RESONANCE THEORY OF DECOHERENCE

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Abstract

We present a rogorous analysis of the phenomenon of decoherence for general N-level systems coupled to reservoirs of free massless bosonic fields. We apply our general results to the specific case of the qubit. Our approach does not involve master equation approximations and applies to a wide variety of systems which are not explicitly solvable.

Results presented here are obtained in collaboration with I.M. Sigal and G.P. Berman:

• *Decoherence and Thermalization*, Phys. Rev. Lett. **98**, 130401 (2007), quant-ph/0608181 (2006)

• *Resonance theory of decoherence and thermalization*, to appear in Ann. Phys. (2007), quant-ph/0702207.

1 INTRODUCTION

Open quantum system $\mathrm{S}+\mathrm{R}$:

Hilbert space: $\mathfrak{H} = \mathfrak{H}_{\mathrm{S}} \otimes \mathfrak{H}_{\mathrm{R}}$

Hamiltonian: $H = H_{\mathrm{S}} + H_{\mathrm{R}} + \lambda v$

$\lambda \in \mathbb{R}$: coupling constant

v: interaction between S and R

Reservoir R is spatially infinitely extended $\Longrightarrow$ we have to interpret $\mathfrak{H}_{\mathrm{R}}$ and H in an appropriate limit sense (thermodynamic limit or limit of continuous modes).

Density matrix of total system: ρ_t

Reduced density matrix of S: $\overline{\rho}_t = \mathrm{Tr}_{\mathrm{R}}(\rho_t)$

(trace taken over $\mathfrak{H}_{\mathrm{R}}$)

Let $\{\varphi\}_{j=1}^{N}$ be fixed basis of $\mathfrak{H}_{\mathrm{S}}$, denote matrix elements of $\overline{\rho}$ as

$$[\overline{\rho}_t]_{m,n} := \langle \varphi_m, \overline{\rho}_t \varphi_n \rangle$$

A definition of decoherence: the vanishing of off-diagonals in the limit of large times,

$$\lim_{t\to\infty} [\overline{\rho}_t]_{m,n} = 0, \quad \forall m \neq n.$$

This is a *basis dependent* notion of disappearance of correlations,

$$\overline{\rho}_t = \sum_{m,n} c_{m,n}(t)|\varphi_m\rangle\langle\varphi_n| \longrightarrow \sum_m p_m(t)|\varphi_m\rangle\langle\varphi_m|,$$

as $t \to \infty$.

Example. N-level system with **energy-conserving coupling** to Bose field (see [1] for qubit case, $N = 2$).

$$\mathfrak{H}_{\mathrm{S}} = \mathbb{C}^N, \qquad H_{\mathrm{S}} = \mathrm{diag}(E_1, \ldots, E_N)$$

Interaction operator

$$\begin{aligned} v &= G \otimes \varphi(g) \\ G &= \mathrm{diag}(\gamma_1, \ldots, \gamma_N) \\ \varphi(g) &= \frac{1}{\sqrt{2}}[a^*(g) + a(g)] \end{aligned}$$

$a^\#(g)$: usual bosonic creation and annihilation operators, smeared out with *form factor* $g \in L^2(\mathbb{R}^3, \mathrm{d}^3k)$

$[H_{\mathrm{S}}, H] = [H_{\mathrm{S}}, H_{\mathrm{S}} + H_{\mathrm{R}} + \lambda v] = 0 \Longrightarrow$ energy of small system is conserved.

This model is exactly solvable!

[1] M.G. Palma, K.-A. Suominen, A. Ekert: *Quantum computers and dissipation*, Proc. R. Soc. Lond. A **452**, 567-584 (1996)

Solution:

$$[\overline{\rho}_t]_{m,n} = [\overline{\rho}_0]_{m,n} \mathrm{e}^{-\mathrm{i}t(E_m - E_n) + \mathrm{i}\lambda^2 \alpha_{m,n}(t)},$$

where

$$\begin{aligned}
\alpha_{m,n}(t) &= (\gamma_m^2 - \gamma_n^2) S(t) + \mathrm{i}(\gamma_m - \gamma_n)^2 \Gamma(t) \\
\Gamma(t) &= \int_{\mathbb{R}^3} |g(k)|^2 \coth(\beta|k|/2) \frac{\sin^2(|k|t/2)}{|k|^2} \mathrm{d}^3 k \\
S(t) &= \frac{1}{2} \int_{\mathbb{R}^3} |g(k)|^2 \frac{|k|t - \sin(|k|t)}{|k|^2} \mathrm{d}^3 k
\end{aligned}$$

$\Rightarrow$ Populations are constant: $[\overline{\rho}_t]_{m,m} = [\overline{\rho}_t]_{m,m}$.
$\Rightarrow$ Full decoherence occurs only if $\Gamma(t) \to \infty$ as $t \to \infty$, which depends on infrared behaviour of form factor (and the space dimension).

Infrared behaviour characterized by $g(k) \sim |k|^p$ as $|k| \sim 0$. Then

$$\lim_{t\to\infty} \frac{\alpha_{m,n}(t)}{t} = \frac{1}{2}(\gamma_m^2 - \gamma_n^2) \left\langle g, |k|^{-1} g \right\rangle + \mathrm{i}(\gamma_m - \gamma_n)^2 \begin{cases} 0 & \text{if } p > 0 \\ \text{const.} & \text{if } p = -1/2 \\ +\infty & \text{if } p < -1/2 \end{cases}$$

This is a **non-demolition model** (H_{S} conserved: processes of absorption and emission of quanta of the reservoir by the system S are suppressed.

To enable such processes, need $[H_{\mathrm{S}}, v] \neq 0$. But then expect that **thermalization** takes place as well.

$\rho(\beta, \lambda)$: equilibrium state of total system at temperature $T = 1/\beta$

$\rho_{t=0}$: arbitrary initial density matrix (on $\mathfrak{H}$).

Thermalization: for any observable A of total system,

$$\mathrm{Tr}_{\mathrm{S+R}}(\rho_t A) \longrightarrow \mathrm{Tr}_{\mathrm{S+R}}(\rho(\beta, \lambda) A), \qquad \text{as } t \to \infty \tag{1}$$

This implies

$$\overline{\rho}_t \to \overline{\rho}_\infty(\beta, \lambda) := \mathrm{Tr}_{\mathrm{R}}(\overline{\rho}(\beta, \lambda)), \qquad \text{as } t \to \infty$$

Expansion of $\overline{\rho}_\infty(\beta, \lambda)$ in coupling constant:

$$\overline{\rho}_\infty(\beta, \lambda) = \overline{\rho}_\infty(\beta, 0) + O(\lambda)$$

where $\overline{\rho}_\infty(\beta, 0)$ is **Gibbs state** of system S. Now Gibbs state (density matrix) is *diagonal* in energy basis (H_{S}), but correction term $O(\lambda)$ is *not*, in general.

$\Rightarrow$ *Even if* S *is initially in incoherent superposition of energy eigenstates, it will acquire some* "residual coherence" *of order* $O(\lambda)$ *during the process of thermalization.*

$\Rightarrow$ Define decoherence as decay of off-diagonals of $\overline{\rho}_t$ to limit values (= off-diagonals of $\overline{\rho}_\infty(\beta, \lambda)$)

In (vast) literature on this topic we have encountered only

- models with energy-conserving interactions (which are explicitly solvable)
- models with markovian approximations (master equations, Lindblad dynamics, with uncontrolled errors)

Our goal:
Describe decoherence for systems which may also exhibit thermalization, in a rigorous fashion (controlled perturbation expansions)

Main tool: dynamical resonance theory based on complex deformations and recent progress in theory of open quantum systems

2 RESULTS

S: N-level system, energies $\{E_j\}_{j=1}^N$

R: free massless Bose field ($\omega(k) = |k|$, spatially ∞ extended)

Coupling: $\lambda v = \lambda\, G \otimes \varphi(g)$

Assume: a certain regularity condition on form factor $g(k)$ (to be specified below)

For observables A of S we set

$$\begin{aligned}\langle A\rangle_t &:= \mathrm{Tr}_{\mathrm{S}}(\overline{\rho}_t A)\\ \langle\!\langle A\rangle\!\rangle_\infty &:= \lim_{T\to\infty}\frac{1}{T}\int_0^T \langle A\rangle_t \mathrm{d}t\end{aligned}$$

Theorem 1. *There is a $\lambda_0 > 0$ s.t. the following statements hold for $|\lambda| < \lambda_0$.*

1. *$\langle\!\langle A\rangle\!\rangle_\infty$ exists for all A*
2. *We have*

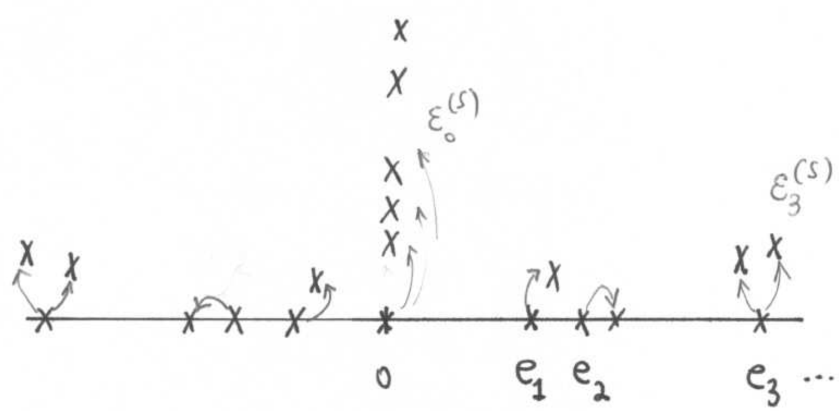
$$\langle A\rangle_t - \langle\!\langle A\rangle\!\rangle_\infty = \sum_{\varepsilon\neq 0} \mathrm{e}^{\mathrm{i}t\varepsilon} R_\varepsilon(A) + O(\lambda^2 \mathrm{e}^{-\omega t}), \quad (2)$$

where the ε are resonance energies*, $0 \le \mathrm{Im}\varepsilon < \omega$, and $R_\varepsilon(A)$ are linear functionals of A which depend on the initial state $\rho_{t=0}$.*

3. *Let e be an eigenvalue of the operator $H_{\rm S}\otimes \mathbb{1}_{\rm S} - \mathbb{1}_{\rm S}\otimes H_{\rm S}$ (acting on $\mathfrak{H}_{\rm S}\otimes\mathfrak{H}_{\rm S}$). For $\lambda = 0$ each ε coincides with one of the e and we have the following expansion for small λ*

$$\varepsilon \equiv \varepsilon_e^{(s)} = e - \lambda^2\delta_e^{(s)} + O(\lambda^4).$$

The $\delta_e^{(s)}\in\mathbb{C}$ are eigenvalues of so-called level shift operators Λ_e, *satisfying* $\mathrm{Im}(\delta_e^{(s)})\le 0$.



Furthermore, we have

$$R_\varepsilon(A) = \sum_{(m,n)\in I_e} \varkappa_{m,n}A_{m,n} + O(\lambda^2),$$

with $I_e = \{(m,n)\mid E_m - E_n = e\}$, and where $A_{m,n}$ is the (m,n)-matrix element of A and the numbers $\varkappa_{m,n}$ depend on the initial state.

We treat the resonances in setting of *spectral deformation*. This requires the following **regularity condition**

(A) The function

$$g_\beta(u,\sigma) := \sqrt{\frac{u}{1-\mathrm{e}^{-\beta u}}}\,|u|^{1/2} \begin{cases} g(u,\sigma) & \text{if } u \geq 0 \\ \mathrm{e}^{\mathrm{i}\phi}\overline{g}(-u,\sigma) & \text{if } u < 0 \end{cases}$$

is such that $\vartheta \mapsto g_\beta(u+\vartheta,\sigma)$ has an analytic continuation, as a map $\mathbb{C} \to L^2(\mathbb{R}\times S^2, \mathrm{d}u \times \mathrm{d}\sigma)$, into $\{|\vartheta| < \omega\}$, for some $\omega > 0$. Here, ϕ is an arbitrary fixed phase.

Examples of admissible g:

$$g(k) = g_1(\sigma)|k|^p \mathrm{e}^{-|k|^2},$$

where $p = -1/2 + n$, $n = 0, 1, 2, \ldots$, and $g_1(\sigma) = \mathrm{e}^{\mathrm{i}\phi}\overline{g}_1(\sigma)$.

Discussion. • Relation (2) gives detailed picture of dynamics. Resonance energies ε and functionals R_ε can be calculated for concrete models, to arbitrary precision (rigorous perturbation theory in λ).

• In absence of interaction ($\lambda = 0$) we have $\varepsilon = e \in \mathbb{R}$. Depending on interaction, each resonance energy ε may migrate into upper complex plane, or it may stay on real axis, as $\lambda \neq 0$.

• Averages $\langle A\rangle_t$ approach their ergodic means $\langle\langle A\rangle\rangle_\infty$ if and only if $\mathrm{Im}\varepsilon > 0$ for all $\varepsilon \neq 0$. In this case, convergence is on time scale $[\mathrm{Im}\varepsilon]^{-1}$. Otherwise $\langle A\rangle_t$ oscillates.

• Sufficient condition for decay: $\mathrm{Im}\delta_e^{(s)} < 0$ (and λ small).

• Two processes drive the decay: energy-exchange processes and energy preserving ones. The former are induced by interactions enabling processes of absorption and emission of field quanta with energies corresponding to the Bohr frequencies of S (**Fermi Golden Rule Condition**). Energy preserving interactions suppress such processes, allowing only for a phase change of the system during the evolution ("phase damping").

• Even if initial density matrix is a product of system and reservoir density matrices, at $t > 0$ it is not of product form. Evolution creates system-reservoir entanglement.

• If system has property of return to equilibrium, then

$$[\overline{\rho}_\infty]_{m,n} = \delta_{m,n}\frac{e^{-\beta E_m}}{\mathrm{Tr}_{\mathrm{S}}(\mathrm{e}^{-\beta H_{\mathrm{S}}})} + O(\lambda^2)$$

$\Rightarrow$ Gibbs distribution is obtained by first letting $t \to \infty$, then $\lambda \to 0$.

3 APPLICATION TO QUBIT (SPIN $1/2$)

$$\mathfrak{H}_{\mathrm{S}} = \mathbb{C}^2, \qquad H_{\mathrm{S}} = \mathrm{diag}(E_1, E_2)$$

Let

$$\Delta = E_2 - E_1 > 0, \qquad \varphi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Coupling operator

$$v = \begin{bmatrix} a & c \\ \overline{c} & b \end{bmatrix} \otimes \varphi(g)$$

Theorem 1 $\Longrightarrow$ For all $t \geq 0$,

$$\begin{aligned}
[\overline{\rho}_t]_{1,1} - \langle\!\langle |\varphi_1\rangle\langle\varphi_1| \rangle\!\rangle_\infty &= \mathrm{e}^{\mathrm{i}t\varepsilon_0(\lambda)}[C_0 + O(\lambda^2)] \\
&\quad + \mathrm{e}^{\mathrm{i}t\varepsilon_\Delta(\lambda)}O(\lambda^2) + \mathrm{e}^{\mathrm{i}t\varepsilon_{-\Delta}(\lambda)}O(\lambda^2) \\
&\quad + O(\lambda^2 \mathrm{e}^{-t\omega})
\end{aligned}$$

$$\begin{aligned}
[\overline{\rho}_t]_{1,2} - \langle\!\langle |\varphi_2\rangle\langle\varphi_1| \rangle\!\rangle_\infty &= \mathrm{e}^{\mathrm{i}t\varepsilon_\Delta(\lambda)}[C_0 + O(\lambda^2)] \\
&\quad + \mathrm{e}^{\mathrm{i}t\varepsilon_0(\lambda)}O(\lambda^2) + \mathrm{e}^{\mathrm{i}t\varepsilon_{-\Delta}(\lambda)}O(\lambda^2) \\
&\quad + O(\lambda^2 \mathrm{e}^{-t\omega})
\end{aligned}$$

C_0, C_Δ: explicit constants, depend on initial state $\rho_{t=0}$

Expansion of resonance energies:

$$\begin{aligned}
\varepsilon_0(\lambda) &= \mathrm{i}\lambda^2|c|^2\xi(\Delta) + O(\lambda^4)\\
\varepsilon_\Delta(\lambda) &= \Delta + \lambda^2 R + \frac{\mathrm{i}}{2}\lambda^2\left[|c|^2\xi(\Delta) + (b-a)^2\xi(0)\right] + O(\lambda^4)\\
\varepsilon_{-\Delta}(\lambda) &= -\overline{\varepsilon_\Delta(\lambda)}
\end{aligned}$$

where

$$\xi(\eta) := \int_{\mathbb{R}^3}\coth\left(\frac{\beta|k|}{2}\right)|g(k)|^2\delta(\eta-|k|)\mathrm{d}^3k$$

and

$$R = \frac{b^2-a^2}{2}\left\langle g, |k|^{-1}g\right\rangle + \frac{|c|^2}{2}\mathrm{P.V.}\int_{\mathbb{R}\times S^2} u^2\coth\left(\frac{\beta|k|}{2}\right)\frac{|g(|u|,\sigma)|^2}{u-\Delta}\mathrm{d}u\,\mathrm{d}\sigma$$

Thermalization time: $\omega_{\mathrm{th}} := [\mathrm{Im}\varepsilon_0(\lambda)]^{-1}$
Decoherence time: $\omega_{\mathrm{dec}} := [\mathrm{Im}\varepsilon_\Delta(\lambda)]^{-1}$

$$\frac{\omega_{\mathrm{dec}}}{\omega_{\mathrm{th}}} = \frac{1}{2}\left[1 + \frac{(b-a)^2}{|c|^2}\frac{\xi(0)}{\xi(\Delta)}\right] + O(\lambda^2),$$

note: $\frac{\xi(0)}{\xi(\Delta)} \sim T$ for small T

Example Spin-Boson model:

$$H_{\mathrm{S}} = -\tfrac{1}{2}\hbar\Delta_0\sigma_x + \tfrac{1}{2}\epsilon\,\sigma_z$$

σ: Pauli matrices, Δ_0: bare tunneling matrix element, ϵ: bias

Coupling operator: $v = \sigma_z \otimes \varphi(g)$

$\Rightarrow$ determines matrix elements a, b, c in general formulation:

$$\frac{(b-a)^2}{|c|^2} = 16\frac{\epsilon^2}{\hbar^2\Delta_0^2}$$

$\Rightarrow$ Decoherence time becomes smaller relative to thermalization time if bias ϵ is decreased, or if tunneling parameter Δ_0 is increased

Remarks 1. If system has property of return to equilibrium, then

$$\begin{aligned}\langle\langle|\varphi_1\rangle\langle\varphi_1|\rangle\rangle_\infty &= \frac{\mathrm{e}^{-\beta E_1}}{Z_{\mathrm{S},\beta}} + O(\lambda^2)\\ \langle\langle|\varphi_2\rangle\langle\varphi_1|\rangle\rangle_\infty &= O(\lambda^2)\end{aligned}$$

we recover the Gibbs law by first taking $t \to \infty$, then $\lambda \to 0$

2. $\xi(0) > 0$ for IR behaviour $g(k) \sim |k|^{-1/2}$, $\xi(0) = 0$ for more regular IR behaviour. Moreover, $\xi(0) \sim T$ and $\xi(\Delta) \sim$ const. > 0, as $T \sim 0$.

4 DYNAMICAL RESONANCE THEORY

Consider observable A of S:

$$\begin{aligned}\langle A\rangle_t &= \mathrm{Tr}_{\mathrm{S}}\,[\overline{\rho}_t\; A] \\ &= \mathrm{Tr}_{\mathrm{S+R}}\,[\rho_t\; A\otimes \mathbb{1}_{\mathrm{R}}] \\ &= \left\langle \psi_0, \mathrm{e}^{\mathrm{i}tL}\,[A\otimes \mathbb{1}_{\mathrm{S}}\otimes \mathbb{1}_{\mathrm{R}}]\,\mathrm{e}^{-\mathrm{i}tL}\psi_0\right\rangle \qquad (3)\end{aligned}$$

In last step, we pass to the *representation Hilbert space* of system (the GNS Hilbert space), where initial density matrix is represented by a *vector* ψ_0; L is (standard) Liouville operator

Explicitly: $\mathcal{H} = \mathfrak{H}_{\mathrm{S}}\otimes\mathfrak{H}_{\mathrm{S}}\otimes\mathcal{F}\otimes\mathcal{F}$

Take $\psi_0 = \Omega_{\mathrm{S},\beta}\otimes\Omega_{\mathrm{R},\beta}$, where $\Omega_{\mathrm{S/R},\beta}$ are equilibrium states of S,R at temperature $T = 1/\beta$

Dynamics is implemented by $\mathrm{e}^{\mathrm{i}tL}\cdot\mathrm{e}^{-\mathrm{i}tL}$

Trick from analysis of open systems far from equilibrium: there is a (non self-adjoint) generator K s.t.

$$\begin{aligned}\mathrm{e}^{\mathrm{i}tL}\cdot\mathrm{e}^{-\mathrm{i}tL} &= \mathrm{e}^{\mathrm{i}tK}\cdot\mathrm{e}^{-\mathrm{i}tK} \qquad \text{and} \\ K\psi_0 &= 0\end{aligned}$$

$\Rightarrow$ replace propagators in (3) by $\mathrm{e}^{\pm\mathrm{i}tK}$, use $\mathrm{e}^{-\mathrm{i}tK}\psi_0 = \psi_0$, and (formal) relation

$$\mathrm{e}^{\mathrm{i}tK} = \frac{-1}{2\pi\mathrm{i}} \int_{\mathbb{R}-\mathrm{i}} (K - z)^{-1} \mathrm{e}^{\mathrm{i}tz} \mathrm{d}z$$

$\Rightarrow$ we obtain resolvent representation

$$\langle A \rangle_t = \frac{-1}{2\pi\mathrm{i}} \int_{\mathbb{R}-\mathrm{i}} \left\langle \psi_0, (K_\lambda - z)^{-1} \left[A \otimes \mathbb{1}_\mathrm{S} \otimes \mathbb{1}_\mathrm{R} \right] \psi_0 \right\rangle \mathrm{e}^{\mathrm{i}tz} \mathrm{d}z \tag{4}$$

Uncovering resonances:

Notation: second quantization of a one-body operator O acting on single-particle wave functions of variable $k \in \mathbb{R}^3$:

$$\mathrm{d}\Gamma(O) = \int_{\mathbb{R}^3} a^*(k) O a(k)\, \mathrm{d}^3 k$$

Total number operator:

$$N = \mathrm{d}\Gamma(\mathbb{1}) \otimes \mathbb{1}_\mathrm{R} + \mathbb{1}_\mathrm{R} \otimes \mathrm{d}\Gamma(\mathbb{1})$$

acts on $\mathfrak{H}_\mathrm{R} = \mathcal{F}(L^2(\mathbb{R}^3, \mathrm{d}^3 k)) \otimes \mathcal{F}(L^2(\mathbb{R}^3, \mathrm{d}^3 k))$

Deformation transformation: $U(\omega) = \mathrm{e}^{-\mathrm{i}\omega D}$, generator

$$D = \mathrm{d}\Gamma(\vartheta) \otimes \mathbb{1}_\mathrm{R} - \mathbb{1}_\mathrm{R} \otimes \mathrm{d}\Gamma(\vartheta),$$

where $\vartheta = \frac{\mathrm{i}}{2}(\hat{k} \cdot \nabla + \nabla \cdot \hat{k})$, with $\hat{k} = \frac{k}{|k|}$.

Transformed generator $K_\lambda(\omega) = U(\omega) K_\lambda U(\omega)^{-1}$:

$$\begin{aligned} K_\lambda(\omega) &= L_0 + \omega N + \lambda I(\omega) \\ L_0 &= H_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{S}} - \mathbb{1}_{\mathrm{S}} \otimes H_{\mathrm{S}} \\ &\quad + \mathrm{d}\Gamma(|k|) \otimes \mathbb{1}_{\mathrm{R}} - \mathbb{1}_{\mathrm{R}} \otimes \mathrm{d}\Gamma(|k|) \end{aligned}$$

(have explicit formula also for $I(\omega)$)

$U(\omega)$ unitary for $\omega \in \mathbb{R} \Rightarrow$ spec(K_λ) =spec$(K_\lambda(\omega))$

$K_\lambda(\omega)$ analytic for $\omega \in \mathbb{C}$, $|\mathrm{Im}\ \omega| < 2\pi/\beta$.

spec$(K_\lambda(\omega))$ varies as Im(ω) does $\Rightarrow$ **spectral deformation**

$U(\omega)\psi_0 = \psi_0$ & analyticity of $K_\lambda(\omega)$ & (4) $\Rightarrow$

$$\langle A \rangle_t = \frac{-1}{2\pi \mathrm{i}} \int_{\mathbb{R}-\mathrm{i}} \left\langle \psi_0, (K_\lambda(\omega) - z)^{-1} \left[A \otimes \mathbb{1}_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}} \right] \psi_0 \right\rangle \mathrm{e}^{\mathrm{i}tz} \mathrm{d}z \tag{5}$$

May take $\omega = \mathrm{i}\omega'$, $\omega' > 0$

The point: spectrum of $K_\lambda(\omega)$ much easier to analyze than that of K_λ! $K_0(\omega) = L_0 + \mathrm{i}\omega' N$:

$$\mathrm{spec}(K_0(\omega)) = (\{E_i - E_j\}_{i,j=1,\ldots,N}) \cup_{n \geq 1} (\mathrm{i}\omega' n + \mathbb{R}).$$

Eigenvalues $E_i - E_j$: eigenvectors $\varphi_i \otimes \varphi_j \otimes \Omega_{\mathrm{R},\beta}$

lines $\mathrm{i}\omega' n + \mathbb{R}$: continuous spectrum

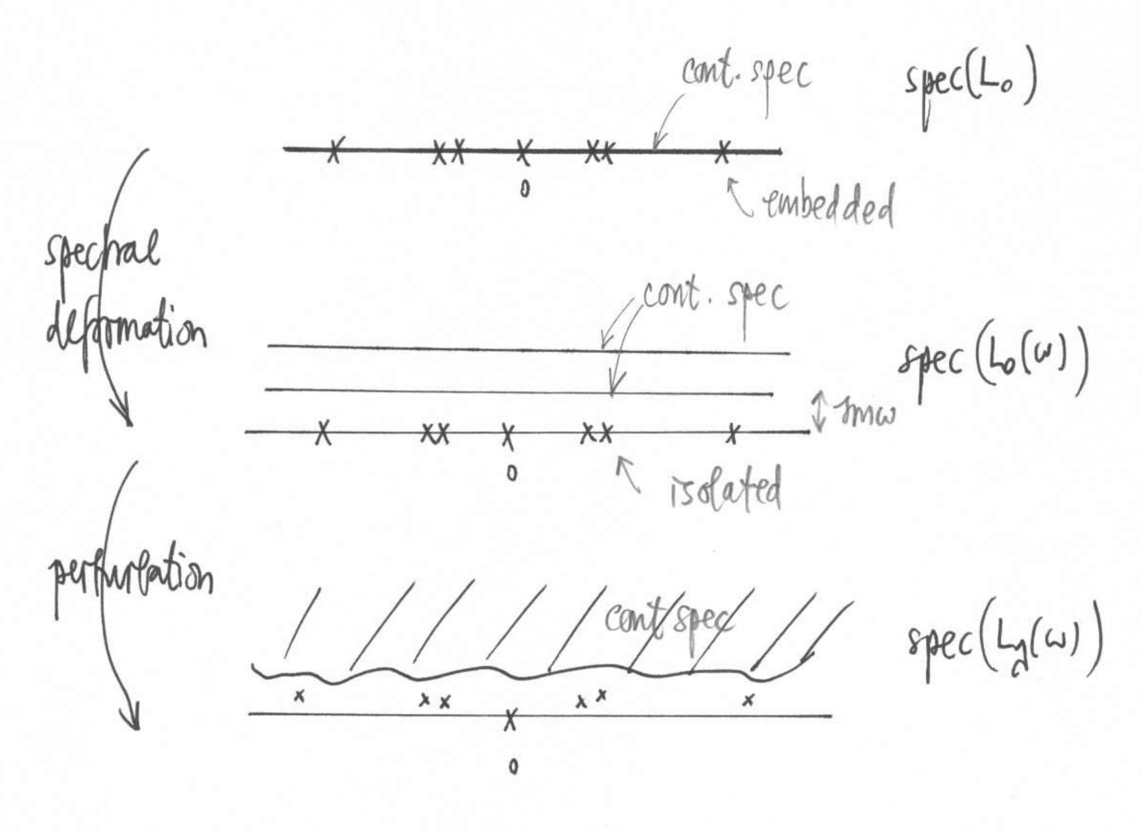
Gap of size ω' separating eigenvalues from the continuous spectrum of $K_0(\omega)$ $\Rightarrow$ can follow location of eigenvalues by simple (analytic) perturbation theory, provided λ is small compared to ω'

Theorem 1.1 *Fix $\omega' > 0$. There is a constant $c_0 > 0$ s.t. if $|\lambda| \leq c_0/\beta$ then, for all ω with $\mathrm{Im}\omega > \omega'$, the spectrum of $K_\lambda(\omega)$ in the complex half-plane $\{\mathrm{Im} z < \omega'/2\}$ is independent of ω and consists purely of the distinct eigenvalues*

$$\{\varepsilon_e^{(s)}(\lambda) \mid e \in \mathrm{spec}(L_{\mathrm{S}}), s = 1, \ldots, \nu(e)\},$$

where $1 \leq \nu(e) \leq \mathrm{mult}(e)$ counts the splitting of the eigenvalue e. Moreover, we have $\lim_{\lambda\to 0} |\varepsilon_e^{(s)}(\lambda) - e| = 0$ for all $s = 1, \ldots, \nu(e)$, and we have $\mathrm{Im}\varepsilon_e^{(s)}(\lambda) \geq 0$. Also, the continuous spectrum of $K_\lambda(\omega)$ lies in the region $\{\mathrm{Im} z \geq 3\omega'/4\}$.

By construction: $K_\lambda(\omega)\psi_0 = 0$ (so set $\varepsilon_0^{(1)} = 0$)

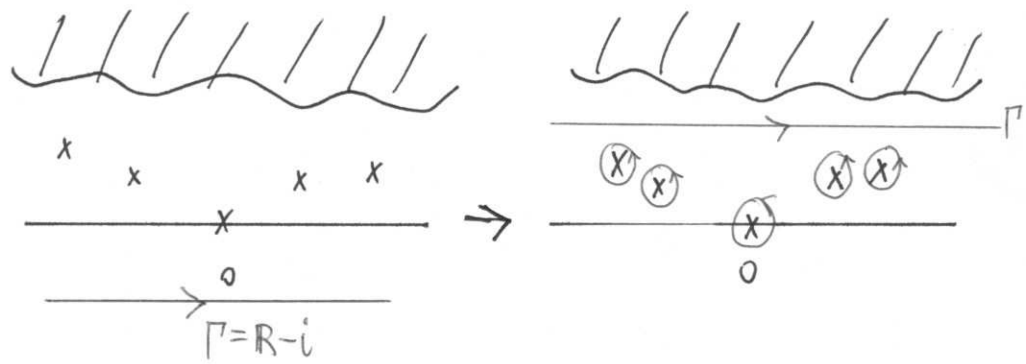


Pole approximation: deform contour

$$z = \mathbb{R} - \mathrm{i} \;\mapsto\; z = \mathbb{R} + \mathrm{i}\omega'/2$$

$\Rightarrow$ pick up residues of poles of integrand, sitting at the resonance energies $\varepsilon_e^{(s)}(\lambda)$

$\mathcal{C}_e^{(s)}$: small circle around $\varepsilon_e^{(s)}$ not enclosing any other point of the spectrum of $K_\lambda(\omega)$



$$\int_{\Gamma} dz \quad \longrightarrow \quad \sum_j \int_{\mathcal{C}_j} dz + \int_{\Gamma'} dz$$

$$\langle A\rangle_t = \tag{6}$$
$$\frac{-1}{2\pi\mathrm{i}}\sum_e\sum_{s=1}^{\nu(e)}\int_{\mathcal{C}_e^{(s)}}\mathrm{e}^{\mathrm{i}tz}\left\langle\psi_0,(K_\lambda(\omega)-z)^{-1}[A\otimes\mathbb{1}_{\mathrm{S}}\otimes\mathbb{1}_{\mathrm{R}}]\psi_0\right\rangle\mathrm{d}z+R$$

where

$$R=\frac{-1}{2\pi\mathrm{i}}\int_{\mathbb{R}+\mathrm{i}\omega'/2}\mathrm{e}^{\mathrm{i}tz}\left\langle\psi_0,(K_\lambda(\omega)-z)^{-1}(A\otimes\mathbb{1}_{\mathrm{S}}\otimes\mathbb{1}_{\mathrm{R}})\psi_0\right\rangle\mathrm{d}z$$

One shows: $R=O(\lambda^2\mathrm{e}^{-t\omega'/2})$

Can replace $\mathrm{e}^{\mathrm{i}tz}$ by $\mathrm{e}^{\mathrm{i}t\varepsilon_e^{(s)}}$ in (6)

$$
\begin{aligned}
\langle A \rangle_t \; &= \; \sum_e \sum_{s=1}^{\nu(e)} \mathrm{e}^{\mathrm{i}t\varepsilon_e^{(s)}} \left\langle \psi_0, Q_e^{(s)}[A \otimes \mathbb{1}_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}}]\psi_0 \right\rangle \\
&\quad +O(\lambda^2 \mathrm{e}^{-\omega' t/2})
\end{aligned} \tag{7}
$$

where $Q_e^{(s)}$ are (non-orthogonal) projections

$$
Q_e^{(s)} = Q_e^{(s)}(\omega, \lambda) = \frac{-1}{2\pi \mathrm{i}} \int_{\mathcal{C}_e^{(s)}} (K_\lambda(\omega) - z)^{-1} \mathrm{d}z
$$

If $\varepsilon_e^{(s)}$ is simple eigenvalue of $K_\lambda(\omega)$:

$$
Q_e^{(s)} = |\chi_e^{(s)}\rangle \langle \widetilde{\chi}_e^{(s)}|
$$

where vectors $\chi_e^{(s)}$ and $\widetilde{\chi}_e^{(s)}$ satisfy

$$
K_\lambda(\omega)\chi_e^{(s)} = \varepsilon_e^{(s)}\chi_e^{(s)} \;\; \text{and} \;\; (K_\lambda(\omega))^* \widetilde{\chi}_e^{(s)} = \overline{\varepsilon_e^{(s)}} \widetilde{\chi}_e^{(s)}
$$

and are normalized as $\left\langle \chi_e^{(s)}, \widetilde{\chi}_e^{(s)} \right\rangle = 1$.

Finally

$$
\begin{aligned}
\langle\langle A \rangle\rangle_\infty \; &:= \; \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle A \rangle_t \mathrm{d}t \\
&= \; \sum_{s':\ \varepsilon_0^{(s')}=0} \left\langle \psi_0, Q_0^{(s')}(A \otimes \mathbb{1}_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}})\psi_0 \right\rangle
\end{aligned}
$$

All the other terms vanish in the ergodic mean limit.

If $\langle A\rangle_t$ has limit as $t \to \infty$ ($\mathrm{I}m\, \varepsilon_e^{(s)}(\lambda) > 0$ for $\varepsilon_e^{(s)}(\lambda) \neq 0$) then $\langle\langle A\rangle\rangle_\infty$ is just that limit.

It may happen that $\langle A\rangle_t$ does not have limit, but $\langle\langle A\rangle\rangle_\infty$ always exists.

$\Rightarrow$ Limit term in expansion (7) is $\langle\langle A\rangle\rangle_\infty$ and we obtain desired result.

In specific models (like qubit), one can calculate (perturbatively in λ, to any order) resonance energies $\varepsilon_e^{(s)}$ and projection operators $Q_e^{(s)}$, and one obtains estimates on difference $\langle A\rangle_t - \langle\langle A\rangle\rangle_\infty$.

Evolution of reduced density matrix $[\overline{\rho}_t]_{m,n}$ is obtained from these formulas by using $A = |\varphi_n\rangle\langle\varphi_m|$.