mathcal{C}(E_m-E_n)} A_t(m,n;p,q)A_s(p,q;k,l)$$

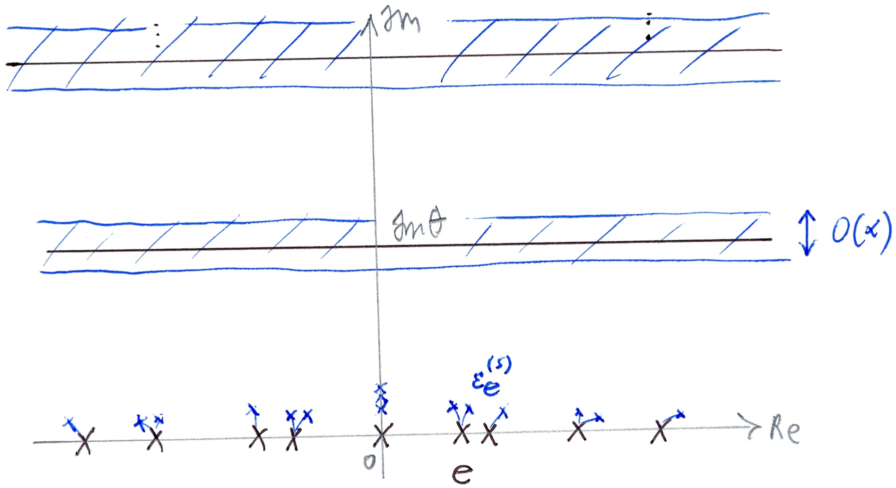
and they have the resonance representation

$$A_t(m,n;k,l) = \sum_{s=1}^{\mathrm{mult}(E_n-E_m)} \mathrm{e}^{\mathrm{i}t\varepsilon^{(s)}_{E_n-E_m}} \langle \Phi_l \otimes \Phi_k, \eta^{(s)}_{E_n-E_m}\rangle \langle \widetilde{\eta}^{(s)}_{E_n-E_m}, \Phi_n \otimes \Phi_m\rangle.$$

Here, $\varepsilon^{(s)}_{E_n-E_m} \in \mathbf{C}$ *are resonance energies and* $\eta^{(s)}_{E_n-E_m}, \widetilde{\eta}^{(s)}_{E_n-E_m} \in \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{S}}$ *are resonance vectors.*

• Leading dynamics: distinct spectral subspaces of $L_{\rm S} = H_{\rm S}\otimes \mathbb{1} - \mathbb{1}\otimes H_{\rm S}$ evolve independently, dynamics within each subspace is markovian.

• To calculate resonance data $\varepsilon_e^{(s)}, \eta_e^{(s)}, \widetilde{\eta}_e^{(s)}$, use spectral deformation of Liouville operator $K_\alpha(\theta) = L_0(\theta) + W_\alpha(\theta)$ and perturbation theory in coupling strength α.



$L_0(\theta) = L_{\rm S} + L_{\rm R} + \theta N$ sum of commuting selfadjoint operators; continuous spectrum separated from point spectrum by $\mathrm{Im}\,\theta$.
Resonance energies
$\varepsilon_e^{(s)} = e + \delta_e^{(s)} + O(\alpha^4)$
$\delta_e^{(s)} = O(\alpha^2)$

III Open Spin Systems

Model: N spins $1/2$ coupled to local and collective reservoirs

$$
\begin{aligned}
H &= \sum_{n=1}^{N} \omega_n S_n^z + \sum_{n=1}^{N} H_{\mathrm{R},n} + H_{\mathrm{R}} \\
&\quad + \sum_{n=1}^{N} \lambda_n S_n^x \otimes \varphi_c(g_c) + \sum_{n=1}^{N} \kappa_n S_n^z \otimes \varphi_c(f_c) \\
&\quad + \sum_{n=1}^{N} \mu_n S_n^x \otimes \varphi_n(g_n) + \sum_{n=1}^{N} \nu_n S_n^z \otimes \varphi_n(f_n).
\end{aligned}
$$

$\omega_n > 0$: frequency of spin n; H_{R}: Hamiltonian of single bosonic reservoir,

$$
S^z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad S^x = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad H_{\mathrm{R}} = \int_{\mathrm{R}^3} |k| a^*(k) a(k) \mathrm{d}^3 k
$$

Form factors $f_c(k)$, $f_n(k)$, coupling constants $\kappa_n, \lambda_n, \mu_n, \nu_n$

Results on:

• Decoherence: N spins, collective reservoirs
(with G.P. Berman and I.M. Sigal, *Phys. Rev. Lett.* (2007), *Annals of Physics* (2008), *Annals of Physics* (2008))

• Entanglement: 2 spins, collective and local reservoirs
(with G.P. Berman, F. Borgonovi, K. Gebresellasie, *submitted* 2010)

Work in progress:

• Magnetization: N spins, collective and local reservoirs
(with G.P. Berman and T. Redondo)

Evolution of collective decoherence

Palma-Suominen-Ekert ['96]: pure dephasing

$$H = \mathrm{diag}(E_1, \dots, E_N) + H_\mathrm{R} + \mathrm{diag}(\gamma_1, \dots, \gamma_N) \otimes \varphi(g)$$

Explicit solution:

$$[\rho_t]_{m,n} = [\rho_0]_{m,n}\ \mathrm{e}^{-\mathrm{i}t(E_m - E_n)}\ \mathrm{e}^{\mathrm{i}(\gamma_m^2 - \gamma_n^2)S(t)}\ \mathrm{e}^{-(\gamma_m - \gamma_n)^2 \Gamma(t)}$$

where

$$\begin{aligned} \Gamma(t) &= \int_{\mathrm{R}^3} |g(k)|^2 \coth(\beta\omega/2) \frac{\sin^2(\omega t/2)}{\omega^2} \mathrm{d}^3 k \\ S(t) &= \frac{1}{2} \int_{\mathrm{R}^3} |g(k)|^2 \frac{\omega t - \sin \omega t}{\omega^2} \mathrm{d}^3 k \end{aligned}$$

Model with dephasing and energy-exchange

N-qubit register *collectively* coupled to single bosonic reservoir

$$H = \sum_{j=1}^{N} B_j S_j^z + H_{\rm R} + \lambda_1 \sum_{j=1}^{N} S_j^z \otimes \phi(g_1) + \lambda_2 \sum_{j=1}^{N} S_j^x \otimes \phi(g_2).$$

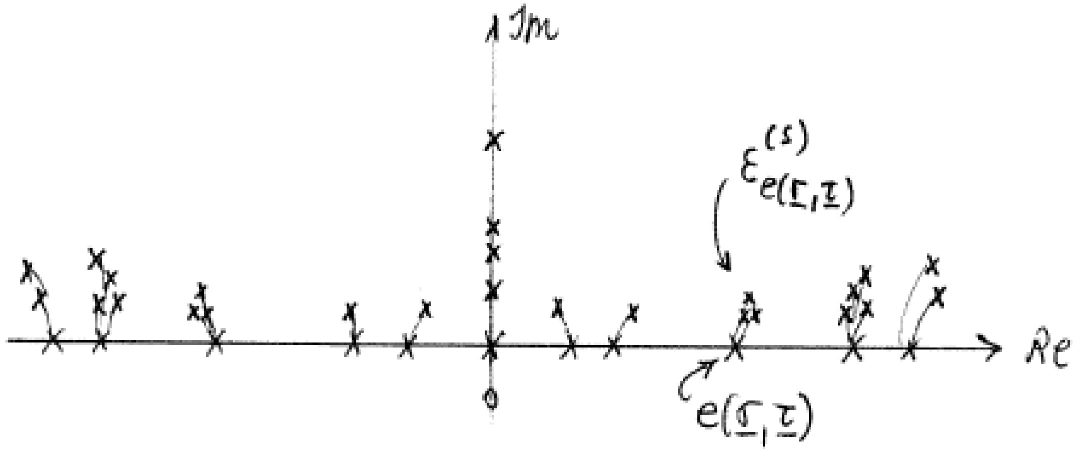
$B_j > 0$: magnetic field at location of spin j, collective energy conserving and energy exchange interaction; spin config. $\underline{\sigma} = (\sigma_1, \ldots, \sigma_N)$, $\sigma_j = \pm 1$

- Energy basis: $H_{\rm S}\varphi_{\underline{\sigma}} = E(\underline{\sigma})\varphi_{\underline{\sigma}}$, $E(\underline{\sigma}) = \sum_{j=1}^{N} \frac{1}{2} B_j \sigma_j$
- Bohr energies: $e(\underline{\sigma}, \underline{\tau}) = E(\underline{\sigma}) - E(\underline{\tau})$
- Matrix element clusters: $\mathcal{C}(\underline{\sigma}, \underline{\tau}) = \{(\underline{\sigma}', \underline{\tau}') \ : \ e(\underline{\sigma}, \underline{\tau}) = e(\underline{\sigma}', \underline{\tau}')\}$
- Assume *uncorrelated* magnetic field: $n_j, n_j' \in \{-1, 0, 1\}$

$$\left\{ \sum_{j=1}^{N} B_j (n_j - n_j') = 0 \right\} \Rightarrow \{ n_j = n_j' \ \text{ for all } j \}$$

- Resonance representation

$$[\rho_t]_{\underline{\sigma},\underline{\tau}} = \sum_{(\underline{\sigma}',\underline{\tau}')\in\mathcal{C}(\underline{\sigma},\underline{\tau})} \sum_{s=1}^{\mathrm{mult}(e(\underline{\sigma},\underline{\tau}))} \exp\{\mathrm{i}t\varepsilon^{(s)}_{e(\underline{\sigma}',\underline{\tau}')}\}\ C(\underline{\sigma},\underline{\tau};\underline{\sigma}',\underline{\tau}')\ [\rho_0]_{\underline{\sigma}',\underline{\tau}'} + O(\lambda_1^2+\lambda_2^2)$$



- Perturbation expansion: $\varepsilon^{(s)}_e = e + \delta^{(s)}_e + O(\lambda_1^4+\lambda_2^4)$
- Remainder not uniform in N

Cluster decoherence rates

- Each cluster (e) has own decay rate: cluster decoherence rate

$$\gamma_e = \min\left\{\mathrm{Im}\varepsilon_e^{(s)} : \ s = 1, \ldots, \mathrm{mult}(e)\right\}$$

- Thermalization rate:

$$\gamma_{\mathrm{therm}} = \min\left\{\mathrm{Im}\varepsilon_0^{(s)} : \ s = 1, \ldots, \mathrm{mult}(0) \ \ \text{with } \mathrm{Im}\,\varepsilon_0^{(s)} \neq 0\right\}$$

- Explicitly solvable energy-conserving model (Palma-Suominen-Ekert)

$$\gamma_e \propto [\textstyle\sum_{j=1}^{N}(\sigma_j - \tau_j)]^2$$

$\exists$ Decoherence-free subspaces

Explicit form of decoherence rates

$$\gamma_e = \left\{ \begin{array}{ll} \lambda_2^2 y_0, & e=0 \\ \lambda_1^2 y_1(e) + \lambda_2^2 y_2(e) + y_{12}(e), & e \neq 0 \end{array} \right\} + O(\lambda_1^4 + \lambda_2^4)$$

• $y_0 = \pi \min_{1\le j\le N}\{B_j^2 \mathcal{G}_2(B_j) \coth(\beta B_j/2)\}$
energy exchange, $\mathcal{G}_2(x) \propto |g_2(x)|^2$

• $y_1(e) = \frac{\pi}{2\beta}[e_0(e)]^2 \gamma_+$
energy conserving, $e_0(e) = \sum_{j=1}^N (\sigma_j - \tau_j)$, $\gamma_+ = \lim_{|k|\to 0} |k| \mathcal{G}_1(k)$

• $y_2(e) = \frac{1}{2}\pi \sum_{j:\sigma_j\neq\tau_j} B_j^2 \mathcal{G}_2(B_j) \coth(\beta B_j/2)$
energy exchange

• $y_{12}(e) \geq 0$: more complicated expression, dep. on both interactions, $O(\lambda_1^2 + \lambda_2^2)$
$y_{12}(e) > 0$ unless λ_1 or λ_2 or $e_0(e)$ or γ_+ vanish; $y_{12}(e)$ approaches constant values as $T \to 0, \infty$

• **Full decoherence** ($\gamma_e > 0$ for all $e \neq 0$): If $\lambda_2 \neq 0$, $g_2(B_j) \neq 0$ for all j
No decoherence-free subspaces!

Dependence on register size N

- Thermalization: y_0 independent of N
- Assume distribution of magnetic field $\langle\ \rangle$;

$$\langle y_1\rangle = y_1 \propto [e_0(e)]^2, \quad \langle y_2\rangle \propto D(e), \quad \langle y_{12}\rangle \propto N_0(e),$$

$$\text{where} \begin{cases} e_0(e) = \sum_{j=1}^N (\sigma_j - \tau_j) \\ D(e) = \sum_{j=1}^N |\sigma_j - \tau_j| \quad \textbf{Hamming distance} \\ N_0(e) = \{\# j : \sigma_j = \tau_j\} \end{cases}$$

∘ Pure energy-cons. interaction: $\gamma_e \propto \lambda_1^2[e_0(e)]^2$ as large as $O(\lambda_1^2 N^2)$
∘ Pure energy exchange interaction: $\gamma_e \propto \lambda_2^2 D(e) \leq O(\lambda_2^2 N)$
∘ Both interactions: additional term $\langle y_{12}\rangle = O((\lambda_1^2 + \lambda_2^2)N)$

- Local, energy-conserving interaction $\Rightarrow$ fastest decoherence rate $O(\lambda_1^2 N)$
- Assumption $\tau_{\rm S} \ll \tau_{\rm relax} \Leftrightarrow \lambda_{1,2}^2 N^2 \ll \Delta_N := \min^*_{\underline{\sigma},\underline{\tau}} |E(\underline{\sigma}) - E(\underline{\tau})|$
- Magnetic field roughly constant $B_j \sim B \Rightarrow \Delta_N \sim B$ indep. of N

Evolution of Entanglement

Von Neumann entropy of quantum state ρ: $S(\rho) = -\mathrm{Tr}\big(\rho \ln \rho\big) \geq 0$
Entanglement of pure state $\psi \in \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$ [Bennet et al. PhysRevA'96]

$$\mathcal{E}(\psi) := S\big(\mathrm{Tr}_B |\psi\rangle\langle\psi|\big) \geq 0$$

Property: $\mathcal{E}(\psi) = 0 \Leftrightarrow \mathrm{Tr}_B|\psi\rangle\langle\psi|$ pure $\Leftrightarrow \psi = \psi_A \otimes \psi_B$
Entanglement of mixed state ρ of $\mathrm{A}+\mathrm{B}$

$$\mathcal{E}(\rho) := \inf_{\mathcal{R}(\rho)} \sum_j p_j \mathcal{E}(\psi_j) \geq 0$$

$$\mathcal{R}(\rho) := \Big\{ (\psi_j, p_j) \ : \ \psi_j \in \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}},\ \|\psi_j\| = 1,\ 0 \leq p_j \leq 1,\ \textstyle\sum_j p_j = 1 \\ \text{s.t. } \rho = \sum_j p_j |\psi_j\rangle\langle\psi_j| \Big\}$$

Property: $\mathcal{E}(\rho) = 0 \Leftrightarrow \rho = \sum_j p_j |\psi_j^A\rangle\langle\psi_j^A| \otimes |\psi_j^B\rangle\langle\psi_j^B|$ (separable state)

Representation of $\mathcal{E}(\rho)$ if $\mathrm{A} = \mathrm{B} = \mathrm{spin}\ 1/2$ [Wootters PRL97]

$$\mathcal{E}(\rho) = h\big(C(\rho)\big), \quad C(\rho) \in [0,1] \ \textbf{concurrence}, \ h \text{ increasing } 0...1$$

$\Rightarrow$ concurrence is good measure of entanglement; explicit form

$$C(\rho) = \max\big\{0, D(\rho)\big\}, \qquad D(\rho) = \sqrt{\nu_1} - \big[\sqrt{\nu_2} + \sqrt{\nu_3} + \sqrt{\nu_4}\big]$$

where $\nu_1 \geq \nu_2 \geq \nu_3 \geq \nu_4 \geq 0$ are eigenvalues of matrix

$$\xi := \rho(\sigma^y \otimes \sigma^y)\overline{\rho}(\sigma^y \otimes \sigma^y)$$

with

$$\sigma^y = \left[\begin{array}{cc} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{array}\right]$$

Some previous results

(A) [Yu-Eberly PRL'04] $S_1 + R_1 \,||\, S_2 + R_2$
$R_{1,2}$ zero temperature cavities $(\sum_k \omega_k a_k^\dagger a_k)$, local energy exchange, Markovian master equation approximation
Results
- Decay of entanglement: $C(\rho(t)) \leq \mathrm{e}^{-\gamma t} C(\rho(0))$
- Entanglement sudden death: $\exists \rho(0)$ s.t. $C(\rho(0)) > 0$ but $C(\rho(t)) = 0$ $\forall t \geq t_d$
- $\exists \rho(0)$ s.t. $C(\rho(t)) > 0 \ \forall t < \infty$

([Yu-Eberly PhysRevB'03] S_1, S_2 in classical noises: $\exists$ decay of entanglement and $\exists$ disentanglement free subspaces.)

(B) [Bellomo et al. PRL'07] $S_1 + R_1 \,||\, S_2 + R_2$
Non-markovian regime (reservoir correl. time = 100 system relax. time)
Results
- Death and revival of entanglement: initial entanglement dies and stays zero for a while, then reappears and builds up to maximum, decreases and dies, reappears and so on.

(C) [Braun PRL'02] $S_1 + S_2 + R$
R harm. osci. heat bath $T > 0$, collective energy conserving interaction (explicitly solvable model)
Results
• Creation of entanglement: $\exists\ \rho(0)$ s.t. $C(\rho(0)) = 0$ but $C(\rho(t)) > 0$ for small times

(D) [Paz et al. PRL'08] Spin $\longrightarrow$ harm. osci. in environment of harm. osci.

Observe: Thermalization $\Rightarrow$ sudden death of entanglement.

Thermalization: $\lim_{t\to\infty} \rho_t = \rho_\infty = \rho^\beta + O(\lambda)$, where $\rho^\beta = Z_\beta^{-1} \mathrm{e}^{-\beta H_{\mathrm{S}}}$

ρ^β has neighbourhood of non-entangled states of size $O(1/\mathrm{Tr} \mathrm{e}^{-\beta H_{\mathrm{S}}})$

$\longrightarrow$ Temperature fixed, λ small: sudden death

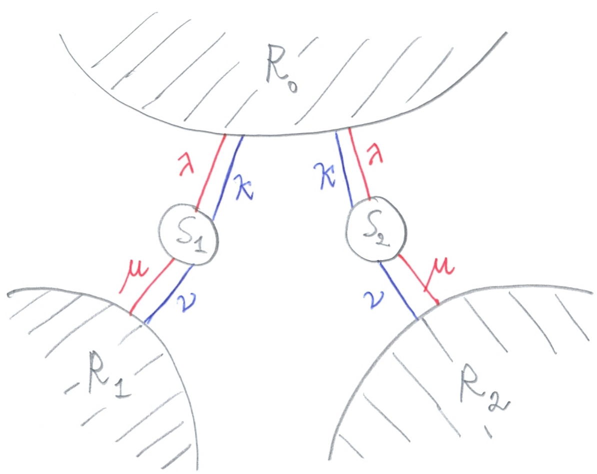
(!) However, λ fixed and T sufficiently small: entanglement can persist for all times [Paz et al. PRL'08]

Goal: Estimate entanglement death times (1st scenario).

Model

Two spins $1/2$ coupled to local and collective reservoirs,

$$H = B_1 S_1^z + B_2 S_2^z + H_{\mathrm{R}_1} + H_{\mathrm{R}_2} + H_{\mathrm{R}_0} + W$$



$$W = \left.\begin{array}{l} (\lambda_1 S_1^x + \lambda_2 S_2^x) \otimes \varphi_0(g) \\ +(\kappa_1 S_1^z + \kappa_2 S_2^z) \otimes \varphi_0(f) \end{array}\right\} \text{collective}$$

$$\left.\begin{array}{l} +\mu_1 S_1^x \otimes \varphi_1(g) + \mu_2 S_2^x \otimes \varphi_2(g) \\ +\nu_1 S_1^z \otimes \varphi_1(f) + \nu_2 S_2^z \otimes \varphi_2(f) \end{array}\right\} \text{local}$$

energy exchange terms λ, μ, energy conserving terms κ, ν

Magnetic fields: $0 < B_1 < B_2$ s.t. $B_2 \neq 2B_1$ (avoids degeneracies)

- Transition energies: $\{0, \pm B_1, \pm B_2, \pm(B_2 - B_1), \pm(B_1 + B_2)\}$
- Matrix element clusters: $\mathcal{C}_1, \ldots, \mathcal{C}_5$

$$\begin{bmatrix} * & \bullet & \bullet & \diamond \\ & * & \diamond & \bullet \\ & & * & \bullet \\ & & & * \end{bmatrix}$$

(& hermitian)

$$\begin{aligned} \gamma_{\mathrm{therm}} &= \min_{j=1,2} \left\{ (\lambda_j^2 + \mu_j^2)\sigma_g(B_j) \right\} + O(\alpha^4) \\ \gamma_2 &= \tfrac{1}{2}(\lambda_1^2 + \mu_1^2)\sigma_g(B_1) + \tfrac{1}{2}(\lambda_2^2 + \mu_2^2)\sigma_g(B_2) \\ &\qquad -Y_2 + (\kappa_1^2 + \nu_1^2)\sigma_f(0) + O(\alpha^4) \\ \gamma_5 &= (\lambda_1^2 + \mu_1^2)\sigma_g(B_1) + (\lambda_2^2 + \mu_2^2)\sigma_g(B_2) \\ &\qquad + \left[(\kappa_1 + \kappa_2)^2 + \nu_1^2 + \nu_2^2 \right] \sigma_f(0) + O(\alpha^4) \end{aligned}$$

$\sigma_f(\omega) = \coth(\beta\omega/2) J_f(\omega)$, $J_f(\omega) \propto \omega^2 \int_{S^2} |f(\omega, \Sigma)|^2 \mathrm{d}\Sigma$ spectral density

$Y = Y(\kappa, \mu, \sigma(B), r)$ complicated function

• Thermalization rate depends on energy-exchange coupling only.

• Purely energy-exchange interactions: $\kappa_j = \nu_j = 0 \Rightarrow$ rates depend symmetrically on local and collective influence through $\lambda_j^2 + \mu_j^2$.

• Purely energy-conserving interactions: $\lambda_j = \mu_j = 0 \Rightarrow$ rates do not depend symmetrically on local and collective terms.

• Dominant dynamics: only initially populated clusters have nontrivial dynamics

• Pure initial state $\psi_0 = a|++\rangle + b|--\rangle$

$$\rho_0 = \begin{bmatrix} p & 0 & 0 & u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \overline{u} & 0 & 0 & 1-p \end{bmatrix} \Rightarrow \rho_t = \begin{bmatrix} x_1(t) & 0 & 0 & u(t) \\ 0 & x_2(t) & 0 & 0 \\ 0 & 0 & x_3(t) & 0 \\ \overline{u}(t) & 0 & 0 & x_4(t) \end{bmatrix} + O(\alpha^2)$$

- Initial concurrence: $C(\rho_0) = 2\sqrt{p(1-p)}$
- Dynamics

$$\begin{aligned} x_1(t) &= pA_t(11;11) + (1-p)A_t(11;44) \\ x_2(t) &= pA_t(22;11) + (1-p)A_t(22;44) \\ &\vdots \\ u(t) &= \mathrm{e}^{\mathrm{i}t\varepsilon_{2(B_1+B_2)}} u(0) \end{aligned}$$

$A_t(kk;ll) \leftarrow$ resonance energies bifurcating out of $e = 0$. Leading terms:

$$\delta_2 = (\lambda_1^2 + \mu_1^2)\sigma_g(B_1), \quad \delta_3 = (\lambda_2^2 + \mu_2^2)\sigma_g(B_2), \quad \delta_4 = \delta_2 + \delta_3$$

Leading term of $\mathrm{Im}\,\varepsilon_{2(B_1+B_2)}$:

$$\delta_5 = \delta_2 + \delta_3 + [(\kappa_1 + \kappa_2)^2 + \nu_1^2 + \nu_2^2]\sigma_f(0)$$

Theorem. *Take coupling s.t.* $\delta_2, \delta_3 > 0$ *(thermalization). There is a positive constant* α_0 *(independent of* p*) s.t. if* $0 < \alpha \leq \alpha_0\sqrt{p(1-p)}$*, then we have the following.*

(A) Entanglement survival: *Concurrence* $C(\rho_t) > 0$ *for all* $t \leq t_A$*,*

$$t_A := \frac{1}{\max\{\delta_2, \delta_3\}} \ln\left[1 + C_A\alpha^2\right],$$

for some constant $C_A > 0$ *(independent of* p*,* α*).*

(B) Entanglement death: *Concurrence* $C(\rho_t) = 0$ *for all* $t \geq t_B$*,*

$$t_B := \max\left\{\frac{1}{\delta_5}\ln\left[C_B\frac{\sqrt{p(1-p)}}{\alpha^2}\right], \frac{1}{\delta_2+\delta_3}\ln\left[C_B\frac{p(1-p)}{\alpha^2}\right]\right\},$$

for some constant $C_B > 0$ *(independent of* p*,* α*).*

Discussion

• Result gives disentanglement bounds for true dynamics of qubits

• Disentanglement time *finite* since $\delta_2, \delta_3 > 0$ (implying thermalization). If system does not thermalize then it can happen that entanglement stays nonzero for all times (it may decay or stay constant).

• Rates δ are of order α^2. Both t_A and t_B increase with decreasing coupling strength.

• Bounds are not optimal. Disentanglement bound depends on both kinds of couplings and each coupling decreases t_B (the bigger the noise the quicker disentanglement dies). Entanglement survival time bound does not depend on the energy-conserving couplings.

Entanglement creation

Braun [PRL 02]: energy conserving collective coupling, initial unentangled pure state

$$\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \otimes \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$$

Explicitly solvable model: concurrence creation, death and revival (Peres-Horodecki criterion)

Dynamics in resonance approximation:

• *Purely energy-exchange coupling*

In resonance approx., $[\rho_t]_{mn}$ depends on $\lambda^2 + \mu^2$ only $\Rightarrow$ *Creation of entanglement under collective and local energy-exchange dynamics is* same *in this approx.* But no concurrence creation for purely local interaction. So true concurrence is $O(\lambda^2)$ for all times.

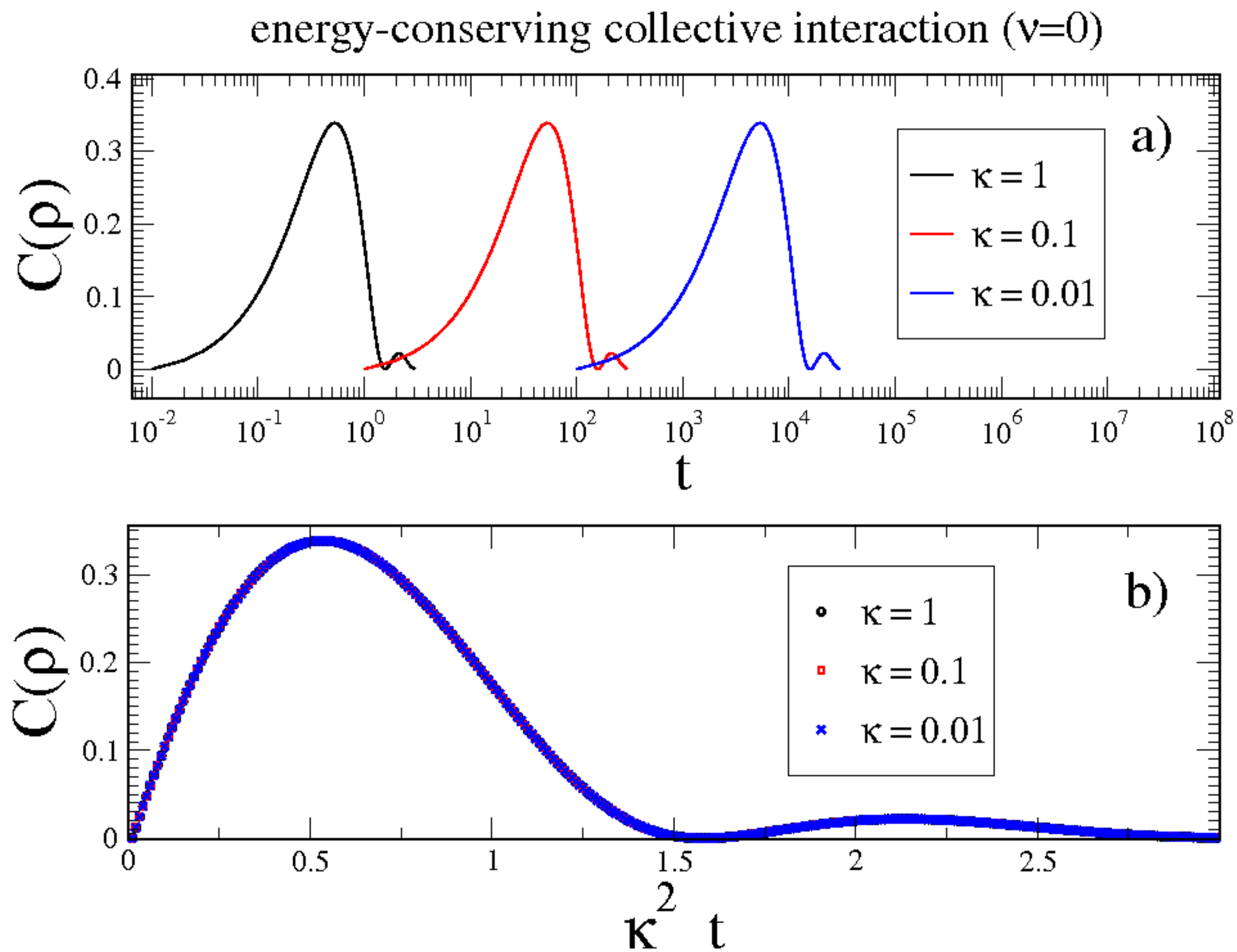
• *Purely energy-conserving coupling*

Can expect creation of concurrence (solvable model)

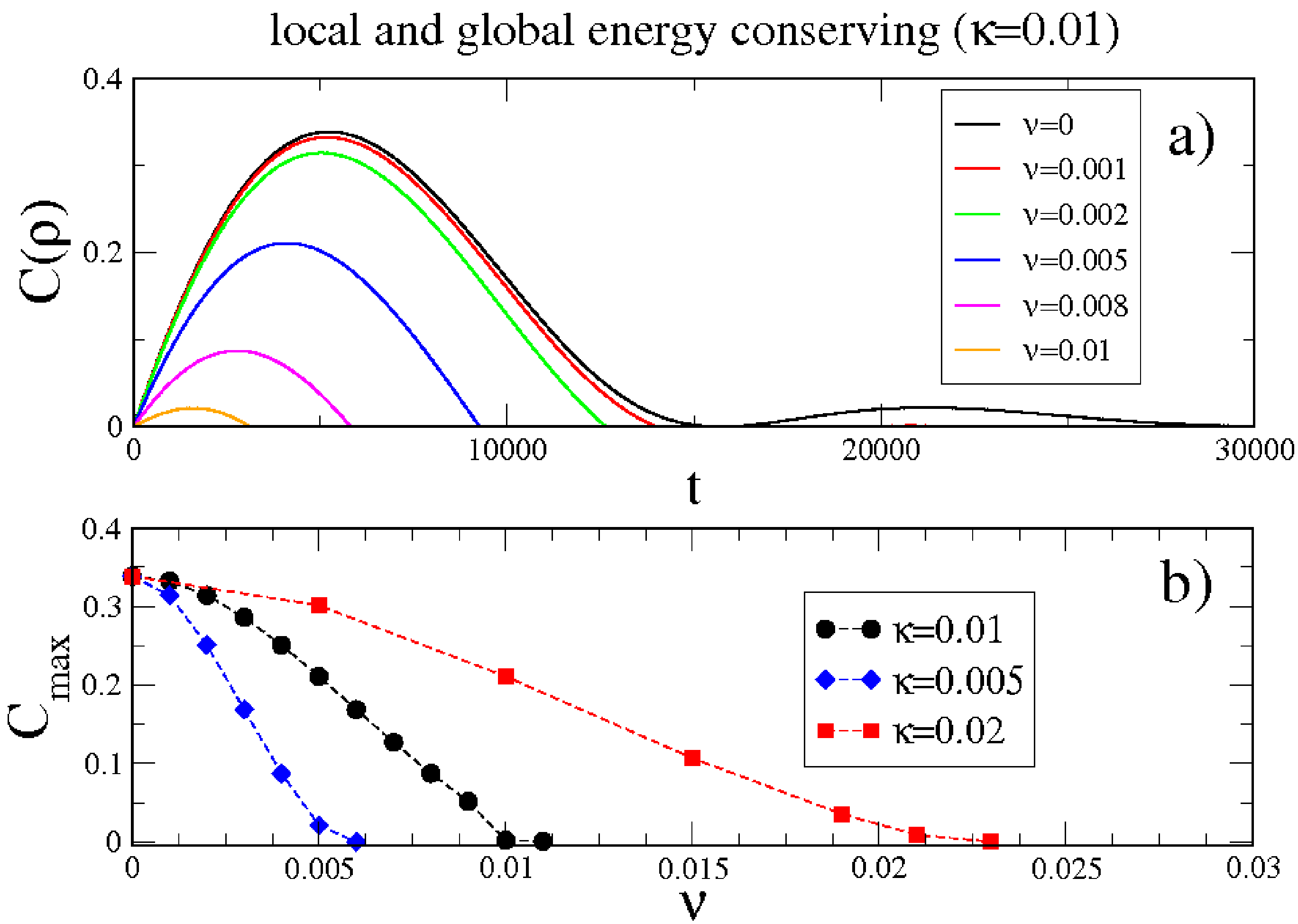
• *Full coupling*

Matrix elements evolve as complicated functions of all coupling parameters, effects of different interactions are correlated.

Numerical results on concurrence creation

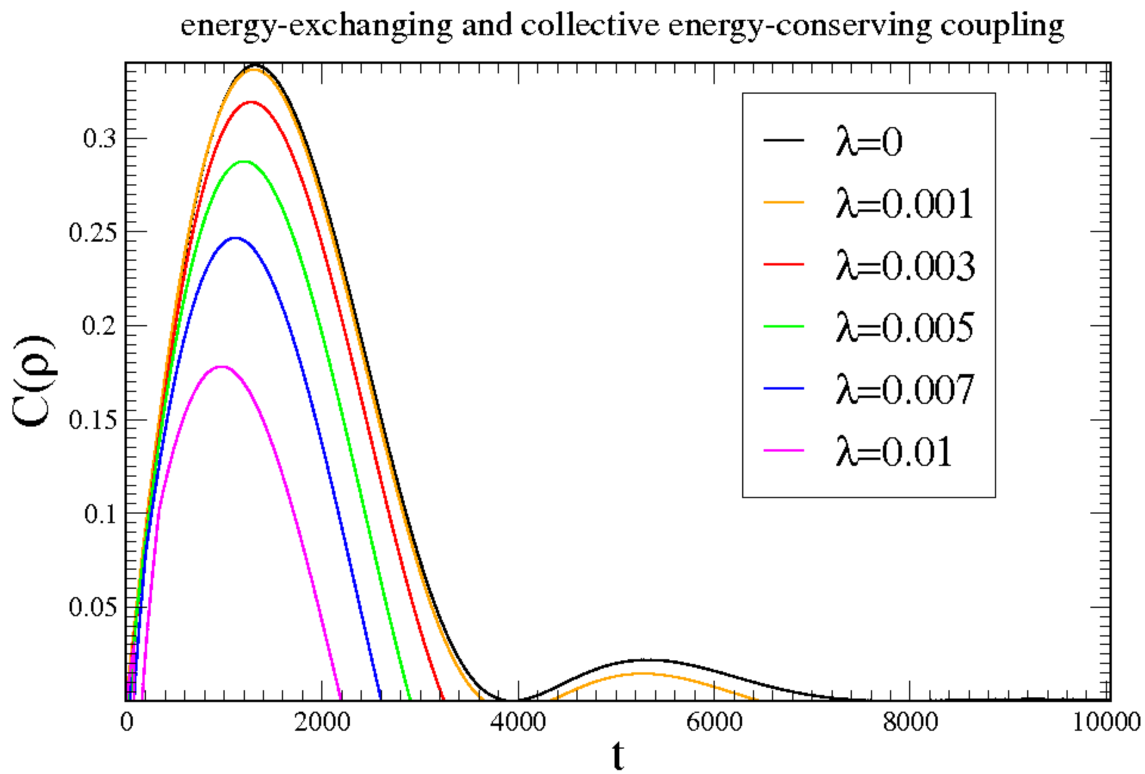


Amount of entanglement created is *independent* of coupling κ; peak at $t_0 \approx 0.5\kappa^{-2}$; revival of entanglement $t_1 \approx 2.1\kappa^{-2}$



Switching on local (energy conserving) coupling:

- creation of entanglement *reduced*
- if local coupling exceeds collective one $\Rightarrow$ *no* concurrence is created



Energy-exchange collective and local interactions: $\lambda = \mu$ (symmetric); $\kappa = 0.02$ (collective, conserving), $\nu = 0$ (local, conserving)

• entanglement creation is *reduced* and peak time t_0 slightly reduced

• revival suppressed for increasing λ

• small times: density matrix in resonance approx. has partly negative eigenvalues (as Caldeira-Legget, Unruh-Zurek); numerics not reliable (non-smooth behavior in λ)

Outline of resonance approach

Consider observable $A \in B(\mathcal{H}_{\rm S})$. Initial density matrix is represented by the vector ψ_0 in GNS space $\langle A\rangle_0 = \langle\psi_0, A\otimes \mathbf{1}_{\rm S}\otimes \mathbf{1}_{\vec{\rm R}}\ \psi_0\rangle$. Full dynamics implemented by group $\mathrm{e}^{\mathrm{i}tL_\alpha}\cdot \mathrm{e}^{-\mathrm{i}tL_\alpha}$. The self-adjoint generator

$$L_\alpha = L_{\rm S} + L_{\vec{\rm R}} + W_\alpha = L_0 + W_\alpha$$

is called the *Liouville operator*.

$$\langle A\rangle_t = \langle\psi_0, \mathrm{e}^{\mathrm{i}tL_\alpha}\left[A\otimes \mathbf{1}_{\rm S}\otimes \mathbf{1}_{\vec{\rm R}}\right]\mathrm{e}^{-\mathrm{i}tL_\alpha}\psi_0\rangle.$$

Convenient trick:

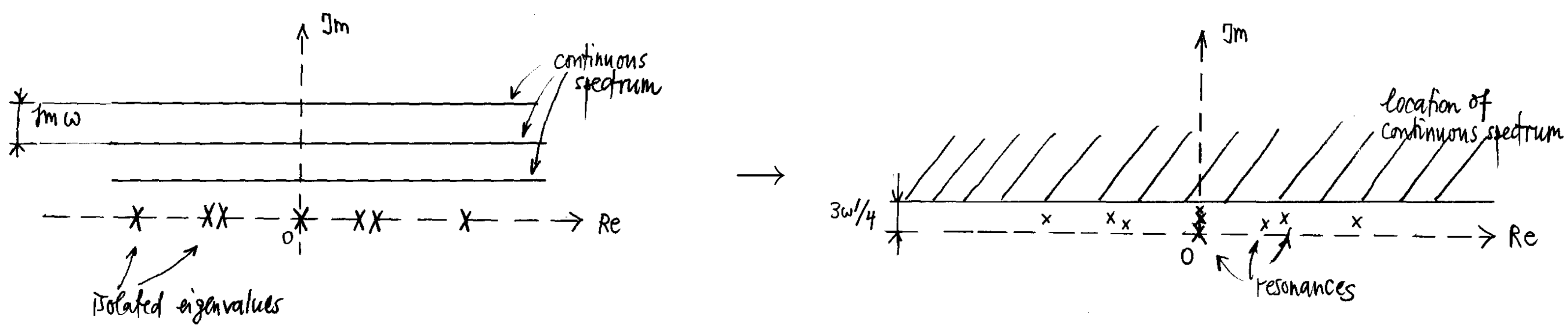
$$\exists K_\alpha \ \text{ s.t. } \ \mathrm{e}^{\mathrm{i}tL_\alpha}A\mathrm{e}^{-\mathrm{i}tL_\alpha} = \mathrm{e}^{\mathrm{i}tK_\alpha}A\mathrm{e}^{-\mathrm{i}tK_\alpha} \ \text{ and } \ K_\alpha\psi_0 = 0.$$

Standard way of constructing K_α given L_α, observable algebra and reference vector ψ_0 (modular theory of von Neumann algebras).

$$\langle A\rangle_t = -\frac{1}{2\pi \mathrm{i}}\int_{\mathbf{R}-\mathrm{i}} \mathrm{e}^{\mathrm{i}tz}\ \langle\psi_0, (K_\alpha(\theta)-z)^{-1}\left[A\otimes 1\!\!1_{\mathrm{S}}\otimes 1\!\!1_{\vec{\mathrm{R}}}\right]\psi_0\rangle \mathrm{d}z,$$

where $\theta\mapsto K_\alpha(\theta)$ is a spectral deformation (translation) of K_α:

$$K_\alpha(\theta) = U(\theta)K_\alpha U(\theta)^{-1} = L_0 + \theta N + I_\alpha(\theta).$$



• Uncovering resonances: $\mathrm{Im}\,\theta > 0$ fixed, $\alpha \ll \mathrm{Im}\,\theta$, then eigenvalues $\varepsilon_e^{(s)}$ bifurcating (α) out of real eigenvalues of L_0 are independent of θ.

• Analytic perturbation theory: $\varepsilon_e^{(s)} = e + \delta_e^{(s)} + O(\alpha^4)$, where $\delta_e^{(s)}$ are eigenvalues of *Level Shift Operator* Λ_e,

$$\Lambda_e \eta_e^{(s)} = \delta_e^{(s)} \eta_e^{(s)}$$

where $\Lambda_e = -P_e I_\alpha \bar{P}_e (L_0 - e + \mathrm{i}0)^{-1} \bar{P}_e I_\alpha P_e$.

• Γ: simple closed contour enclosing all $\varepsilon_e^{(s)}$ but no continuous spectrum, associated Riesz projection

$$Q = \frac{-1}{2\pi\mathrm{i}} \int_\Gamma (K_\alpha(\theta) - z)^{-1} \mathrm{d}z$$

• $K_\alpha(\theta)$ reduced by Q, finite-dimensional block $QK_\alpha(\theta)Q$,

$$\langle A \rangle_t = \langle \psi_0, \mathrm{e}^{\mathrm{i}tQK_\alpha(\theta)Q} [A \otimes \mathbb{1}_{\mathrm{S}} \otimes \mathbb{1}_{\vec{\mathrm{R}}}] \psi_0 \rangle + O(\alpha^2 \mathrm{e}^{-\gamma t})$$

with $\gamma = \mathrm{Im}\,\theta - O(\alpha) > \max \mathrm{Im}\,\varepsilon_e^{(s)}$

• $\psi_0 = \psi_{\mathrm{S}} \otimes \psi_{\vec{\mathrm{R}}}$. Set $\widetilde{V}(t) = \mathrm{Tr}_{\vec{\mathrm{R}}}\big[|\psi_{\vec{\mathrm{R}}}\rangle\langle\psi_{\vec{\mathrm{R}}}|\ \mathrm{e}^{\mathrm{i}tQK_\alpha(\theta)Q}\big]$, then

$$\langle A\rangle_t = \langle\psi_{\mathrm{S}}, \widetilde{V}(t)[A\otimes 1\!\!1_{\mathrm{S}}]\psi_{\mathrm{S}}\rangle + O(\alpha^2 \mathrm{e}^{-\gamma t})$$

• WOLOG consider ψ_{S} trace state (cyclic and separating):

$$[V(t)A]\otimes 1\!\!1_{\mathrm{S}}\ \psi_{\mathrm{S}} = \widetilde{V}(t)[A\otimes 1\!\!1_{\mathrm{S}}]\psi_{\mathrm{S}}$$

defines reduced Heisenberg dynamics V of S (but $V(t)$ not semigroup):

$$\big|\omega_{\mathrm{S}}^t(A) - \omega_{\mathrm{S}}^0(V(t)A)\big| \leq C\alpha^2 \mathrm{e}^{-\gamma t}$$

• All resonance energies simple $\Rightarrow$

$$\mathrm{e}^{\mathrm{i}tQK_\alpha(\theta)Q} = \sum_e \sum_{s=1}^{\mathrm{mult}(e)} \mathrm{e}^{\mathrm{i}t\varepsilon_e^{(s)}} |\chi_e^{(s)}\rangle\langle\widetilde{\chi}_e^{(s)}|$$

with $K_\alpha(\theta)\chi_e^{(s)} = \varepsilon_e^{(s)}\chi_e^{(s)}$, $[K_\alpha(\theta)]^*\widetilde{\chi}_e^{(s)} = [\varepsilon_e^{(s)}]^*\widetilde{\chi}_e^{(s)}$, $\langle\chi_e^{(s)}, \widetilde{\chi}_{e'}^{(s')}\rangle = \delta_{e,e'}\delta_{s,s'}$

- Perturbation expansion

$$\widetilde{V}(t) = \sum_e \sum_{s=1}^{\mathrm{mult}(e)} \mathrm{e}^{\mathrm{i}t\varepsilon_e^{(s)}} \big[|\eta_e^{(s)}\rangle\langle\widetilde{\eta}_e^{(s)}| + O(\alpha^2)\big]$$

where $\eta_e^{(s)}$, $\widetilde{\eta}_e^{(s)}$ are eigenvectors of level shift operators.

• Action of reduced Heisenberg dynamics (Φ_j energy basis of the spins)

$$V(t)|\Phi_n\rangle\langle\Phi_m|$$

$$= \sum_e \sum_{s=1}^{\mathrm{mult}(e)} \mathrm{e}^{\mathrm{i}t\varepsilon_e^{(s)}} \left[\sum_{k,l} \langle \Phi_l \otimes \Phi_k, \eta_e^{(s)} \rangle \langle \widetilde{\eta}_e^{(s)}, \Phi_n \otimes \Phi_m \rangle |\Phi_l\rangle\langle\Phi_k| + O(\alpha^2) \right]$$

scalar products vanish unless $E_l - E_k = e = E_n - E_m$, so

$$\begin{aligned} \omega_{\mathrm{S}}(V(t)|\Phi_n\rangle\langle\Phi_m|) &= \sum_{s=1}^{\mathrm{mult}(E_n-E_m)} \mathrm{e}^{\mathrm{i}t\varepsilon_{E_n-E_m}^{(s)}} \sum_{(k,l)\in\mathcal{C}(E_m-E_n)} \langle \Phi_l \otimes \Phi_k, \eta_{E_n-E_m}^{(s)} \rangle \\ &\quad \times \langle \widetilde{\eta}_{E_n-E_m}^{(s)}, \Phi_n \otimes \Phi_m \rangle \ \omega_{\mathrm{S}}(|\Phi_l\rangle\langle\Phi_k|) + O(\alpha^2) \\ &= \sum_{(k,l)\in\mathcal{C}(E_n-E_m)} A_t(m,n;k,l)\omega_{\mathrm{S}}(|\Phi_l\rangle\langle\Phi_k|) + O(\alpha^2). \end{aligned}$$