Quantum Multi–time Measurements on Scattered Particles

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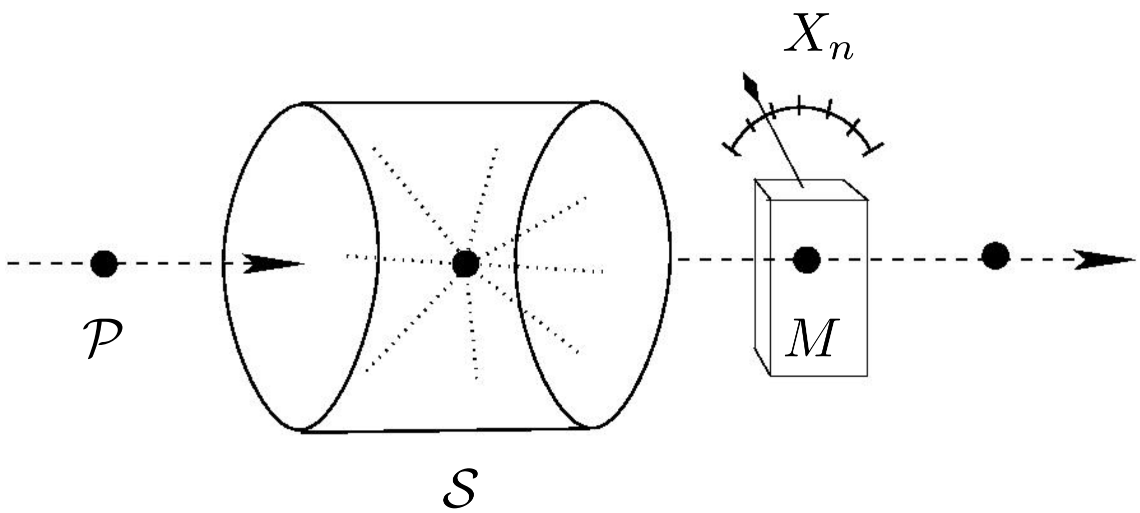
The Problem



$\left.\begin{array}{lll} \mathcal{P} & : & \text{probe} \\ \mathcal{S} & : & \text{scatterer} \\ M & : & \text{measurement (operator)} \end{array}\right\}$ $\begin{array}{c} \{X_n\}_{n\geq 1} \text{ measurement outcomes} \\ X_n \in \mathrm{spec}(M) = \{m_1, \ldots, m_r\} \end{array}$

Questions

- Asymptotics: $P(X_n$ converges$) =?$, $P(X_n \in A$ eventually$) =?$
- Input–output relationship?
- Asymptotic state of $\mathcal{S}$? Convergence speed?

Basic mechanism

• Interaction: τ (time), V (operator)

• Before scattering: probes are independent

• At scattering: probe n becomes correlated with $\mathcal{S}$, which is correlated with probes k, $1 \leq k \leq n-1 \Rightarrow X_n$ are dependent; quantum entanglement

Ergodicity assumption

In absence of measurement the interaction is *effective*: $\mathcal{S}$ is driven to an asymptotic state (large times). This is observed in laboratory & is shown by theory to hold generically.

Consequence

Scatterer 'loses memory': $\mathcal{S}$ initiates convergence to asymptotic state during $m-l \Rightarrow$ random variables X_l and X_m are weakly correlated if $m-l \gg 1$.

Results

Theorem (Correlation decay). *There are constants* $c > 0$, $\gamma > 0$ *such that, for all* $A \in \sigma(X_k, \ldots, X_l)$, $B \in \sigma(X_m, \ldots, X_n)$, $1 \leq k \leq l < m \leq n \leq \infty$, *we have*

$$|P(A \cap B) - P(A)P(B)| \leq c\mathrm{e}^{-\gamma(m-l)}P(A).$$

• Decaying correlations $\Rightarrow$ **Zero-One Law** (Kolmogorov): Any event in tail sigma-algebra

$$\mathcal{T} = \bigcap_{k \geq 1} \sigma(X_k, X_{k+1}, \ldots),$$

$A \in \mathcal{T}$, satisfied $P(A) = 0$ or $P(A) = 1$.

• Example: $A = \{X_k \text{ converges}\} \in \mathcal{T}$, so $P(X_k \text{ converges})$ is either zero or one. WHICH ONE IS IT?

$P = 0$ or $P = 1$: Perturbative approach

- V small ($\|V\| \ll 1$), m a fixed possible measurement outcome
- $P(X_n = m) = P_{\rm in}(m) + O(V)$
- $P(X_k = m_k, X_l = m_l) = P(X_k = m_k)P(X_l = m_l) + O(V)$

$$\begin{aligned} P(X_{n+1} = X_n) &= \sum_m P(X_{n+1} = m, X_n = m) \\ &= \sum_m P(X_{n+1} = m)P(X_n = m) + O(V) \\ &= \sum_m P_{\rm in}(m)^2 + O(V) \end{aligned}$$

So

$$\{0,1\} \ni P(X_{n+1} = X_n \text{ eventually}) \le \sum_m P_{\rm in}(m)^2 + O(V)$$

Conclusion: $P(X_n \text{ converges}) = 0$ if the in-state is not localized with respect to M (if M has nonvanishing variance) and V is small.

Frequencies

• m: possible measurement outcome. Frequency of m:

$$f_m = \lim_{n\to\infty} \frac{1}{n} \mathbf{E}\big[\#k \in \{1,\ldots,n\}: \ X_k = m\big]$$

• What is the effect of the scattering process on frequencies? Define

$$\overline{E}_m(\tau) = \frac{1}{\tau}\int_0^\tau \mathrm{e}^{\mathrm{i}sH} E_m \mathrm{e}^{-\mathrm{i}sH} \mathrm{d}s,$$

the time-averaged eigenprojection associated to $m \in \mathrm{spec}(M)$.

Theorem (Frequencies). *The first order correction (in V) to the frequency f_m is the flux of the averaged eigenprojection associated to m,*

$$f_m = P_{\mathrm{in}}(m) + \tau \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \omega_{\mathrm{in}} \otimes \omega_{\mathcal{S}}\Big(\mathrm{e}^{\mathrm{i}tH}\overline{E}_m(\tau)\mathrm{e}^{-\mathrm{i}tH}\Big) + O(V^2).$$

Note: the derivative term equals $\mathrm{i}\tau\omega_{\mathrm{in}} \otimes \omega_{\mathcal{S}}\Big([V,\overline{E}_m(\tau)]\Big)$.

Mean

Mean value

$$\overline{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$$

Theorem (Law of large numbers). *There is a* μ_∞ *such that*

$$\lim_{n\to\infty} P(\overline{X}_n - \mu_\infty) = 0$$

Approach is constructive and non-perturbative. We can answer more subtle questions, *e.g.*, questions of large deviation type:

$$P(|\overline{X}_n - \mu_\infty| > \epsilon) \sim \mathrm{e}^{-n\rho(\epsilon)} \qquad (n \to \infty),$$

with explicit $\rho(\epsilon)$.

Mathematical setup

• Hilbert space of pure states $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{P}} \otimes \mathcal{H}_{\mathcal{P}} \otimes \cdots$

• Initial state $\rho_0 = \rho_{\mathcal{S}} \otimes \rho_{\text{in}} \otimes \rho_{\text{in}} \otimes \cdots$

$\rho_{\#}$: **density matrices** (trace-class, non-negative operators on $\mathcal{H}_{\#}$)

• Dynamics at time-step n

$$H_n = \sum_{k=1}^{\infty} H_{\mathcal{P},k} + H_{\mathcal{S}} + V_n$$

V_n: interaction operator acting on $\mathcal{S}$ and n-th $\mathcal{P}$

• Measurement observable $M \in \mathcal{B}(\mathcal{H}_{\mathcal{P}})$, self-adjoint. Eigenvalues m_j, spectral projections E_{m_j}.

• From principles of quantum mechanics:

$$\begin{aligned} &P(X_1 = m_1, \ldots, X_n = m_n) \\ &\quad = \mathrm{Tr}\Big(E_{m_n} \mathrm{e}^{-\mathrm{i}\tau H_n} \cdots E_{m_1} \mathrm{e}^{-\mathrm{i}\tau H_1} \rho_0 \mathrm{e}^{\mathrm{i}\tau H_1} E_{m_1} \cdots \mathrm{e}^{\mathrm{i}\tau H_n} E_{m_n}\Big) \end{aligned}$$

Theorem (Representation of joint probabilities). *We have*

$$P(X_1 = m_1, \dots, X_n = m_n) = \langle \psi, T_{m_1} \cdots T_{m_n} \psi \rangle,$$

where T_m is a "reduced dynamics operator" (no measurement: T), the inner product is that of $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{S}$ and $\psi \in \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{S}$ represents the initial state ρ_{S}. (Gelfand-Naimark-Segal, or Liouville Hilbert space.)

$$\begin{aligned} P(X_n = m \text{ eventually}) &= P(\cup_{n\geq 1} \cap_{k\geq n} \{X_k = m\}) \\ &= \lim_{n\to\infty} \lim_{k\to\infty} P(X_n = m, X_{n+1} = m, \dots, X_k = m) \\ &= \lim_{n\to\infty} \lim_{k\to\infty} \langle \psi, T^{n-1} (T_m)^{k-n} \psi \rangle \\ &= \langle \psi, \Pi\ \Pi_m \psi \rangle \end{aligned}$$

Π, Π_m: **Riesz spectral projections** of T, T_m associated to eigenvalue one

Proving correlation decay

- From above theorem

$$P(X_l \in A, X_m \in B) = \langle \psi, T^{l-1} T_A T^{m-l-1} T_B \psi \rangle$$

- Ergodicity assumption implies that, for μ large,

$$T^\mu = |\psi\rangle\langle\psi^*| + O(\mathrm{e}^{-\gamma\mu})$$

for some $\gamma > 0$ and where $T\psi = \psi$, $T^*\psi^* = \psi^*$, $\langle \psi, \psi^* \rangle = 1$. Therefore

$$P(X_l \in A, X_m \in B) = \langle \psi, T^{l-1} T_A \psi \rangle \langle \psi^*, T_B \psi \rangle + O(\mathrm{e}^{-\gamma(m-l)})$$

- First factor on right side is $P(X_l \in A)$, second one is

$$\langle \psi, (|\psi\rangle\langle\psi^*|) T_B \psi \rangle = \langle \psi, T^{m-1} T_B \psi \rangle + O(\mathrm{e}^{-\gamma m}) = P(X_m \in B) + O(\mathrm{e}^{-\gamma m})$$

$$\Rightarrow P(X_l \in A, X_m \in B) = P(X_l \in A) P(X_m \in B) + O(\mathrm{e}^{-\gamma(m-l)})$$