ERGODICITY OF THE SPIN-BOSON MODEL FOR ARBITRARY COUPLING STRENGTH

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Abstract. We prove that the spin-boson system is ergodic, for arbitrary strengths of the coupling between the spin and the boson bath, provided the spin tunneling matrix element is small enough.

1. Introduction and main result

The Hilbert space of pure states of the spin-boson system is $\mathbb{C}^2 \otimes \mathcal{F}$, where

$$\mathcal{F} = \bigoplus_{n\geq 0} L^2_{\rm sym}(\mathbb{R}^{3n}, d^{3n}k) \tag{1.1}$$

is the symmetric Fock space over the one-particle (momentum representation) space $L^2(\mathbb{R}^3, d^3k)$. The spin-boson Hamiltonian is the self-adjoint operator (see [25], equation (1.4))

$$H = -\tfrac{1}{2}\Delta\sigma_x + \tfrac{1}{2}\varepsilon\sigma_z + H_R + \tfrac{1}{2}q_0\sigma_z \otimes \phi(h), \tag{1.2}$$

where σ_x and σ_z are Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.3}$$

$\Delta \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$ are the 'tunneling matrix element' and the 'detuning parameter', respectively. We are using units in which $\hbar$ takes the value one. The free field Hamiltonian is given by

$$H_R = \int_{\mathbb{R}^3} |k| a^*(k) a(k) d^3k, \tag{1.4}$$

where the creation and annihilation operators satisfy the canonical commutation relations $[a(k), a^*(l)] = \delta(k-l)$ (Dirac delta distribution). $q_0 \in \mathbb{R}$ is the coupling constant, and $\phi(h)$ is the field operator, smeared out with a test function $h \in L^2(\mathbb{R}^3, d^3k)$,

$$\phi(h) = \frac{1}{\sqrt{2}}\left(a^*(h) + a(h)\right) = \frac{1}{\sqrt{2}}\int_{\mathbb{R}^3} \big(h(k)a^*(k) + \overline{h}(k)a(k)\big)\, d^3k. \tag{1.5}$$

In [25], Leggett et al. consider (among many other things) the average of σ_z at time $t \geqslant 0$, when the spin starts (at $t=0$) in the state 'up' and the environment starts in its thermal equilibrium. They call this quantity $P(t)$. For arbitrary q_0 fixed, they perform formal time-dependent perturbation theory in Δ (small) and establish the formula ((3.37) in [25])

$$P(t) = P(\infty) + [1 - P(\infty)] \exp -t/\tau, \tag{1.6}$$

where $P(\infty) = -\tanh(\beta\varepsilon/2)$ is the equilibrium value and

$$\tau^{-1} = \Delta^2 \int_0^\infty dt \cos(\varepsilon t) \cos\left[\frac{q_0^2}{\pi} Q_1(t)\right] e^{-\frac{q_0^2}{\pi} Q_2(t)}. \tag{1.7}$$

Here,

$$Q_1(t) = \int_0^\infty d\omega \frac{J(\omega)}{\omega^2} \sin(\omega t), \tag{1.8}$$

$$Q_2(t) = \int_0^\infty d\omega \frac{J(\omega)(1-\cos(\omega t))}{\omega^2} \coth(\beta\omega/2), \tag{1.9}$$

where the *spectral density* of the reservoir is defined by

$$J(\omega) = \tfrac{\pi}{2}\omega^2 \int_{S^2} |h(\omega,\Sigma)|^2 d\Sigma, \qquad \omega \geq 0, \tag{1.10}$$

the integral being taken over the angular part in $\mathbb{R}^3$. The function h is the form factor in (1.2).[1] Of course, it is assumed in [25] that the integral in (1.7) does not vanish, so that $\tau < \infty$ is a finite relaxation time. Assuming this as well in the present paper, we show in Corollary 1.2 that the spin-boson system has the property of return to equilibrium, for arbitrary q_0 and small Δ. Our main result, Theorem 1.1, implies the corollary. It describes completely the spectrum of the generator of dynamics, which is purely absolutely continuous covering $\mathbb{R}$, except for a simple eigenvalue at the origin.

The spin-boson system is a W^*-dynamical system $(\mathcal{H}, \mathfrak{M}, \alpha)$, where $\mathfrak{M}$ is a von Neumann algebra of observables acting on a Hilbert space $\mathcal{H}$ and where α^t is a group of $*$automorphisms of $\mathfrak{M}$. The "positive temperature Hilbert space" is given by

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathcal{F}_\beta, \tag{1.11}$$

where $\mathcal{F}_\beta$ is the Fock space

$$\mathcal{F}_\beta = \bigoplus_{n\geq 0} L^2_{\rm sym}((\mathbb{R}\times S^2)^n, (du \times d\Sigma)^n). \tag{1.12}$$

It differs from the 'zero-temperature' Fock space (1.1) in that the single-particle space at positive temperature is the 'glued' space $L^2(\mathbb{R}\times S^2, du\times d\Sigma)$ [22] ($d\Sigma$ is the uniform measure on S^2). $\mathcal{F}_\beta$ carries a representation of the CCR algebra. The represented Weyl operators are given by $W(f_\beta) = e^{i\phi(f_\beta)}$, where $\phi(f_\beta) = \frac{1}{\sqrt{2}}(a^*(f_\beta) + a(f_\beta))$. Here, $a^*(f_\beta)$ and $a(f_\beta)$ denote creation and annihilation operators on $\mathcal{F}_\beta$, smoothed out with the function

$$f_\beta(u,\Sigma) = \sqrt{\frac{u}{1-e^{-\beta u}}}\, |u|^{1/2} \begin{cases} f(u,\Sigma), & u \geq 0 \\ -\overline{f}(-u,\Sigma), & u < 0 \end{cases} \tag{1.13}$$

belonging to $L^2(\mathbb{R}\times S^2, du\times d\Sigma)$. It is easy to see that the CCR are satisfied, namely,

$$W(f_\beta)W(g_\beta) = e^{-\frac{i}{2}\mathrm{Im}\langle f\,|\,g\rangle} W(f_\beta + g_\beta). \tag{1.14}$$

[1]The spectral density is related to the Fourier transform of the reservoir correlation function $C(t) = \omega_{R,\beta}(e^{itH_R}\varphi(h)e^{-itH_R}\varphi(h))$ by $J(\omega) = \sqrt{\pi/2}\tanh(\beta\omega/2)[\widehat{C}(\omega) + \widehat{C}(-\omega)]$.

The vacuum vector Ω represents the infinite-volume equilibrium state of the free Bose field, determined by the formula

$$\langle\, \Omega \,|\, W(f_\beta)\Omega \,\rangle = \exp\left\{-\tfrac{1}{4}\langle\, f \mid \coth(\beta|k|/2)f \,\rangle\right\}, \tag{1.15}$$

see also [3]. The CCR algebra is represented on (1.12) as $W(f) \mapsto W(f_\beta)$, for $f \in L^2(\mathbb{R}^3)$ such that $\langle\, f \mid \coth(\beta|k|/2)f \,\rangle < \infty$. We denote the von Neumann algebra of the represented Weyl operators by $\mathcal{W}_\beta$.

The doubled spin Hilbert space in (1.11) allows to represent any (pure or mixed) state of the two-level system by a vector, again by the GNS construction. This construction is as follows. Let ρ be a density matrix on $\mathbb{C}^2$. When diagonalized it takes the form $\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$, to which we associate the vector $\Psi_\rho = \sum_i \sqrt{p_i}\varphi_i \otimes \overline{\varphi}_i \in \mathbb{C}^2 \otimes \mathbb{C}^2$ (complex conjugation in any fixed basis – we will choose the eigenbasis of H_S given after (1.18) below). Then $\mathrm{Tr}(\rho A) = \langle\, \Psi_\rho \,|\, (A \otimes \mathbb{1}_S)\Psi_\rho \,\rangle$ for all $A \in \mathcal{B}(\mathbb{C}^2)$ and where $\mathbb{1}_S$ is the identity in $\mathbb{C}^2$. This is the GNS representation of the state given by ρ [8, 27]. The von Neumann algebra of observables is

$$\mathfrak{M} = \mathcal{B}(\mathbb{C}^2) \otimes \mathbb{1}_S \otimes \mathcal{W}_\beta \subset \mathcal{B}(\mathcal{H}). \tag{1.16}$$

The dynamics of the spin-boson system is given by

$$\alpha^t(A) = e^{itL} A e^{-itL}, \qquad A \in \mathfrak{M}. \tag{1.17}$$

It is generated by the self-adjoint Liouville operator acting on $\mathcal{H}$,

$$L = L_0 + \tfrac{1}{2}q_0 V - \tfrac{1}{2}q_0 JVJ \tag{1.18}$$

$$L_0 = L_S + L_R, \tag{1.19}$$

where $L_S = H_S \otimes \mathbb{1}_S - \mathbb{1}_S \otimes H_S$ with $H_S = -\frac{1}{2}\Delta\sigma_x + \frac{1}{2}\varepsilon\sigma_z$ the free two-level part and $L_R = d\Gamma(u)$ is the second quantization of multiplication by the radial variable u, i.e. the free Bose part. The interaction operator in (1.18) is

$$V = \sigma_z \otimes \mathbb{1}_S \otimes \phi(h_\beta), \tag{1.20}$$

where h_β is the image of the form factor h of (1.2) under the mapping (1.13). The operator J in (1.18) is the modular conjugation, which acts as

$$J(A \otimes \mathbb{1}_S \otimes W(f_\beta(u,\Sigma)))J = \mathbb{1}_S \otimes \overline{A} \otimes W(\overline{f}_\beta(-u,\Sigma)), \tag{1.21}$$

where $\overline{A}$ is the matrix obtained from A by taking entrywise complex conjugation (matrices are represented in the eigenbasis of H_S). Note that by (1.13), we have $\overline{f}_\beta(-u,\Sigma) = -e^{-\beta u/2} f_\beta(u,\Sigma)$. By the Tomita-Takesaki theorem [8], conjugation by J maps the von Neumann algebra of observables (1.16) into its commutant. In particular, V and JVJ commute. For more detail about this well-known setup we refer to [22, 5, 27] and references therein.

The vector representing the uncoupled ($q_0 = 0$) KMS state is

$$\Omega_{0,\mathrm{KMS}} = \Omega_{S,\beta} \otimes \Omega, \tag{1.22}$$

where $\Omega_{S,\beta}$ is the vector representative of the Gibbs density matrix $\rho_{S,\beta} \propto e^{-\beta H_S}$. For $\Delta = 0$, we have

$$\Omega_{S,\beta,\Delta=0} = \frac{e^{-\beta\varepsilon/4}\varphi_{++} + e^{\beta\varepsilon/4}\varphi_{--}}{\sqrt{e^{-\beta\varepsilon/2} + e^{\beta\varepsilon/2}}}. \tag{1.23}$$

According to Araki's perturbation theory, the (α^t, β)-KMS state on $\mathfrak{M}$ is

$$\Omega_{\rm KMS} = \frac{e^{-\beta(L_0+\frac12 q_0 V)/2}\Omega_{0,\rm KMS}}{\|e^{-\beta(L_0+\frac12 q_0 V)/2}\Omega_{0,\rm KMS}\|}. \tag{1.24}$$

One shows that $\Omega_{0,\rm KMS}$ is in the domain of $e^{-\beta(L_0+\frac12 q_0 V)/2}$ for any $\Delta, q_0 \in \mathbb{R}$ (see e.g. [13, 5, 8]).

Our analysis requires a regularity assumption on the form factor h. Let $\alpha \geqslant 0$. We say h satisfies the **Condition** (A_α) if

$$(1+|i\partial_u|^\alpha)(ih/u)_\beta \in L^2(\mathbb{R}\times S^2, du\times d\Sigma), \tag{1.25}$$

where $(h/u)_\beta$ is obtained from h/u via (1.13).

Theorem 1.1. *The spectrum of L is all of $\mathbb{R}$, for arbitrary $q_0, \Delta \in \mathbb{R}$. For any $q_0 \in \mathbb{R}$, $q_0 \neq 0$, there is a constant Δ_0 such that if $0 < |\Delta| \leqslant \Delta_0$, then we have the following.*

(a) *If* (A_α), (1.25), *holds for some $\alpha > 3/2$, then L has no eigenvalues except for a simple one at the origin, and $L\Omega_{\rm KMS} = 0$.*

(b) *If* (A_α), (1.25), *holds for some $\alpha > 2$, then the absolutely continuous spectrum of L is all of $\mathbb{R}$ and the singular continuous spectrum of L is empty.*

Admissible form factors satisfying (A_α) with $\alpha > 2$ are for instance $h(u) = u^{1/2}e^{-u^2}$, $h(u) = u^p\, e^{-u}$ or $h(u) = u^p\, e^{-u^2}$ with $p > 3$. We mention that the 'glueing' of the function f into f_β given in (1.13) can be done in various ways. In particular, the minus sign in the second line ($u < 0$) can be changed into an arbitrary phase $e^{i\phi}$. This phase can be chosen to accommodate different form factors to satisfy (A_α). A discussion of this has been given in [18].

The spectral properties of L given in Theorem 1.1 imply readily that any initial state converges to the equilibrium state, see e.g. [22, 5].

Corollary 1.2 (Return to equilibrium). *Assume the conditions of Theorem 1.1, (b). For any normal state ω of $\mathfrak{M}$ and any $A \in \mathfrak{M}$, we have*

$$\lim_{t\to\infty} \omega(\alpha^t(A)) = \langle\, \Omega_{\rm KMS} \,|\, A\Omega_{\rm KMS}\,\rangle.$$

Remarks. 1. Here, a state ω of $\mathfrak{M}$ is called normal if it is represented by a vector $\psi \in \mathcal{H}$, $\omega(A) = \langle\, \psi \,|\, A\psi\,\rangle$ (see [8] for more detail).

2. The corollary shows that $\lim_{t\to\infty} P(t) = P(\infty) + O(\Delta)$, in accordance with Leggett et al.'s formula (1.6) (they only exhibit the lowest order term in Δ).

Outline of the strategy. The Liouvillean L (1.18) is unitarily equivalent to $\mathcal{L}$ (2.1). We describe this transformation, inspired by [25], in Section 2. The advantage of working with $\mathcal{L}$ is that the coupling constant q_0 appears in $\mathcal{L}$ in a uniformly bounded way as opposed to a linear function as in L (see (2.3)-(2.5)). This enables us to obtain results for all $q_0 \in \mathbb{R}$.

We analyze the eigenvalues of $\mathcal{L}$ in Section 3, using the conjugate operator method. We take for the conjugate operator A_ν a regularized version of the translation generator $A = d\Gamma(-i\partial_u)$. It is important to note that the "spectral deformation" technique cannot be applied here. This is so since the interaction is essentially given by (a spin operator times) a Weyl operator $W(f) = e^{i\phi(f)}$. When applying a spectral translation with parameter $\theta \in \mathbb{C}$ to the interaction, the Weyl operator transforms into $W_\theta(f) = e^{i\theta A} W(f) e^{-i\theta A} = e^{\frac{i}{\sqrt{2}}(a^*(f_\theta)+a(f_{\bar\theta}))}$. The operator $a^*(f_\theta)+a(f_{\bar\theta})$ is not self-adjoint for $\theta \notin \mathbb{R}$ and hence the interaction becomes huge and is not relatively bounded with respect to the number operator N. It is not known how to show analyticity of $(\theta, z) \mapsto e^{i\theta A}(\mathcal{L}-z)^{-1}e^{-i\theta A} \in \mathcal{B}(\mathcal{H})$ in this situation. The idea is then to assume, instead of analyticity in θ, that only the first few real derivatives $\partial_t^\alpha|_{t=0} W_t(f)$ exist (we manage with $\alpha = 1, 2$). The α-th derivative is the α-fold commutator of W with A, which is relatively bounded w.r.t. $N^{\alpha/2}$, becoming more singular with increasing α. This presents a difficulty we have to overcome in our analysis, which is not present in previous works, since there the interaction term is linear in field operators.

Using a positive commutator argument, we show in Theorem 3.6 that $\mathcal{L}$ has no eigenvalues except for a simple one at zero, with corresponding eigenvector the KMS state ψ_{KMS}. Two important ingredients of the proof are: a regularity result on eigenvectors of $\mathcal{L}$ with the ensuing virial identity (Theorem 3.3) and a usually called a Fermi Golden Rule Condition on the effectiveness of the coupling. The latter is expressed here by the fact that Leggett et al.'s "relaxation time" τ is finite (which is also assumed in [25]).

We show in Section 4 that the continuous spectrum of $\mathcal{L}$ is purely absolutely continuous. To do so, we control the boundary values of the resolvent $(\mathcal{L}-z)^{-1}$, as $\mathrm{Im}z \to 0_+$ (see (4.1)). More precisely, we show that $\langle\, \varphi \,|\, (\mathcal{L}-z)^{-1}\psi \,\rangle$ is bounded as $\mathrm{Im}z \to 0_+$, for any φ, ψ in a dense set, in the following way. Using the Feshbach map, we relate the resolvent to a "reduced resolvent" $(\bar{\mathcal{L}}-z)^{-1}$ and a "Feshbach part" $\mathfrak{F}(z)^{-1}$, see (4.3). The reduced resolvent acts on the reduced Hilbert space $\mathrm{Ran}\bar{P}_\Omega$, while $\mathfrak{F}(z)^{-1}$ is an operator on $\mathrm{Ran}P_\Omega$ (of dimension four). The control (boundedness) of the boundary values of $(\mathcal{L}-z)^{-1}$ is implied by that of $(\bar{\mathcal{L}}-z)^{-1}$ and $\mathfrak{F}(z)^{-1}$, shown in Theorems 4.1 and 4.2, respectively. To prove Theorem 4.1, we analyze the reduced resolvent based on a suitable approximation $(\bar{\mathcal{L}}(\eta)-z)^{-1}$, $\eta > 0$, with $\bar{\mathcal{L}}(0) = \bar{\mathcal{L}}$. We show that $\partial_z(\bar{\mathcal{L}}(\eta)-z)^{-1}$ is Hölder continuous in $\eta > 0$, weakly on a dense set of vectors and uniformly in $\mathrm{Im}\, z > 0$. This implies that $(\bar{\mathcal{L}}-z)^{-1}$ has a bounded extension to $\mathrm{Im}\, z = 0_+$. In order to prove Theorem 4.2, namely boundedness of the boundary values of $\mathfrak{F}(z)^{-1}$, we first use the proven regularity of $(\bar{\mathcal{L}}-z)^{-1}$ to derive the existence of boundary values of $\mathfrak{F}(z)$, as $\mathrm{Im}\, z \to 0_+$. We then show the invertibility of $\mathfrak{F}(x)$, $x \in \mathbb{R}\backslash\{0\}$, by using the fact that the only eigenvalue of $\mathcal{L}$ is zero and is simple.

Related works. As mentioned in the outline of our strategy above, the spectral deformation method is not applicable to the problem at hand, so we do not further discuss the associated literature. The Mourre-, positive commutator-, or conjugate operator method originated in [29]. It was further developed in [1, 7, 11, 12, 16, 17, 18, 21, 26, 28]. A detailed exposition is given in the book [1]. These works differ from ours in that the system-reservoir interaction in the other works is linear in field operators, while ours is proportional to a Weyl operator. Weyl operators are bounded and field operators are only relatively $N^{1/2}$-bounded. However, the technique requires the control of commutators of the interaction with the conjugate operator (A). For the field operator, those commutators of all orders are as well $N^{1/2}$-bounded, but for Weyl operators, they become more singular with each commutation with A (see above). This results in the breakdown of the analytic spectral deformation technique since the latter amounts to taking infinitely many commutators. It also requires a modification of the conjugate operator method, relative to the above works. In particular, to show a limiting absorption principle, we use the Feshbach map, which was also done in [11]. In the latter work, the authors establish existence of $\partial_\eta^{\alpha_1}\partial_z^{\alpha_2}(\mathcal{L}(\eta)-z)^{-1}$ (in a certain topology and for $\mathrm{Im} z \geqslant 0$), for general $\alpha_{1,2}=0,1,2,\dots$ In the present work, we only show and use the existence of the derivatives $\partial_\eta^{\alpha_1}\partial_z^{\alpha_2}(\mathcal{L}(\eta)-z)^{-1}$ for $\alpha_{1,2}\in\{0,1\}$. This, however, suffices to show the limiting absorption principle, or the absence of singular continuous spectrum of $\mathcal{L}$. We need 'two plus epsilon' derivatives of the (unitarily transformed) form factors to be square integrable (condition (A_α) with $\alpha>2$), while only 'one plus epsilon' derivatives of the form factor need to be square integrable for the linearly coupled system in [11].

Some results for Pauli-Fierz systems for arbitrary coupling strength have been given in [28]. It is shown in [28] that the spectrum of L is all of $\mathbb{R}$ (we also give a short Weyl-sequences argument at the beginning of Section 4 to show this). Regularity of eigenvectors based on positive commutator estimates (and for arbitrary coupling constants) has been shown before for Pauli-Fierz type models, see e.g. [17] and afterwards also in [28] (the infra-red regularity of form factors required in the latter paper is the same we require in the present work, namely that $h(k)$ should decay at least as $|k|^{1+\epsilon}$, and it is better than that in the original [17], which was $|k|^{2+\epsilon}$).

Our approach to showing instability of eigenvalues under perturbation (Theorem 3.6) via a positive commutator argument is inspired by [6, 26].

Regularizations of $\mathcal{L}$ of the type $\mathcal{L}(\eta)$, that we use in the analysis of the absolutely continuous spectrum, are often used in Mourre theory. They have been introduced in [29] and have also been used in [1, 7, 11, 16, 21].

An alternative treatment of the dynamics of spin-boson (Pauli-Fierz) type models has been given in [14, 15]. The spectral analysis is traded in for a time-discretization, a Dyson series expansion of the unitary propagator, time ordering and subsequent combinatorial arguments. It also needs an analysis of the weak coupling limit. An advantage of this method is that regularity conditions on the form factors coming from spectral deformation or Mourre theory techniques are replaced by a condition of integrability of correlations. This leads to a generally milder condition on the form factors (see the discussion in section 1.6.4. of [14]). It would be interesting to see if this method can be modified to treat the problem at hand, but this has not been done yet, to our knowledge.

2. Unitary transformation

By a suitable unitary transformation, the Hamiltonian (1.2) with $\Delta=0$ can be diagonalized explicitly, see (3.28) of [25]. We modify this idea for application to the Liouville operator (1.18). The unitarily transformed Liouville operator is

$$\mathcal{L} = ULU^* = \mathcal{L}_0 + \Delta I \tag{2.1}$$

$$\mathcal{L}_0 = \mathcal{L}_S + \mathcal{L}_R = \tfrac{\varepsilon}{2}(\sigma_z\otimes\mathbb{1}_S - \mathbb{1}_S\otimes\sigma_z) + L_R \tag{2.2}$$

$$I = -\tfrac{1}{2}(\mathcal{V} - J\mathcal{V}J) \tag{2.3}$$

$$\mathcal{V} = \sigma_+\otimes\mathbb{1}_S\otimes W(2f_\beta) + \sigma_-\otimes\mathbb{1}_S\otimes W(-2f_\beta). \tag{2.4}$$

The raising and lowering operators are given by

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and

$$f_\beta = (-\tfrac{i}{2} q_0 h/u)_\beta, \tag{2.5}$$

where h and q_0 are the form factor and coupling constant given in the interaction in (1.2), with $f \mapsto f_\beta$ given in (1.13). Note that $\mathcal{V}$ is self-adjoint and bounded and satisfies $\mathcal{V}^2 = \mathbb{1}$. Since $\|\mathcal{V}\| = 1$, we have $\|I\| \leqslant 1$. Define the unitary operator

$$U = \exp i \big[\sigma_z \otimes \mathbb{1}_S \otimes \phi(f_\beta) - J\{\sigma_z \otimes \mathbb{1}_S \otimes \phi(f_\beta)\}J\big], \tag{2.6}$$

where the action of J is given in (1.21). Note that U depends on the coupling parameter q_0. For the uncoupled system $q_0 = 0$, we have $U = \mathbb{1}$. The KMS vector associated to $\mathcal{L}_0$ is

$$\psi_{0,\mathrm{KMS}} = \psi_{S,\beta} \otimes \Omega = U\Omega_{0,\mathrm{KMS}} \quad \text{where} \quad \psi_{S,\beta} := \Omega_{S,\beta,\Delta=0}. \tag{2.7}$$

By Araki's perturbation theory, the KMS vector associated to $\mathcal{L}$ is

$$\psi_{\mathrm{KMS}} = \frac{e^{-\beta(\mathcal{L}_0 - \frac{1}{2}\Delta\mathcal{V})/2}\psi_{0,\mathrm{KMS}}}{\|e^{-\beta(\mathcal{L}_0 - \frac{1}{2}\Delta\mathcal{V})/2}\psi_{0,\mathrm{KMS}}\|} = U\Omega_{\mathrm{KMS}}. \tag{2.8}$$

Theorem 2.1. *The spectrum of $\mathcal{L}$ is all of $\mathbb{R}$, for arbitrary $q_0, \Delta \in \mathbb{R}$. For any $q_0 \in \mathbb{R}$, $q_0 \neq 0$, there is a constant Δ_0 such that if $0 < |\Delta| \leqslant \Delta_0$, then we have the following.*

(a) *If* (A_α), (1.25), *holds for some $\alpha > 3/2$, then $\mathcal{L}$ has no eigenvalues except for a simple one at the origin, and $\mathcal{L}\psi_{\mathrm{KMS}} = 0$.*

(b) *If* (A_α), (1.25), *holds for some $\alpha > 2$, then the absolutely continuous spectrum of $\mathcal{L}$ is all of $\mathbb{R}$ and the singular continuous spectrum of $\mathcal{L}$ is empty.*

The proof of Theorem 1.1 follows immediately from this result and relation (2.1).

3. Proofs: Eigenvalues of $\mathcal{L}$

3.1. **Conjugate operator.** We will assume throughout this section that (1.25) is satisfied for $\alpha > 3/2$. Let $0 < \nu \leqslant 1$, $0 < \epsilon < \alpha - 3/2$ and set

$$w_\nu(u) = \int_0^u \frac{ds}{(\nu|s| + 1)^{1+\epsilon}}, \qquad u \in \mathbb{R}.$$

The derivative $w_\nu'(u) = (\nu|u| + 1)^{-1-\epsilon}$ is strictly positive and converges to the constant function one as ν tends to zero. We abbreviate

$$w_\nu = w_\nu(-i\partial_u), \quad w_\nu' = w_\nu'(-i\partial_u).$$

The ϵ is arbitrary but fixed, determined by the regularity of the form factor, see (1.25). We define the self-adjoint operators

$$A_\nu = d\Gamma(w_\nu), \quad \mathcal{C}_\nu = d\Gamma(w_\nu').$$

The domains of both operators contain that of $N = d\Gamma(\mathbf{1})$, and the inequalities $0 < \mathcal{C}_\nu \leqslant N$ and $\pm A_\nu \leqslant N/\epsilon\nu$ hold in the sense of quadratic forms on $\mathrm{dom}(N)$. Moreover, A_ν, $\mathcal{C}_\nu$ and N commute on $\mathrm{dom}(N)$ and as a quadratic form on $\mathrm{dom}(N) \cap \mathrm{dom}(\mathcal{L}_R)$, we have

$$i[A_\nu, \mathcal{L}_R] = \mathcal{C}_\nu. \tag{3.1}$$

Lemma 3.1. *1. For $g \in \mathrm{dom}((w_\nu')^{-1/2})$ and $\psi \in \mathrm{dom}(\mathcal{C}_\nu^{1/2})$,*

$$\|a(g)\psi\|^2 \leqslant \|(w_\nu')^{-1/2} g\|^2 \, \|\mathcal{C}_\nu^{1/2}\psi\|^2, \tag{3.2}$$
$$\|a^*(g)\psi\|^2 \leqslant \|(w_\nu')^{-1/2} g\|^2 \, \|\mathcal{C}_\nu^{1/2}\psi\|^2 + \|g\|^2 \, \|\psi\|^2.$$

2. For $\psi \in \mathrm{dom}(N)$,

$$|\langle \psi \,|\, i[A_\nu, I]\psi \rangle| \leqslant c_1 \|\mathcal{C}_\nu^{1/2}\psi\| \, \|\psi\| + c_2\|\psi\|^2, \tag{3.3}$$

where $c_1 = 16 \cdot 2^{\epsilon/2} \|(1 + |i\partial_u|^{3/2+\epsilon/2}) f_\beta\|$ and $c_2 = c_1(1 + \|f_\beta\|)$.

The inequality (3.3) implies that for all $\alpha > 0$, $i[A_\nu, I] \geqslant -\alpha c_1 \mathcal{C}_\nu - (\frac{c_1}{4\alpha} + c_2)$, as a quadratic form on $\mathrm{dom}(N)$. In combination with (3.1) we obtain that for any $\alpha > 0$,

$$i[A_\nu, \mathcal{L}] \geqslant (1 - \alpha|\Delta| c_1)\mathcal{C}_\nu - |\Delta|(\tfrac{c_1}{4\alpha} + c_2), \tag{3.4}$$

as a quadratic form on $\mathrm{dom}(N) \cap \mathrm{dom}(\mathcal{L}_R)$.

Proof of Lemma 3.1. 1. The relative bounds (3.2) are most easily obtained by applying the Fourier transform (so that functions of $-i\partial_u$ become multiplication operators in the Fourier variable). Their derivation is standard, see e.g. [4].

2. Let D be a self-adjoint operator on $L^2(\mathbb{R} \times S^2)$ and let $f \in \mathrm{dom}(D)$. As a quadratic form on $\mathrm{dom}(d\Gamma(D)) \cap \mathrm{dom}(N)$, we have

$$[d\Gamma(D), W(f)] = W(f)\big(\phi(iDf) + \tfrac{1}{2}\langle f \,|\, Df \rangle\big), \tag{3.5}$$

where ϕ is the field operator. This relation is readily obtained by taking the derivative $-i\partial_t|_{t=0}$ of $e^{itd\Gamma(D)} W(f) e^{-itd\Gamma(D)} = W(e^{itD} f)$. According to (2.3), (2.4) the interaction I consists of four similar terms. We treat the part $-\frac{1}{2}\sigma_+ \otimes \mathbb{1}_S \otimes W(2f_\beta)$, the others are dealt with in the same way. Taking into account (3.5) with $D = w_\nu$, we obtain for $\psi \in \mathrm{dom}(N)$

$$\tfrac{1}{2}|\langle \psi \,|\, \sigma_+ \otimes \mathbb{1}_S \otimes [A_\nu, W(2f_\beta)]\psi \rangle| \leqslant \|\psi\| \left(\|\phi(iw_\nu f_\beta)\psi\| + \| \, |w_\nu|^{1/2} f_\beta\|^2 \, \|\psi\|\right). \tag{3.6}$$

Using (3.2) gives

$$\|\phi(iw_\nu f_\beta)\psi\| \leqslant \sqrt{2}\|w_\nu (w_\nu')^{-1/2} f_\beta\| \, \|\mathcal{C}_\nu^{1/2}\psi\| + \|w_\nu f_\beta\| \|\psi\|.$$

Next, by the definition of w_ν given above and since $\nu \leqslant 1$,

$$|w_\nu(u)(w_\nu'(u))^{-1/2}| \leqslant \big(\nu|u| + 1\big)^{3/2+\epsilon/2} \leqslant 2^{3/2+\epsilon/2} \max(|u|^{3/2+\epsilon/2}, 1) \leqslant 2^{3/2+\epsilon/2}(|u|^{3/2+\epsilon/2} + 1)$$

and $|w_\nu(u)| \leqslant |u|$. So both norms $\|w_\nu(w_\nu')^{-1/2} f_\beta\|$ and $\|w_\nu f_\beta\|$ are bounded above by $2^{3/2+\epsilon/2}\|(1 + |i\partial_u|^{3/2+\epsilon/2}) f_\beta\|$. Furthermore,

$$\| \, |w_\nu|^{1/2} f_\beta\|^2 \leqslant \|f_\beta\| \, \| \, |i\partial_u| f_\beta\| \leqslant \|f_\beta\| \, \|(1 + |i\partial_u|)^{3/2+\epsilon/2} f_\beta\|.$$

This shows (3.3) and concludes the proof of Lemma 3.1. $\square$

3.2. **Regularity of eigenvectors of $\mathcal{L}$.** Let $0 \leqslant \chi \leqslant 1$ be a smooth function which satisfies $\chi(x) = 1$ for $|x| \leqslant 1/2$ and $\chi(x) = 0$ for $|x| \geqslant 1$. We set

$$\chi_\mu = \chi((N+1)/\mu), \qquad \chi_\mu^{(n)} = \chi^{(n)}((N+1)/\mu),$$

where $\chi^{(n)}$ denotes the n-th derivative of χ, and where $\mu \geqslant 1$.

Lemma 3.2. *1. The k-fold commutator ($k \geqslant 1$) of N with $\mathcal{L}$, $ad_N^{(k)}(\mathcal{L}) = [N, [N, \dots [N, \mathcal{L}] \dots]]$, is relatively $N^{k/2}$ bounded, and*

$$\|ad_N^{(k)}(\mathcal{L})(N+1)^{-k/2}\| \leqslant |\Delta| c(k)(1 + \|f_\beta\|)^{2k}, \tag{3.7}$$

where $c(k)$ is independent of f_β.

2. On $\mathrm{dom}(\mathcal{L}_R)$,

$$[\chi_\mu, \mathcal{L}] = \mu^{-1}\chi_\mu'\,[N, \mathcal{L}] - \tfrac{1}{2}\mu^{-2}\chi_\mu''\,[N, [N, \mathcal{L}]] + \Delta\mu^{-3/2} R_\mu, \tag{3.8}$$

with $\sup_{\mu \geqslant 1} \|R_\mu\| < \infty$.

Proof. 1. The operators N and $\mathcal{L}_0$ commute, only the interaction contributes to the commutator. Using repeatedly (3.5) with $D = \mathbb{1}$, together with the form equality $[N, a^*(f)] = a^*(f)$ (and its adjoint), one readily sees that $ad_N^{(k)}(\mathcal{L})$ is a sum of four terms, each of the form $S \otimes W(f_\beta) T_k$, where S is one of $\sigma_\pm \otimes \mathbb{1}_S$ or $\mathbb{1}_S \otimes \sigma_\pm$, and T_k is a polynomial in $a^*(f_\beta)$, $a(f_\beta)$ (of maximal joint degree k). The relative bound follows.

2. By means of the Helffer-Sjöstrand formula [10],

$$\chi_\mu^{(n)} = (-1)^n n! \int_{\mathbb{C}} \partial_{\bar z}\widetilde{\chi}(z)\left(\tfrac{N+1}{\mu} - z\right)^{-1-n} dz \tag{3.9}$$

for $n = 0, 1, 2 \dots$ We have, strongly on $\mathrm{dom}(\mathcal{L}) = \mathrm{dom}(\mathcal{L}_R)$,

$$[\chi_\mu, \mathcal{L}] = \Delta \int_{\mathbb{C}} \partial_{\bar z}\widetilde{\chi}(z)\left[\left(\tfrac{N+1}{\mu} - z\right)^{-1}, I\right] dz.$$

Using the relations (3.9) and $[A^{-1}, B] = A^{-1}[B, A]A^{-1}$, we arrive at

$$[\chi_\mu, \mathcal{L}] = \Delta\mu^{-1}\chi_\mu'\,[N, I] + \tfrac{1}{2}\Delta\mu^{-2}\chi_\mu''\,[N, [N, I]] + \Delta\mu^{-3/2} R_\mu, \tag{3.10}$$

where

$$R_\mu = \mu^{-3/2} \int_{\mathbb{C}} \partial_{\bar z}\widetilde{\chi}(z)\left(\tfrac{N+1}{\mu} - z\right)^{-3} ad_N^{(3)}(I)\left(\tfrac{N+1}{\mu} - z\right)^{-1} dz. \tag{3.11}$$

Invoking the relative bound (3.7) and that $|\mathrm{Re}z|$, $|\mathrm{Im}z| \leqslant 2$ (since z is in the support of the almost-analytic extension $\partial_{\bar z}\widetilde{\chi}(z)$), we get

$$\left\|\left(\tfrac{N+1}{\mu} - z\right)^{-3} ad_N^{(3)}(I)\left(\tfrac{N+1}{\mu} - z\right)^{-1}\right\| \leqslant C\mu^{3/2}|\mathrm{Im}z|^{-4},$$

with a constant C independent of μ and of z. However, $|\partial_{\bar z}\widetilde{\chi}(z)| \leqslant C'|\mathrm{Im}z|^4$ for some constant C' and so $\sup_{\mu \geqslant 1} \|R_\mu\| < \infty$. $\square$

Theorem 3.3 (Regularity of eigenvectors)**.** *Let ψ be a normalized eigenvector of $\mathcal{L}$. Then $\psi \in \mathrm{dom}(N^{1/2})$ and for every $0 < \xi < 1$,*

$$\|N^{1/2}\psi\|^2 \leqslant \frac{\xi^{-1}\Delta^2 c_1^2/4 + |\Delta| c_2}{1-\xi}. \tag{3.12}$$

Let $A \equiv A_{\nu=0} \equiv d\Gamma(-i\partial_u)$. The commutator $i[A,\mathcal{L}]$ is well defined as a quadratic form on $\mathrm{dom}(N^{1/2})$ and we have the virial identity

$$\langle\, \psi \,|\, i[A,\mathcal{L}]\psi \,\rangle = 0. \tag{3.13}$$

Remarks. 1. In (3.13), the commutator $i[A,\mathcal{L}]$ is understood as the closure of the sesquilinear form, defined on $\mathrm{dom}(A)\cap\mathrm{dom}(\mathcal{L})$ by $i\langle\, A\varphi \,|\, \mathcal{L}\psi \,\rangle - i\langle\, \mathcal{L}\varphi \,|\, A\psi \,\rangle$. The self-adjoint operator associated to the closure of this form is $i[A,\mathcal{L}] = N + \Delta i[A,I]$, where

$$\begin{aligned} i[A,I] \;=\; & -i\sigma_+ \otimes \mathbb{1}_S \otimes W(2f_\beta)\left(\phi(f_\beta') - i\langle\, f_\beta \,|\, f_\beta' \,\rangle\right) \\ & + i\ \mathbb{1}_S \otimes \sigma_+ \otimes J_R W(2f_\beta)\left(\phi(f_\beta') - i\langle\, f_\beta \,|\, f_\beta' \,\rangle\right) J_R \\ & + \text{adjoint}. \end{aligned} \tag{3.14}$$

The virial relation (3.13) needs a proof since ψ is generally not in $\mathrm{dom}(A)$.

2. This result does not require Δ to be small.

Proof of Theorem 3.3. Since the operator $A_{\nu,\mu} := \chi_\mu A_\nu \chi_\mu$ is self-adjoint and bounded and $\psi \in \mathrm{dom}(\mathcal{L})$, we have the virial identity

$$0 = \langle\, \psi \,|\, i[A_{\nu,\mu},\mathcal{L}]\psi \,\rangle = t_1 + t_2, \tag{3.15}$$

with $t_1 = \langle\, \chi_\mu\psi \,|\, i[A_\nu,\mathcal{L}]\chi_\mu\psi \,\rangle$ and $t_2 = 2\mathrm{Re}\, i\langle\, \psi \,|\, [\chi_\mu,\mathcal{L}]A_\nu\chi_\mu\psi \,\rangle$. Choosing $\alpha = \xi(|\Delta|c_1)^{-1}$ in (3.4) gives the lower bound

$$t_1 \geqslant (1-\xi)\,\langle\, \chi_\mu\psi \,|\, \mathcal{C}_\nu\chi_\mu\psi \,\rangle - \tfrac{\Delta^2 c_1^2}{4\xi} - |\Delta| c_2. \tag{3.16}$$

The expansion (3.8), together with the bound $\|A_\nu(N+1)^{-1}\| \leqslant 1/\epsilon\nu$ implies that

$$\begin{aligned} |t_2| \leqslant\, & 2|\Delta|\ \mu^{-1}\ |\langle\, \psi \,|\, \chi_\mu'[N,I]A_\nu\chi_\mu\psi \,\rangle| \\ & + 2|\Delta|(\epsilon\nu)^{-1}\, c(2)\ (1+\|f_\beta\|)^4\ \|\chi_\mu''\psi\| + 2|\Delta|(\epsilon\nu\mu^{1/2})^{-1}\|R_\mu\|. \end{aligned}$$

Recall that $c(k)$ is defined in Lemma 3.2. Proceeding as in the proof of that lemma, point 1., one shows that for all $\varphi \in \mathrm{dom}(N^{1/2})$, $\|[N,I]\varphi\| \leqslant 8\|(\phi(if_\beta) + \|f_\beta\|^2)\varphi\|$. Combining this estimate with (3.2) gives

$$\begin{aligned} & |\langle\, \psi \,|\, \chi_\mu'[N,I]A_\nu\chi_\mu\psi \,\rangle| \leqslant \|\chi_\mu'\psi\|\ \|[N,I]A_\nu\chi_\mu\psi\| \\ & \leqslant 8\sqrt{2}(\|f_\beta'\| + \|f_\beta\|)\ \|\chi_\mu'\psi\|\ \|\mathcal{C}_\nu^{1/2}A_\nu\chi_\mu\psi\| + 8\mu(\epsilon\nu)^{-1}\|\chi_\mu'\psi\|\ \|f_\beta\|(1+\|f_\beta\|). \end{aligned}$$

The $\mathcal{C}_\nu$, χ_μ and A_ν commute and $\|\mathcal{C}_\nu^{1/2}\chi_\mu A_\mu\psi\| \leqslant \mu(\epsilon\nu)^{-1}\|\mathcal{C}_\nu^{1/2}\chi_\mu\psi\|$. We make use of

$$\|\chi_\mu'\psi\|\,\|\mathcal{C}_\nu^{1/2}\chi_\mu\psi\| \leqslant \widetilde{\alpha}\langle\, \chi_\mu\psi \,|\, \mathcal{C}_\nu\chi_\mu\psi \,\rangle + (4\widetilde{\alpha})^{-1}\|\chi_\mu'\psi\|^2,$$

with $\widetilde{\alpha} = \kappa\epsilon\nu[16\sqrt{2}|\Delta|^{-1}(\|f_\beta\| + \|f'_\beta\|)]^{-1}$, for an arbitrary $\kappa > 0$. This gives

$$\begin{aligned} |t_2| &\leqslant \kappa\langle\, \chi_\mu\psi \,|\, \mathcal{C}_\nu\chi_\mu\psi \,\rangle + C|\Delta|(\epsilon\nu)^{-1}\|\chi''_\mu\psi\| \\ &\quad + C|\Delta|(\epsilon\nu)^{-1}\|\chi'_\mu\psi\|(|\Delta|(\epsilon\nu\kappa)^{-1}\|\chi'_\mu\psi\| + 1) + C|\Delta|(\epsilon\nu\mu^{1/2})^{-1}, \end{aligned} \tag{3.17}$$

where C is a constant independent of Δ, μ, ν, κ. The spectral support of the operators χ'_μ, χ''_μ is contained in $\mu/2 \leq N+1 \leq \mu$. Thus we have $\lim_{\mu\to\infty} \|\chi'_\mu\psi\| = 0 = \lim_{\mu\to\infty} \|\chi''_\mu\psi\|$. It follows from (3.17) that there exists a $\mu_0(\nu,\kappa)$ such that for $\mu \geqslant \mu_0$, we have

$$|t_2| \leqslant \kappa\langle\, \chi_\mu\psi \,|\, \mathcal{C}_\nu\chi_\mu\psi \,\rangle + \kappa. \tag{3.18}$$

Combining (3.15), (3.16) and (3.18) gives

$$\langle\, \chi_\mu\psi \,|\, \mathcal{C}_\nu\chi_\mu\psi \,\rangle \leqslant a \equiv \frac{\xi^{-1}\Delta^2 c_1^2/4 + |\Delta|c_2 + \kappa}{1 - \xi - \kappa}$$

whenever $\mu \geq \mu_0$. Note that $\mathcal{C}_\nu$ is self-adjoint and positive. Since a does not depend on μ, one easily shows, by taking $\mu \to \infty$, that $\psi \in \mathrm{dom}(\mathcal{C}_\nu^{1/2})$ and $\|\mathcal{C}_\nu^{1/2}\psi\| \leqslant \sqrt{a}$. Next we take $\nu \downarrow 0$. According to the decomposition of Fock space into a direct sum of n-particle sectors, we have

$$\langle\, \psi \,|\, \mathcal{C}_\nu\psi \,\rangle = \sum_{n\geq 1}\sum_{j=1}^{n} \langle\, \psi_n \,|\, [w'_\nu]_j\ \psi_n \,\rangle,$$

where $[w'_\nu]_j$ is the operator $(\nu|i\partial_{u_j}|+1)^{-1-\epsilon}$, acting on the j-th radial variable, u_j, of n-particle sector $\psi_n(u_1,\Sigma_1,\ldots,u_n,\Sigma_n)$. Since $[w'_\nu]_j \uparrow 1$ as $\nu \downarrow 0$ we invoke the monotone convergence theorem to conclude that $\lim_{\nu\downarrow 0}\langle\, \psi \,|\, \mathcal{C}_\nu\psi \,\rangle = \langle\, \psi \,|\, N\psi \,\rangle \leqslant a$. Upon taking $\kappa \to 0$ we obtain the bound (3.12).

Next we prove (3.13). We know from (3.18) that $|t_2| \leq \kappa(a+1)$, provided $\mu \geqslant \mu_0$. Taking first $\mu \to \infty$ and then $\kappa \to 0$ in (3.15) gives

$$\lim_{\mu\to\infty} \langle\, \chi_\mu\psi \,|\, i[A_\nu, \mathcal{L}]\chi_\mu\psi \,\rangle = 0. \tag{3.19}$$

We have $i[A_\nu, \mathcal{L}] = \mathcal{C}_\nu + i\Delta[A_\nu, I]$ and we know from the above that

$$\lim_{\nu\to 0}\lim_{\mu\to\infty} \langle\, \chi_\mu\psi \,|\, \mathcal{C}_\nu\chi_\mu\psi \,\rangle = \langle\, \psi \,|\, N\psi \,\rangle.$$

Furthermore, as $[A_\nu, I]$ is a well-defined operator on $\mathrm{dom}(N^{1/2})$ (see Lemma 3.2) and has the strong limit (3.14) for $\nu \to 0$, relation (3.13) follows from (3.19) by first taking $\mu \to \infty$ and then $\nu \to 0$. □

3.3. **Eigenvalues of $\mathcal{L}$.**

Proposition 3.4. *1. Let Δ be arbitrary and suppose ψ is a normalized eigenvector of $\mathcal{L}$ with eigenvalue e. Then*

$$\|\bar{P}_\Omega\psi\| \leqslant 10c_2|\Delta|, \tag{3.20}$$

$$\mathrm{dist}\big(e, \mathrm{spec}(\mathcal{L}_S)\big) \leqslant \tfrac{2}{\sqrt{3}}|\Delta|(1 - \|\bar{P}_\Omega\psi\|^2)^{-1/2}, \tag{3.21}$$

where c_2 is given in Lemma 3.1.

2. Suppose that Δ is small such that $\frac{2}{\sqrt{3}}|\Delta|(1-\|\bar{P}_\Omega\psi\|^2)^{-1/2}<\varepsilon/2$, where ε is the distance between the nearest eigenvalues of $\mathcal{L}_S$. Then, by (3.21), there is a unique $e_0\in\operatorname{spec}(\mathcal{L}_S)$ which is closest to e. Let P_{e_0} be the eigenprojection associated to this e_0 and denote $\bar{P}_{e_0}=\mathbb{1}_S-P_{e_0}$. Then (writing $P_{e_0}P_\Omega$ for $P_{e_0}\otimes P_\Omega$)

$$\|\bar{P}_{e_0}P_\Omega\psi\|\leqslant 2|\Delta|\varepsilon^{-1}. \tag{3.22}$$

Remark. In point 2., which e_0 is closest to e may depend on Δ, and we are not proving that e is continuously varying in Δ.

Proof. 1. Note that $P_\Omega i[A,I]P_\Omega=0$ since $A\Omega=d\Gamma(-i\partial_u)\Omega=0$. The virial identity (3.13) implies

$$0=\langle\,\bar{P}_\Omega\psi\,|\,(N+\Delta i[A,I])\bar{P}_\Omega\psi\,\rangle+2\mathrm{Re}\,\langle\,\bar{P}_\Omega\psi\,|\,\Delta i[A,I]P_\Omega\psi\,\rangle. \tag{3.23}$$

Using $\|\phi(h)N^{-1/2}\bar{P}_\Omega\|\leqslant(1+1/\sqrt{2})\|h\|$ and relation (3.14) one obtains the bound

$$\|[A,I]N^{-1/2}\bar{P}_\Omega\|\leqslant 8(1+1/\sqrt{2})\|f_\beta'\|\,(1+\|f_\beta\|)\leqslant 4c_2.$$

It follows that

$$\begin{aligned}|\langle\,\bar{P}_\Omega\psi\,|\,\Delta i[A,I]\bar{P}_\Omega\psi\,\rangle| &\leqslant 4|\Delta|c_2\|\bar{P}_\Omega\psi\|\,\|N^{1/2}\bar{P}_\Omega\psi\|,\\ 2\mathrm{Re}\,|\langle\,\bar{P}_\Omega\psi\,|\,\Delta i[A,I]P_\Omega\psi\,\rangle| &\leqslant 8|\Delta|c_2\|P_\Omega\psi\|\,\|N^{1/2}\bar{P}_\Omega\psi\|.\end{aligned}$$

We combine the last two inequalities with (3.23) to arrive at

$$0\geqslant(1-\alpha)\|N^{1/2}\bar{P}_\Omega\psi\|^2-24\alpha^{-1}\Delta^2c_2^2,$$

for any $\alpha>0$. The choice $\alpha=1/2$ gives (3.20).

Next we show (3.21). For any eigenvalue e_0 of $\mathcal{L}_S$, set $Q_{e_0}:=\bar{P}_{e_0}P_\Omega$. Projecting $\mathcal{L}\psi=e\psi$, $\|\psi\|=1$, onto the range of Q_{e_0} gives $Q_{e_0}\psi=-\Delta(\mathcal{L}_S-e)^{-1}Q_{e_0}I\psi$. (The result to be proven is clearly true if $e=e_0$ so we may assume $e\neq e_0$.) Therefore, for any eigenvalue e_0 of $\mathcal{L}_S$,

$$\|Q_{e_0}\psi\|\leqslant\frac{|\Delta|}{\operatorname{dist}(e,\operatorname{spec}(\mathcal{L}_S)\backslash\{e_0\})}. \tag{3.24}$$

Since $\sum_{e_0\in\operatorname{spec}(\mathcal{L}_S)}Q_{e_0}=3P_\Omega$ we have $3\|P_\Omega\psi\|^2=\sum_{e_0\in\operatorname{spec}(\mathcal{L}_S)}\|Q_{e_0}\psi\|^2\leqslant 4\|Q_{e_*}\psi\|^2$, where e_* is an eigenvalue of $\mathcal{L}_S$ maximizing the norm $\|Q_{e_0}\psi\|$. Using the latter bound in (3.24) gives

$$\operatorname{dist}(e,\operatorname{spec}(\mathcal{L}_S)\backslash\{e_*\})\leqslant\frac{|\Delta|}{\|Q_{e_*}\psi\|}\leqslant\frac{2|\Delta|}{\sqrt{3}\sqrt{1-\|\bar{P}_\Omega\psi\|^2}}.$$

Since $\operatorname{dist}(e,\operatorname{spec}(\mathcal{L}_S))\leqslant\operatorname{dist}(e,\operatorname{spec}(\mathcal{L}_S)\backslash\{e_*\})$, we have shown (3.21).

2. We have $\operatorname{dist}(e,\operatorname{spec}(\mathcal{L}_S)\backslash\{e_0\})>\varepsilon/2$ and (3.22) follows from (3.24). This concludes the proof of Proposition 3.4. □

Instability of eigenvalues of $\mathcal{L}_0$ under the perturbation ΔI can be shown provided a ("Fermi Golden Rule"-)condition of effective coupling is satisfied.

Proposition 3.5. *Let Π_0 be the rank-two spectral projection onto the kernel of $\mathcal{L}_0$ and set $\bar{\Pi}_0 = \mathbb{1} - \Pi_0$. The operator*

$$\Lambda_0 \equiv \Pi_0 I \bar{\Pi}_0 (\mathcal{L}_0 - i0_+)^{-1} I \Pi_0 \equiv \lim_{\eta\to 0_+} \Pi_0 I \bar{\Pi}_0 (\mathcal{L}_0 - i\eta)^{-1} I \Pi_0$$

exists and is anti self-adjoint (it equals i times a self-adjoint operator). The eigenvalues are $\mathrm{spec}(\Lambda_0) = \{0, i\Delta^{-2}\tau^{-1}\}$, *where τ^{-1} is given in* (1.7). *Moreover, $\Lambda_0\psi_{S,\beta} = 0$ (see* (2.7)*).*

Proof of Proposition 3.5. We identify Λ_0 with a 2×2 matrix relative to the orthonormal basis $\{\varphi_{++}\otimes\Omega, \varphi_{--}\otimes\Omega\}$ of $\mathrm{Ran}\Pi_0$. Here, $\varphi_{++} = \varphi_+\otimes\varphi_+$ and $\sigma_z\varphi_\pm = \pm\varphi_\pm$. We caclulate explicitly

$$\begin{aligned}
4\Lambda_0\varphi_{++} \;=\;& \varphi_{++}\left\langle W(2f_\beta)(\mathcal{L}_R - \varepsilon - i0_+)^{-1} W(2f_\beta)^*\right\rangle\\
&+\varphi_{++}\left\langle JW(2f_\beta)J(\mathcal{L}_R + \varepsilon - i0_+)^{-1} JW(2f_\beta)^*J\right\rangle\\
&-\varphi_{--}\left\langle W(2f_\beta)^*(\mathcal{L}_R + \varepsilon - i0_+)^{-1} JW(2f_\beta)^*J\right\rangle\\
&-\varphi_{--}\left\langle JW(2f_\beta)^*J(\mathcal{L}_R - \varepsilon - i0_+)^{-1} W(2f_\beta)^*\right\rangle. \qquad (3.25)
\end{aligned}$$

Here, $\langle\,\cdot\,\rangle = \langle\Omega, \cdot\,\Omega\rangle$. Since $J\Omega = \Omega$, $Je^{-\beta\mathcal{L}_R/2}W(h_\beta)\Omega = W(h_\beta)^*\Omega$ (by properties of the modular conjugation J and the modular operator $e^{-\beta\mathcal{L}_R/2}$) and since $Je^{-\beta\mathcal{L}_R/2} = e^{\beta\mathcal{L}_R/2}J$, we have $\langle W(g_\beta)JW(h_\beta)J\rangle = \langle W(g_\beta)e^{-\beta\mathcal{L}_R/2}W(h_\beta)^*\rangle$. A term $\langle W(g_\beta)JW(h_\beta)J\rangle$ can thus be calculated as the holomorphic continuation of $\mathbb{R}\ni t\mapsto \langle W(g_\beta)e^{it\mathcal{L}_R}W(h_\beta)^*\rangle$ at $t = i\beta/2$. For real values of t, the latter average is easy to calculate using that (1) the exponential generates a Bogoliubov dynamics ($t\mapsto e^{iut}h_\beta$), (2) the CCR (1.14) and (3) that the thermal average is given by (1.15). The result is

$$\langle W(g_\beta)JW(h_\beta)J\rangle = e^{\frac14\left(\langle g\,|\,e^{-\beta|k|/2}h\rangle - \langle h\,|\,e^{\beta|k|/2}g\rangle\right)}\ e^{-\frac14\left(\langle g\,|\,cg\rangle + \langle h\,|\,ch\rangle - \langle h\,|\,ce^{\beta|k|/2}g\rangle - \langle g\,|\,ce^{-\beta|k|/2}h\rangle\right)},$$

where, for short,

$$c = \coth(\beta|k|/2). \qquad (3.26)$$

Using the representation $(\mathcal{L}_R - \varepsilon - i0_+)^{-1} = i\lim_{\eta\downarrow 0}\int_0^\infty e^{it(\varepsilon+i\eta)}e^{-it\mathcal{L}_R}dt$, we cast (3.25) in the form $\Lambda_0\varphi_{++} = x(\varepsilon)\varphi_{++} + z(\varepsilon)\varphi_{--}$, where

$$\begin{aligned}
x(\varepsilon) &= \tfrac12 i\,\mathrm{Re}\int_0^\infty e^{it\varepsilon}\, e^{-2i\langle f\,|\,\sin(|k|t)f\rangle}\, e^{-2\langle f\,|\,c(1-\cos(|k|t))f\rangle}\,dt\\
z(\varepsilon) &= -\tfrac12 i\int_0^\infty \cos(\varepsilon t)e^{-2\langle f\,|\,\{c - \frac{2\cos(|k|t)}{e^{\beta|k|/2}-e^{-\beta|k|/2}}\}f\rangle}\,dt, \qquad (3.27)
\end{aligned}$$

with c given in (3.26). The symmetry $\sigma_x\otimes\sigma_x\ \Lambda_0(\varepsilon, f)\ \sigma_x\otimes\sigma_x = \Lambda_0(-\varepsilon, -f)$ (where we display the dependence on ε and f explicitly) implies immediately that $\Lambda_0\varphi_{--} = z(\varepsilon)\varphi_{++} + x(-\varepsilon)\varphi_{--}$. (Note that $z(\varepsilon) = z(-\varepsilon)$.) Therefore, the level shift operator takes the matrix form

$$\Lambda_0 = \begin{pmatrix} x(\varepsilon) & z(\varepsilon)\\ z(\varepsilon) & x(-\varepsilon)\end{pmatrix}. \qquad (3.28)$$

By a deformation of the path of integration, it is not hard to verify that $x(\varepsilon) = -e^{\beta\varepsilon/2}z(\varepsilon)$ (see also Appendix E of [25]). This implies that the Gibbs state $\psi_{S,\beta}$, (2.7), is in the kernel of Λ_0.

The other eigenvalue of Λ_0 is hence its trace,

$$\mathrm{Tr}\Lambda_0 = x(\varepsilon) + x(-\varepsilon) = i\int_0^\infty \cos(\varepsilon t)\cos\left(2\langle\, f \mid \sin(|k|t)f\,\rangle\right) e^{-2\langle\, f\,|\,c(1-\cos(|k|t))f\,\rangle} dt.$$

Using the relation (2.5) shows that $\mathrm{Tr}\Lambda_0 = i\Delta^{-2}\tau^{-1}$, see (1.7). This completes the proof of Proposition 3.5. □

Theorem 3.6. *Suppose* $0 < |\Delta| < \Delta_0$, *for some constant* Δ_0 *given in* (3.38). *Then* $\mathcal{L}$ *has no eigenvalues except for a simple one at the origin. Moreover,* $\mathcal{L}\psi_{\mathrm{KMS}} = 0$, *where* ψ_{KMS} *is the coupled KMS state* (2.8).

Proof. Let e be an eigenvalue of $\mathcal{L}$ with associated normalized eigenvector ψ, and define, for $\eta > 0$,

$$X_\eta = \eta\left((\mathcal{L}_0 - e)^2 + \eta^2\right)^{-1} = \mathrm{Im}\,(\mathcal{L}_0 - e - i\eta)^{-1}.$$

We derive an upper bound and a lower bound for

$$q_e(\psi) = \Delta^2\langle\, P_\Omega\psi \mid I\bar{P}_\Omega\, X_\eta I P_\Omega\psi\,\rangle.$$

Upper bound. Since $\Delta\bar{P}_\Omega I P_\Omega\psi = \bar{P}_\Omega(\mathcal{L}-e)P_\Omega\psi = -\bar{P}_\Omega(\mathcal{L}-e)\bar{P}_\Omega\psi$, we have

$$q_e(\psi) = -\Delta\langle\, \bar{P}_\Omega\psi \,|\, (\mathcal{L}_0 - e)\bar{P}_\Omega\, X_\eta I P_\Omega\psi\,\rangle - \Delta^2\langle\, \bar{P}_\Omega\psi \,|\, I\bar{P}_\Omega\, X_\eta I P_\Omega\psi\,\rangle. \tag{3.29}$$

The bounds $\|I\| \leqslant 1$, $\|X_\eta^{1/2}\| \leqslant \eta^{-1/2}$ and $\|X_\eta^{1/2}(\mathcal{L}_0 - e)\| \leqslant \eta^{1/2}$ then imply that

$$q_e(\psi) \leqslant \eta^{1/2}\,\|\bar{P}_\Omega\psi\|\, q_e(\psi)^{1/2} + |\Delta|\eta^{-1/2}\|\bar{P}_\Omega\psi\|\, q_e(\psi)^{1/2}. \tag{3.30}$$

Dividing by $q_e(\psi)^{1/2}$ and squaring gives

$$q_e(\psi) \leqslant (\eta + 2|\Delta| + \eta^{-1}\Delta^2)\,\|\bar{P}_\Omega\psi\|^2. \tag{3.31}$$

The lower bound. Let $e = 0$. With $\Pi_0 = P_0 P_\Omega$ (recall the notation P_0 from Proposition 3.4) we get the lower bound

(3.32)
$$\begin{aligned} q_0(\psi) &\geqslant \Delta^2\|\bar{P}_\Omega X_\eta^{1/2} I\Pi_0\psi\|^2 + \Delta^2\|\bar{P}_\Omega X_\eta^{1/2} I\bar{P}_0 P_\Omega\psi\|^2 - 2\Delta^2\|\bar{P}_\Omega X_\eta^{1/2} I\Pi_0\psi\|\;\|\bar{P}_\Omega X_\eta^{1/2} I\bar{P}_0 P_\Omega\psi\| \\ &\geqslant \tfrac12\Delta^2\|\bar{P}_\Omega X_\eta^{1/2} I\Pi_0\psi\|^2 - \Delta^2\|\bar{P}_\Omega X_\eta^{1/2} I\bar{P}_0 P_\Omega\psi\|^2 \\ &\geqslant \tfrac12\Delta^2\langle\, \Pi_0\psi \,|\, I\bar{P}_\Omega\, X_\eta I\Pi_0\psi\,\rangle - \eta^{-1}\Delta^2\|\bar{P}_0 P_\Omega\psi\|^2. \end{aligned}$$

We link the first term on the right side to the level shift operator Λ_0 given in Proposition 3.5. Recalling that $X_\eta = \mathrm{Im}(\mathcal{L}_0 - i\eta)^{-1}$ and $\bar{P}_\Omega = \bar{\Pi}_0 + \bar{P}_0 P_\Omega$ we see that

$$\|\Pi_0 I\bar{P}_\Omega X_\eta I\Pi_0 - \mathrm{Im}\,\Pi_0 I\bar{\Pi}_0(\mathcal{L}_0 - i\eta)^{-1} I\Pi_0\| \leqslant \eta\varepsilon^{-2}, \tag{3.33}$$

since $\|\mathrm{Im}\,\Pi_0 I\bar{P}_0 P_\Omega(\mathcal{L}_0 - i\eta)^{-1} I\Pi_0\| \leqslant \|\mathrm{Im}\,\bar{P}_0(\mathcal{L}_S - i\eta)^{-1}\| \leqslant \eta\varepsilon^{-2}$, where ε is the gap in the spectrum of $\mathcal{L}_S$. It follows from (3.33) and the definition of Λ_0 given in Proposition 3.5 that $\mathrm{Im}\,\Lambda_0 = \lim_{\eta\to 0_+}\Pi_0 I\bar{P}_\Omega X_\eta I\Pi_0$. The convergence speed is estimated in Lemma 4.5, (4.17). Namely,

$$\left|\langle\, \Pi_0\psi \,|\, I\bar{P}_\Omega X_\eta I\Pi_0\psi\,\rangle - \mathrm{Im}\,\langle\, \psi \,|\, \Lambda_0\psi\,\rangle\right| \leqslant c\eta^{1/3}\|(1+\bar{A}^2)^{1/2}\bar{P}_\Omega I\Pi_0\psi\|^2 \equiv c_5\eta^{1/3},$$

where c_5 does not depend on ψ (which is normalized). Combining the last bound with (3.32) and with

$$\mathrm{Im}\,\Lambda_0 = \Delta^{-2}\tau^{-1}\Pi_0(1-|\psi_{S,\beta}\rangle\langle\psi_{S,\beta}|) = \Delta^{-2}\tau^{-1}(\Pi_0 - |\psi_{0,\mathrm{KMS}}\rangle\langle\psi_{0,\mathrm{KMS}}|)$$

(see Proposition 3.5 and where $\psi_{0,\mathrm{KMS}} = \psi_{S,\beta}\otimes\Omega$ is the unperturbed KMS state, (2.7)), we obtain

$$q_0(\psi) \geqslant \tfrac{1}{2}\tau^{-1}\big(\|\Pi_0\psi\|^2 - |\langle\,\psi_{0,\mathrm{KMS}}\,|\,\psi\,\rangle|^2\big) - \tfrac{1}{2}c_5\Delta^2\eta^{1/3} - \Delta^2\eta^{-1}\|\bar{P}_0P_\Omega\psi\|^2. \tag{3.34}$$

We further decompose $\|\Pi_0\psi\|^2 = \|\psi\|^2 - \|\bar{P}_\Omega\psi\|^2 - \|\bar{P}_0P_\Omega\psi\|^2$. Since ψ is normalized, we arrive at the lower bound

$$q_0(\psi) \geqslant \tfrac{1}{2}\tau^{-1}(1-\|\bar{P}_\Omega\psi\|^2-\|\bar{P}_0P_\Omega\psi\|^2-|\langle\,\psi_{0,\mathrm{KMS}}\,|\,\psi\,\rangle|^2)-\tfrac{1}{2}c_5\Delta^2\eta^{1/3}-\Delta^2\eta^{-1}\|\bar{P}_0P_\Omega\psi\|^2. \tag{3.35}$$

The contradiction. τ^{-1} is proportional to Δ^2, see (1.7). We write $\tau^{-1} = \Delta^2\tau_0^{-1}$, with $\tau_0 < \infty$ independent of Δ. Combining the bounds (3.31) and (3.35) and dividing by Δ^2 gives

$$\begin{aligned}\tfrac{1}{2}\tau_0^{-1} \;\leqslant\;& \tfrac{1}{2}\tau_0^{-1}|\langle\,\psi_{0,\mathrm{KMS}}\,|\,\psi\,\rangle|^2 + [\tfrac{1}{2}\Delta^2\tau_0^{-1} + \eta + 2|\Delta| + \eta^{-1}\Delta^2]\ \Delta^{-2}\|\bar{P}_\Omega\psi\|^2\\ &+[\tfrac{1}{2}\Delta^2\tau_0^{-1} + \eta^{-1}\Delta^2]\ \Delta^{-2}\|\bar{P}_0P_\Omega\psi\|^2 + \tfrac{1}{2}c_5\eta^{1/3}.\end{aligned} \tag{3.36}$$

Suppose that $\mathcal{L}\psi = 0$, $\|\psi\| = 1$ and $\psi\perp\psi_{\mathrm{KMS}}$, where ψ_{KMS} is given in (2.8). Then

$$|\langle\,\psi_{0,\mathrm{KMS}}\,|\,\psi\,\rangle| = |\langle\,\psi_{0,\mathrm{KMS}}-\psi_{\mathrm{KMS}}\,|\,\psi\,\rangle| \leqslant \|\psi_{\mathrm{KMS}}-\psi_{0,\mathrm{KMS}}\| \leqslant |\Delta|c_{\mathrm{KMS}}. \tag{3.37}$$

Here, an upper bound on c_{KMS} is readily obtained by estimating the power series expansion in Δ which relates ψ_{KMS} and $\psi_{0,\mathrm{KMS}}$, see e.g. [2] (c_{KMS} is proportional to β, the inverse temperature). Choosing $\eta = |\Delta|^{3/2}$ and using the bound (3.37) together with (3.20) and (3.22) in (3.36) gives

$$\tfrac{1}{2}\tau_0^{-1} \leqslant \Delta^2\ \tfrac{1}{2}\tau_0^{-1}(c_{\mathrm{KMS}}^2 + c_3^2 + 4/\varepsilon^2) + |\Delta|^{3/2}c_3^2 + 2|\Delta|c_3^2 + |\Delta|^{1/2}(c_3^2 + 4/\varepsilon^2 + \tfrac{1}{2}c_5).$$

The latter inequality is violated for $|\Delta| < \Delta_0$, where

$$\Delta_0 := \min\big\{1, \big[c_{\mathrm{KMS}}^2 + c_3^2 + 4/\varepsilon^2 + 2\tau_0(4c_3^2 + 4/\varepsilon^2 + c_5/2)\big]^{-2}\big\}. \tag{3.38}$$

This shows that $\mathcal{L}$ has a simple kernel if $|\Delta| < \Delta_0$.

To complete the proof one can proceed in two ways. One can either adapt the above argument to show directly instability of all nonzero eigenvalues of $\mathcal{L}_0$ under the perturbation ΔI. Or one can invoke a general result, saying that if $\mathcal{L}$ has a simple kernel, then it does not have any nonzero eigenvalues [23]. □

4. Proofs: Absolutely continuous Spectrum of $\mathcal{L}$

To show that $\mathrm{spec}(\mathcal{L}) = \mathbb{R}$, we can use the Weyl criterion (see e.g. [20], Theorem 5.10): $s\in\mathbb{R}$ is in the spectrum of $\mathcal{L}$ if and only if there is a sequence $\{\psi_n\}_n$ of normalized vectors in the domain of $\mathcal{L}$, satisfying $\lim_{n\to\infty}\|\mathcal{L}\psi_n - s\psi_n\| = 0$. An explicit choice of ψ_n, for any $s\in\mathbb{R}$, is $\psi_n \propto a^*(f_n)\psi_{\mathrm{KMS}}$. Here, ψ_{KMS} is given in (2.8) and $f_n(u,\Sigma) = \sqrt{n/8\pi}\,\mathbb{1}_{[s-1/n,s+1/n]}(u)$.

We now show *absolute continuity.* The spectrum of $\mathcal{L}$ in an interval $(a,b)\subset\mathbb{R}$ is purely absolutely continuous provided that for each vector φ in some dense set, there is a constant

$C(\varphi) < \infty$ such that

$$\liminf_{\epsilon\downarrow 0} \sup_{x\in(a,b)} \langle\, \varphi \,|\, \mathrm{Im}(\mathcal{L} - x - i\epsilon)^{-1}\varphi \,\rangle \le C(\varphi). \tag{4.1}$$

See for instance Proposition 4.1 of [9]. In order to control the boundary values of the resolvent, we expand it using the *Feshbach* map in (4.3) below. For an operator X acting on $\mathcal{H}$ we denote by $\bar{X} = \bar{P}X\bar{P}\restriction_{\mathrm{Ran}\bar{P}}$ its restriction to the range of $\bar{P} = \mathbb{1} - P$, where $P = \mathbb{1}_{\mathbb{C}^2\otimes\mathbb{C}^2} \otimes |\Omega\rangle\langle\Omega|$ and Ω is the vacuum of (1.12). For $z \in \mathbb{C}$ with $\mathrm{Im}z > 0$, we define

$$\mathfrak{F}(z) = P\big(\mathcal{L} - z - \Delta^2 I\bar{P}(\bar{\mathcal{L}} - z)^{-1}\bar{P}I\big)P, \tag{4.2}$$

which we view as an operator on the range of P. The resolvent and reduced resolvent are related by

$$\begin{aligned}(\mathcal{L} - z)^{-1} \;&=\; (\bar{\mathcal{L}} - z)^{-1} + \mathfrak{F}(z)^{-1} + (\bar{\mathcal{L}} - z)^{-1}\bar{P}\mathcal{L}P\mathfrak{F}(z)^{-1}P\mathcal{L}\bar{P}(\mathcal{L} - z)^{-1}\\ &\quad -\mathfrak{F}(z)^{-1}P\mathcal{L}\bar{P}(\bar{\mathcal{L}} - z)^{-1} - (\bar{\mathcal{L}} - z)^{-1}\bar{P}\mathcal{L}P\mathfrak{F}(z)^{-1}.\end{aligned} \tag{4.3}$$

Here, $(\bar{\mathcal{L}} - z)^{-1}$ is interpreted as an operator on $\mathrm{Ran}\bar{P}$. We have $P(\mathcal{L} - z)^{-1}P = \mathfrak{F}(z)^{-1}$.

We introduce the family of norms

$$\|\varphi\|_\kappa = \|\bar{N}^{-1/2}(1 + \bar{A}^2)^{\kappa/2}\varphi\|, \quad \varphi \in \bar{P}\mathcal{H},\ \kappa \geqslant 0. \tag{4.4}$$

Theorem 4.1. *Let $\varphi, \psi \in \mathrm{dom}(\bar{A}^2) \cap \mathrm{dom}(\bar{N}^{1/2})$, where $A = d\Gamma(-i\partial_u)$. Then*

$$\big|\partial_z\langle\, \varphi \,|\, (\bar{\mathcal{L}} - z)^{-1}\psi \,\rangle\big| \leqslant C\big(\|\varphi\|_2 + \|\bar{N}\varphi\|_1\big)\big(\|\psi\|_2 + \|\bar{N}\psi\|_1\big),$$

where C is independent of $z \in \mathbb{C}_+$ and φ, ψ.

Theorem 4.1, together with the fact that $\mathrm{Ran}IP \subset \mathrm{dom}(A^2) \cap \mathrm{dom}(N^{1/2})$, implies that $\mathfrak{F}(z)$, (4.2), extends continuously to $\mathrm{Im}z \geqslant 0$ (as a function with values in the operators on $\mathrm{Ran}P$). We denote its value for $x \in \mathbb{R}$ by $\mathfrak{F}(x)$.

Theorem 4.2. *For any real $x_0 \neq 0$ there exist $r(x_0) > 0$ and $c(x_0) < \infty$ such that*

$$\|\mathfrak{F}(x)^{-1}\| \leqslant c(x_0) \quad \textit{for all } x \textit{ such that } \quad |x - x_0| < r(x_0). \tag{4.5}$$

Theorems 4.1 and 4.2, together with the expansion (4.3), show that (4.1) is satisfied for all (a, b) not containing the origin. This means that the absolutely continuous spectrum of $\mathcal{L}$ (which is a closed set) is $\mathbb{R}$ and that the singular continuous spectrum is empty.

4.1. **Proof of Theorem 4.1.** Let a and b be expressions depending on the quantities $z \in \mathbb{C}_+$, $\eta > 0$, $\Delta \in \mathbb{R}$. We use the notation

$$a \prec b \tag{4.6}$$

to mean that there is a constant c which does not depend on any of the above quantities, such that $a \leqslant cb$. We introduce a regularization $\bar{\mathcal{L}}(\eta)$, $\eta > 0$, defined as follows. The domain is $\mathrm{dom}(\bar{\mathcal{L}}(\eta)) = \mathrm{dom}(\bar{\mathcal{L}}_0) \cap \mathrm{dom}(\bar{N})$ and

$$\bar{\mathcal{L}}(\eta) = \bar{\mathcal{L}}_0 - i\eta\bar{N} + \Delta\bar{I}(\eta), \qquad \text{with} \qquad \bar{I}(\eta) = (2\pi)^{-1/2}\int_{\mathbb{R}} \widehat{f}(s)\tau_{\eta s}(\bar{I})ds, \tag{4.7}$$

and where $\tau_t(X) = e^{it\bar{A}}Xe^{-it\bar{A}}$. Here, f is a Schwartz function satisfying $f(0) = f'(0) = f''(0) = f'''(0) = 1$. In terms of the Fourier transform, this means

$$(2\pi)^{-1/2}\int_{\mathbb{R}}(is)^k\widehat{f}(s)ds = 1, \qquad \text{for } k = 0,1,2,3. \tag{4.8}$$

We show in the next result some properties of $\bar{\mathcal{L}}(\eta)$. In particular, $\bar{\mathcal{L}}(\eta) - z$ is invertible for $\eta > 0$, $\mathrm{Im}z > -\eta/2$. We denote the resolvent by

$$R_z(\eta) := (\bar{\mathcal{L}}(\eta) - z)^{-1}.$$

Lemma 4.3. *There is a $\Delta_0 > 0$, independent of $\eta > 0$, such that for $|\Delta| < \Delta_0$:*

(1) $-2\eta\bar{N} \leqslant \mathrm{Im}\bar{\mathcal{L}}(\eta) \leqslant -\frac{\eta}{2}\bar{N}$. *In particular, any z with $\mathrm{Im}z > -\eta/2$ is in the resolvent set of $\bar{\mathcal{L}}(\eta)$. Moreover, for such z, $\|(\bar{\mathcal{L}}(\eta) - z)^{-1}\| \leqslant (\eta/2 + \mathrm{Im}z)^{-1}$.*

(2) *For $\eta > 0$ and $\mathrm{Im}z > -\eta/2$, we have $\mathrm{Ran}R_z(\eta) \subset \big(\mathrm{dom}(\bar{N}) \cap \mathrm{dom}(\bar{\mathcal{L}}_0)\big)$ and $R_z(\eta)$ leaves $\mathrm{dom}(\bar{A})$ invariant.*

(3) *For all $\psi \in \bar{P}\mathcal{H}$ and all $z \in \mathbb{C}_+$, we have $\lim_{\eta\to 0_+} R_z(\eta)\psi = (\bar{\mathcal{L}} - z)^{-1}\psi$.*

(4) *For all $\psi \in \bar{P}\mathcal{H}$ and all $z \in \mathbb{C}_+$, we have $\|\bar{N}^{1/2}R_z(\eta)\psi\| \leqslant \sqrt{2}\eta^{-1/2}\,|\langle\,\psi\,|\,R_z(\eta)\psi\,\rangle|^{1/2}$ and $\|\bar{N}^{1/2}R_z(\eta)\psi\| \leqslant 2\eta^{-1}\|\bar{N}^{-1/2}\psi\|$. The same estimates hold for $R_z(\eta)$ replaced by $R_z(\eta)^*$.*

Proof of Lemma 4.3. (1) Using (4.8) with $k = 0$ we can write

$$\mathrm{Im}\bar{\mathcal{L}}(\eta) = -\eta\bar{N}^{1/2}\Big(\mathbb{1} + \frac{\Delta}{\sqrt{2\pi}\,\eta}\mathrm{Im}\int_{\mathbb{R}}\widehat{f}(s)\bar{N}^{-1/2}(\tau_{\eta s}(\bar{I}) - \bar{I})\bar{N}^{-1/2}ds\Big)\bar{N}^{1/2}. \tag{4.9}$$

Now $\tau_{\eta s}(\bar{I}) - \bar{I} = \int_0^{\eta s}\partial_{s'}\tau_{s'}(\bar{I})ds' = i\int_0^{\eta s}\tau_{s'}([\bar{A},\bar{I}])ds'$. Thus we have $\|\bar{N}^{-1/2}(\tau_{\eta s}(\bar{I}) - \bar{I})\bar{N}^{-1/2}\| \leqslant \eta|s|\,\|\bar{N}^{-1/2}[A,I]\bar{N}^{-1/2}\|$. The expression (3.14) shows that the latter norm is bounded above by a constant C'. Therefore, (4.9) implies $-\eta(1 + C|\Delta|)\bar{N} \leqslant \mathrm{Im}\bar{\mathcal{L}}(\eta) \leqslant -\eta(1 - C|\Delta|)\bar{N}$ for Δ small and where $C = C'(2\pi)^{-1/2}\int_{\mathbb{R}}|s\widehat{f}(s)|ds$. This gives the bound on $\mathrm{Im}\bar{\mathcal{L}}(\eta)$. Now

$$\|\psi\|\,\|(\bar{\mathcal{L}}(\eta) - z)\psi\| \geqslant |\langle\,\psi\,|\,(\bar{\mathcal{L}}(\eta) - z)\psi\,\rangle| \geqslant |\mathrm{Im}\langle\,\psi\,|\,(\bar{\mathcal{L}}(\eta) - z)\psi\,\rangle| \geqslant (\eta/2 + \mathrm{Im}z)\|\psi\|^2. \tag{4.10}$$

In the same way $\|(\bar{\mathcal{L}}(\eta) - z)^*\psi\| \geqslant (\eta/2 + \mathrm{Im}z)\|\psi\|$. For $\mathrm{Im}z > -\eta/2$, $(\bar{\mathcal{L}}(\eta) - z)^*$ has trivial kernel and so $\mathrm{Ran}(\bar{\mathcal{L}}(\eta) - z)$ is dense. However, due to (4.10) and since $\bar{\mathcal{L}}(\eta) - z$ is a closed operator, $\mathrm{Ran}(\bar{\mathcal{L}}(\eta) - z)$ is also closed, so it is all of $\bar{P}\mathcal{H}$. Therefore, the inverse of $\bar{\mathcal{L}}(\eta) - z$ is defined on the whole space and by the closed graph theorem it is bounded. The bound is obtained from (4.10). This shows (1).

To prove the first part of (2), note that $(\bar{\mathcal{L}}_0 - i\eta\bar{N} - i)R_z(\eta) = \mathbb{1} + (-\Delta I(\eta) + z - i)R_z(\eta)$ is bounded. Hence $\bar{\mathcal{L}}_0R_z(\eta) = \bar{\mathcal{L}}_0(\bar{\mathcal{L}}_0 - i\eta\bar{N} - i)^{-1}(\bar{\mathcal{L}}_0 - i\eta\bar{N} - i)R_z(\eta)$ is bounded as well. In the same way, $\bar{N}R_z(\eta)$ is bounded. It remains to show that $R_z(\eta)$ leaves $\mathrm{dom}(\bar{A})$ invariant. For this, it suffices to prove that the derivative $\partial_t|_{t=0}$ of

$$e^{itA}R_z(\eta)\psi = \Big(\bar{\mathcal{L}}_0 + (t - i\eta)\bar{N} + \Delta\tau_t(\bar{I}(\eta)) - z\Big)^{-1}e^{itA}\psi$$

exists, for any $\psi \in \mathrm{dom}(\bar{A})$. One only needs to check that the derivative of the resolvent, at $t = 0$, is bounded. This can be done easily by writing the derivative as the limit of the difference quotient and using the second resolvent equation for the numerator of the quotient. (2) follows.

(3) It suffices to show the result for any single, fixed z_0 in the upper half plane, e.g. $z_0 = i$. This fact is seen by proceeding as in the proof of Theorem VIII.19 of [30], by expanding the resolvents in a power series in $z - z_0$. (Note that in the above reference, only self-adjoint operators are considered, but all that counts in the argument is the bound on the resolvent which we have established in point (1) of the present lemma). Let us show the result for $z = i$ now. First we note that $(\bar{\mathcal{L}} - i)^{-1}$ leaves $\mathrm{dom}(\bar{N})$ invariant. A proof of this is obtained by expanding $(\bar{\mathcal{L}} - i)^{-1}$ into its Neumann series (in powers of Δ) and using that $\bar{N}$ commutes with $(\bar{\mathcal{L}}_0 - i)^{-1}$ and $\bar{N}^{-1}\bar{I}\bar{N}$ is bounded, so that $\bar{N}^{-1}(\bar{\mathcal{L}} - i)^{-1}\bar{N}$ is bounded. Therefore, for $\psi \in \mathrm{dom}(\bar{N})$,

$$\big((\bar{\mathcal{L}}(\eta) - i)^{-1} - (\bar{\mathcal{L}} - i)^{-1}\big)\psi = (\bar{\mathcal{L}}(\eta) - i)^{-1}\big(i\eta\bar{N} - \Delta\bar{I}(\eta) + \Delta\bar{I}\big)(\bar{\mathcal{L}} - i)^{-1}\psi \to 0, \tag{4.11}$$

as $\eta \to 0_+$. Finally, since $\mathrm{dom}(\bar{N})$ is dense in $\bar{P}\mathcal{H}$ and $\|(\bar{\mathcal{L}}(\eta) - i)^{-1} - (\bar{\mathcal{L}} - i)^{-1}\| \leqslant 2$, (4.11) is valid for all $\psi \in \bar{P}\mathcal{H}$. This proves (3).

(4) Due to (1), we have $\bar{N} \leqslant -2\eta^{-1}\mathrm{Im}\bar{\mathcal{L}}(\eta)$, so

$$\begin{aligned}\|\bar{N}^{1/2}R_z(\eta)\psi\|^2 &\leqslant 2\eta^{-1}\,|\langle\, R_z(\eta)\psi \,|\, \mathrm{Im}\,(\bar{\mathcal{L}}(\eta) - z)R_z(\eta)\psi\,\rangle| \\ &\leqslant 2\eta^{-1}\,|\langle\, \psi \,|\, R_z(\eta)\psi\,\rangle| \leqslant 2\eta^{-1}\,\|\bar{N}^{1/2}R_z(\eta)\psi\|\,\|\bar{N}^{-1/2}\psi\|.\end{aligned}$$

The estimate for $R_z(\eta)$ replaced by $R_z(\eta)^*$ is obtained in the same way. This shows (4) and concludes the proof of Lemma 4.3. □

The operator

$$K(\eta) := [\bar{A}, \bar{I}(\eta)] - \partial_\eta\bar{I}(\eta) = (2\pi)^{-1/2}\int_{\mathbb{R}}(1 - is)\widehat{f}(s)\tau_{\eta s}([\bar{A}, \bar{I}])ds, \tag{4.12}$$

defined on $\mathrm{dom}(\bar{N}^{1/2})$, has the following properties.

Lemma 4.4. *Let $\varphi, \psi \in \bar{P}\mathcal{H}$.*

(a) Assume $\||\partial_u|^\alpha f_\beta\| < \infty$ for some $1 \leqslant \alpha \leqslant 2$. Then

$$|\langle\, \varphi \,|\, K(\eta)\, \psi\,\rangle| \leqslant c\,\eta^{\alpha-1}\,\|\bar{N}^{1/2}\varphi\|\,\|\bar{N}^{1/2}\psi\|. \tag{4.13}$$

(b) Assume $\||\partial_u|^\alpha f_\beta\| < \infty$ for some $2 \leqslant \alpha \leqslant 3$. Then

$$|\langle\, \varphi \,|\, K(\eta)\, \psi\,\rangle| \leqslant c\,\eta^3\,\|\bar{N}^{3/2}\varphi\|\,\|\bar{N}^{1/2}\psi\| + c\,\eta^{\alpha-1}\,\|\bar{N}^{1/2}\varphi\|\,\|\bar{N}^{1/2}\psi\|. \tag{4.14}$$

The constant c does not depend on η. Both (a) and (b) hold if $K(\eta)$ is replaced by $K(\eta)^*$.

Denote

$$G_{\varphi,\psi,z}(\eta) := \langle\, \varphi \,|\, (\bar{\mathcal{L}}(\eta) - z)^{-1}\psi\,\rangle = \langle\, \varphi \,|\, R_z(\eta)\psi\,\rangle.$$

Lemma 4.5. *Assume $\|\,|\partial_u|^\alpha f_\beta\| < \infty$ for some $\alpha > 1$. There is a constant c independent of $z \in \mathbb{C}_+$, $\eta > 0$ and Δ with $|\Delta| < \Delta_0$, such that, for any $\varphi, \psi \in \mathrm{dom}(\bar{A})$,*

$$|G_{\varphi,\psi,z}(\eta)| \leqslant c\,\|\varphi\|_1\,\|\psi\|_1 \tag{4.15}$$

$$\|\bar{N}^{1/2}R_z(\eta)\varphi\|,\ \|\bar{N}^{1/2}R_z(\eta)^*\varphi\| \leqslant c\,\eta^{-1/2}\,\|\varphi\|_1. \tag{4.16}$$

For any $x \in \mathbb{R}$, $\langle \varphi \,|\, (\bar{\mathcal{L}} - x - iy)^{-1}\psi \rangle$ *has a limit as* $y \to 0_+$, *denoted by* $\langle \varphi \,|\, (\bar{\mathcal{L}} - x - i0_+)^{-1}\psi \rangle$, *and*

$$|\langle \varphi \,|\, (\bar{\mathcal{L}} - x - i0_+)^{-1}\psi \rangle - \langle \varphi \,|\, (\bar{\mathcal{L}} - x - iy)^{-1}\psi \rangle| \leqslant c\, y^{\gamma/(1+\gamma)} \|\psi\|_1 \, \|\varphi\|_1, \tag{4.17}$$

uniformly in $x \in \mathbb{R}$, $y \in (0,1)$ *and with* $\gamma = \min\{1/2, \alpha - 1\}$.

Proof of Lemma 4.5. Using Lemma 4.3(2) we obtain

$$\begin{aligned} \partial_\eta G_{\varphi,\psi,z}(\eta) &= -\langle \varphi \,|\, R_z(\eta) \big[\partial_\eta \bar{\mathcal{L}}(\eta)\big] R_z(\eta)\psi \rangle \\ &= \langle \varphi \,|\, [\bar{A}, R_z(\eta)]\psi \rangle + \Delta \langle \varphi \,|\, R_z(\eta)\, K(\eta)\, R_z(\eta)\psi \rangle, \end{aligned} \tag{4.18}$$

where $K(\eta)$ is defined in (4.12). Using the estimate (4.13) in (4.18) yields (recall the notation (4.6))

$$\begin{aligned} |\partial_\eta G_{\varphi,\psi,z}(\eta)| \prec{}& \|\varphi\|_1 \, \|\bar{N}^{1/2} R_z(\eta)\, \psi\| + \|\psi\|_1 \, \|\bar{N}^{1/2} R_z(\eta)^* \, \varphi\| \\ &+ \eta^{\alpha-1} |\Delta| \, \|\bar{N}^{1/2} R_z(\eta)^* \varphi\| \, \|\bar{N}^{1/2} \, R_z(\eta)\psi\|. \end{aligned} \tag{4.19}$$

Using Lemma 4.3(1) in (4.19) gives

$$|\partial_\eta G_{\varphi,\psi,z}(\eta)| \prec \eta^{-1/2}\big(\|\varphi\|_1^2 + \|\psi\|_1^2\big) + (\eta^{-1/2} + \eta^{\alpha-2})\,\big(|G_{\varphi,\varphi,z}(\eta)| + |G_{\psi,\psi,z}(\eta)|\big). \tag{4.20}$$

By Lemma 4.3(4) we have $\|\bar{N}^{1/2} R_z(1) \bar{N}^{1/2}\| \leqslant 2$ and hence $|G_{\varphi,\psi,z}(1)| \leqslant \|\psi\|_0 \, \|\phi\|_0$. Taking $\varphi = \psi$ in (4.20) gives a differential inequality for $G_{\varphi,\varphi,z}(\eta)$ which implies (4.15) for $\varphi = \psi$ by the standard Gronwall estimate [19]. Combining (4.15) for $\varphi = \psi$ with Lemma 4.3(4) gives (4.16). We can now use (4.16) in (4.19) to obtain

$$|\partial_\eta G_{\varphi,\psi,z}(\eta)| \prec (\eta^{-1/2} + \eta^{\alpha-2}) \|\varphi\|_1 \|\psi\|_1. \tag{4.21}$$

Integrating gives (4.15).

We now prove (4.17). Let $0 < \mu \ll \eta$ and $z, w \in \mathbb{C}_+$. By the triangle inequality,

$$|\langle \varphi \,|\, (R_z(\mu) - R_w(\mu))\psi \rangle| \leqslant |\langle \varphi \,|\, (R_z(\eta) - R_w(\eta))\psi \rangle| + \sum_{v=w,z} |G_{\varphi,\psi,v}(\mu) - G_{\varphi,\psi,v}(\eta)|. \tag{4.22}$$

Using the resolvent identity and (4.16) gives $|\langle \varphi \,|\, (R_z(\eta) - R_w(\eta))\psi \rangle| \prec |z - w|\eta^{-1} \|\psi\|_1 \, \|\varphi\|_1$. Next, it follows from (4.21) and (4.15) that

$$|G_{\varphi,\psi,v}(\mu) - G_{\varphi,\psi,v}(\eta)| \leqslant \int_\mu^\eta \big|\partial_\xi G_{\varphi,\psi,v}(\xi)\big| d\xi \;\prec\; \eta^\gamma \|\psi\|_1 \, \|\varphi\|_1.$$

Therefore,

$$|\langle \varphi \,|\, (R_z(\mu) - R_w(\mu))\psi \rangle| \prec (\eta^{-1}|z - w| + \eta^\gamma)\, \|\psi\|_1 \, \|\varphi\|_1. \tag{4.23}$$

Thanks to Lemma 4.3(3) we may send $\mu \to 0$ and choose $\eta = |w - z|^{1/(1+\gamma)}$ to obtain

$$|\langle \varphi \,|\, (\bar{\mathcal{L}} - z)^{-1}\psi \rangle - \langle \varphi \,|\, (\bar{\mathcal{L}} - w)^{-1}\psi \rangle| \prec |z - w|^{\gamma/(1+\gamma)} \, \|\psi\|_1 \, \|\varphi\|_1. \tag{4.24}$$

This shows the existence of $\lim_{y \to 0+} \langle \varphi \,|\, (\bar{\mathcal{L}} - x - iy)^{-1}\psi \rangle$ and proves (4.17). □

For φ, $\psi \in \mathrm{dom}(\bar{A})$, $z \in \mathbb{C}_+$, $\eta > 0$, we define

$$H_{\varphi,\psi,z}(\eta) := \partial_z \langle \varphi \,|\, R_z(\eta)\psi \rangle = \langle \varphi \,|\, R_z(\eta)^2\psi \rangle. \tag{4.25}$$

Due to Lemma 4.3 and (4.12),

$$\partial_\eta H_{\varphi,\psi,z}(\eta) = \langle \varphi \,|\, [\bar{A}, R_z(\eta)^2]\psi \rangle + S_1 + S_2, \tag{4.26}$$

where $S_1 = \langle \varphi \,|\, R_z(\eta)K(\eta)R_z(\eta)^2\psi \rangle$ and $S_2 = \langle \varphi \,|\, R_z(\eta)^2K(\eta)R_z(\eta)\psi \rangle$. Taking into account $\|\bar{N}^{1/2}R_z(\eta)\| \prec \eta^{-1}$ (see Lemma 4.3(4)), (4.4) and (4.16) we have

$$|\langle \varphi \,|\, [\bar{A}, R_z(\eta)^2]\psi \rangle| \prec \eta^{-3/2}\|\varphi\|_1\,\|\psi\|_1. \tag{4.27}$$

Next, due to (4.14),

$$|S_1| \prec \eta^3\|\bar{N}^{3/2}R_z(\eta)^*\varphi\|\,\|\bar{N}^{1/2}R_z(\eta)^2\psi\| + \eta^{\alpha-1}\|\bar{N}^{1/2}R_z(\eta)^*\varphi\|\,\|\bar{N}^{1/2}R_z(\eta)^2\psi\|.$$

Since $\|\bar{N}^{1/2}R_z(\eta)\bar{N}^{1/2}\| \prec \eta^{-1}$ (Lemma 4.3(4)) and using (4.16), we get

$$|S_1| \prec \eta^{3/2}\,\|\bar{N}^{3/2}R_z(\eta)^*\varphi\|\,\|\psi\|_1 + \eta^{\alpha-3}\,\|\varphi\|_1\,\|\psi\|_1. \tag{4.28}$$

A similar upper bound is obtained for $|S_2|$. We show the following result below.

Lemma 4.6. *Let $l \geq 0$, $\eta > 0$, $z \in \mathbb{C}_+$ and $\psi \in \mathrm{dom}(\bar{A}) \cap \mathrm{dom}(\bar{N}^{l/2-1})$. Then*

$$\|\bar{N}^{l/2}R_z(\eta)\psi\| \leqslant C\big(\|\psi\|_1 + \eta^{1/2}\|\bar{N}^{l/2-1}\psi\|\big)\begin{cases}\eta^{-(l+1)/2}, & l \text{ even}\\ \eta^{-l/2}, & l \text{ odd}\end{cases} \tag{4.29}$$

The same statement holds if $R_z(\eta)$ is replaced by $R_z(\eta)^*$.

To shorten notation we set

$$\|\psi\|_1' := \|\psi\|_1 + \|\bar{N}^{1/2}\psi\|. \tag{4.30}$$

Combining (4.29), for $l = 3$, with (4.28) gives

$$|S_1| + |S_2| \prec \eta^{\alpha-3}\,\|\varphi\|_1'\,\|\psi\|_1'. \tag{4.31}$$

With (4.26) and (4.27) we obtain $|\partial_\eta H_{\varphi,\psi,z}(\eta)| \prec (\eta^{-3/2} + \eta^{\alpha-3})\|\varphi\|_1'\,\|\psi\|_1' \prec \eta^{-3/2}\|\varphi\|_1'\,\|\psi\|_1'$ (as $\alpha \geqslant 2$). We integrate this estimate and obtain

$$|H_{\varphi,\psi,z}(\eta)| \prec \eta^{-1/2}\|\varphi\|_1'\,\|\psi\|_1'. \tag{4.32}$$

Finally, we consider again (4.26), but this time we write $\langle \varphi \,|\, [\bar{A}, R_z(\eta)^2]\psi \rangle = H_{\bar{A}\varphi,\psi,z}(\eta) - H_{\varphi,\bar{A}\psi,z}(\eta)$. Then, due to (4.32),

$$|\langle \varphi \,|\, [\bar{A}, R_z(\eta)^2]\psi \rangle| \prec \eta^{-1/2}\|(1+\bar{A}^2)^{1/2}\varphi\|_1'\,\|(1+\bar{A}^2)^{1/2}\psi\|_1'. \tag{4.33}$$

According to (4.4) and (4.30), $\|(1+\bar{A}^2)^{1/2}\varphi\|_1' = \|\varphi\|_2 + \|\bar{N}\varphi\|_1$. We use the improved bound (4.33), together with (4.31), in (4.26) to obtain $|\partial_\eta H_{\varphi,\psi,z}(\eta)| \prec (\eta^{-1/2} + \eta^{\alpha-3})(\|\varphi\|_2 + \|\bar{N}\varphi\|_1)(\|\psi\|_2 + \|\bar{N}\psi\|_1)$. Integration gives the Hölder continuity

$$|H_{\varphi,\psi,z}(\eta) - H_{\varphi,\psi,z}(\eta')| \prec |\eta - \eta'|^{\min\{1/2,\alpha-2\}}\,(\|\varphi\|_2 + \|\bar{N}\varphi\|_1)(\|\psi\|_2 + \|\bar{N}\psi\|_1).$$

It follows that $H_{\varphi,\psi,z}(\eta)$ extends continuously to $\eta = 0$, the extension satisfying $|H_{\varphi,\psi,z}(0)| \leqslant C(\|\varphi\|_2 + \|\bar{N}\varphi\|_1)(\|\psi\|_2 + \|\bar{N}\psi\|_1)$, with C independent of φ, ψ and $z \in \mathbb{C}_+$. By Lemma 4.3(3), the extension is $H_{\varphi,\psi,z}(0) = \langle \varphi \,|\, (\bar{\mathcal{L}} - z)^{-2}\psi \rangle = \partial_z \langle \varphi \,|\, (\bar{\mathcal{L}} - z)^{-1}\psi \rangle$. This concludes the proof of Theorem 4.1, modulo the proofs of Lemmas 4.4 and 4.6, which we give now.

Proof of Lemma 4.4. Due to the definition (4.12) of $K(\eta)$ and the expression (2.3) for I, it is enough to show the estimates (a) and (b) for $\tau_{\eta s}([\bar{A}, \bar{I}])$ in (4.12) replaced by $\tau_{\eta s}([A, W(g)]) = W(e^{\eta s \partial_u} g)\big(\phi(e^{\eta s \partial_u} g') - \frac{i}{2}\langle g \,|\, g' \rangle\big)$, where $\| |\partial_u|^\alpha g\| < \infty$ (see also (3.5)). Hence it suffices to show the bounds (a) and (b) for $\langle \varphi \,|\, \widetilde{K}(\eta)\psi \rangle$, where

$$\widetilde{K}(\eta) = \int_{\mathbb{R}} W(e^{\eta s \partial_u} g)\, \big(\phi(e^{\eta s \partial_u} g') - \tfrac{i}{2}\langle g \,|\, g' \rangle\big) d\mu(s), \tag{4.34}$$

with $d\mu(s) = (2\pi)^{-1/2}(1 - is)\widehat{f}(s)ds$. By (4.8), $\widetilde{K}(0) = 0$, so the value of the integral (4.34) stays the same if we replace the integrand by

$$\mathcal{I} = W(e^{\eta s \partial_u} g)\, \big(\phi(e^{\eta s \partial_u} g') - \tfrac{i}{2}\langle g \,|\, g' \rangle\big) - W(g)\, \big(\phi(g') - \tfrac{i}{2}\langle g \,|\, g' \rangle\big).$$

Proof of (a). We write

$$\mathcal{I} = \big(W(e^{\eta s \partial_u} g) - W(g)\big)\, \big(\phi(e^{\eta s \partial_u} g') - \tfrac{i}{2}\langle g \,|\, g' \rangle\big) + W(g)\ \phi(e^{\eta s \partial_u} g' - g') \tag{4.35}$$

and estimate

$$\|(W(e^{\eta s \partial_u} g)^* - W(g)^*)\varphi\| \leqslant \int_0^{\eta s} \big\|\partial_t W(-e^{t \partial_u} g)\varphi\big\| dt \prec \eta |s| \|\bar{N}^{1/2}\varphi\|. \tag{4.36}$$

The last bound is obtained from $\partial_t W(-e^{t\partial_u} g) = i[A, W(-e^{t\partial_u} g)]$ and an application of (3.5) (with $D = -i\partial_u$). It follows that

$$\big|\langle \varphi \,|\, \big(W(e^{\eta s \partial_u} g) - W(g)\big)\, \big(\phi(e^{\eta s \partial_u} g') - \tfrac{i}{2}\langle g \,|\, g' \rangle\big)\psi \rangle\big| \prec \eta |s| \|\bar{N}^{1/2}\varphi\|\, \|\bar{N}^{1/2}\psi\|. \tag{4.37}$$

Next we consider the remaining term in (4.35). By the spectral theorem,

$$\|\phi(e^{\eta s \partial_u} g' - g')\psi\| \prec \|e^{\eta s \partial_u} g' - g'\|\, \|\bar{N}^{1/2}\psi\| \prec (|s|\eta)^\gamma \sup_{r \neq 0} \tfrac{|e^{ir} - 1|}{|r|^\gamma}\, \| |\partial_u|^\gamma g'\|\, \|\bar{N}^{1/2}\psi\|$$

and thus for any $\gamma \in [0, 1]$, if $\|\, |\partial_u|^{1+\gamma} g\| < \infty$, then

$$\|\phi(e^{\eta s \partial_u} g' - g')\psi\| \prec \eta^\gamma\, |s|^\gamma\ \|\bar{N}^{1/2}\psi\|. \tag{4.38}$$

It follows that $|\langle \varphi \,|\, W(g)\phi(e^{\eta s \partial_u} g' - g')\psi \rangle| \prec \eta^\gamma |s|^\gamma \|\varphi\|\, \|\bar{N}^{1/2}\psi\|$. Combining this last estimate with (4.37) yields $|\langle \varphi \,|\, \widetilde{K}(\eta)\psi \rangle| \prec (\eta + \eta^\gamma)\|\bar{N}^{1/2}\varphi\|\, \|\bar{N}^{1/2}\psi\|$, which proves (4.13).

Proof of (b). We write $\mathcal{I} = T_1 + T_2 + T_3$, with

$$T_1 = \big(W(e^{\eta s \partial_u} g) - W(g)\big)\big(\phi(e^{\eta s \partial_u} g') - \phi(g')\big), \tag{4.39}$$

$$T_2 = \big(W(e^{\eta s \partial_u} g) - W(g)\big)\big(\phi(g') - \tfrac{i}{2}\langle g \,|\, g' \rangle\big), \tag{4.40}$$

$$T_3 = W(g)\big(\phi(e^{\eta s \partial_u} g') - \phi(g')\big). \tag{4.41}$$

Using the bounds (4.36) and (4.38) with $\gamma = 1$ gives $|\langle \varphi \,|\, T_1\psi \rangle| \prec \eta^2 s^2 \|\bar{N}^{1/2}\varphi\|\, \|\bar{N}^{1/2}\psi\|$, so

$$\left|\int_{\mathbb{R}} \langle \varphi \,|\, T_1\psi \rangle d\mu(s)\right| \prec \eta^2 \|\bar{N}^{1/2}\varphi\|\, \|\bar{N}^{1/2}\psi\|. \tag{4.42}$$

Next, since $W(e^{\eta s\partial_u}g)-W(g)=\int_0^{\eta s}\partial_t W(e^{t\partial_u}g)dt=\int_0^{\eta s}W(e^{t\partial_u}g)\{i\phi(e^{t\partial_u}g')+\frac{1}{2}\langle\, g\,|\, g'\,\rangle\}dt$, we obtain

$$\int_{\mathbb{R}}\langle\,\varphi\,|\,T_2\psi\,\rangle d\mu(s)=\int_{\mathbb{R}}d\mu(s)\int_0^{\eta s}dt\;\langle\,(T_2'+T_2'')\varphi\,|\,(\phi(g')-\tfrac{i}{2}\langle\, g\,|\, g'\,\rangle)\psi\,\rangle, \tag{4.43}$$

where

$$T_2' \;=\; W(-e^{t\partial_u}g)\{-i\phi(e^{t\partial_u}g')+i\phi(g')\}, \tag{4.44}$$

$$T_2'' \;=\; \big(W(-e^{t\partial_u}g)-W(-g)\big)\big(-i\phi(g')+\tfrac{1}{2}\langle\, g\,|\, g'\,\rangle\big). \tag{4.45}$$

Note that due to (4.8), $\int_{\mathbb{R}}d\mu(s)\int_0^{\eta s}dt\langle\, W(-g)(-i\phi(g')+\frac{1}{2}\langle\, g\,|\, g'\,\rangle)\varphi\,|\,(\phi(g')-\frac{i}{2}\langle\, g\,|\, g'\,\rangle)\psi\,\rangle=0$. In order to estimate the contribution to (4.43) coming from (4.44), we use (4.38) with $\gamma=1$. The contribution of T_2' to (4.43) is $\prec \eta^2\|\bar N^{1/2}\varphi\|\,\|\bar N^{1/2}\psi\|$. Next we consider the contribution to (4.43) coming from (4.45). The double integral in (4.43) stays unchanged if we replace T_2'' by

$$T_2'''=\int_0^t\big(\partial_r W(-e^{r\partial_u}g)-\partial_r|_{r=0}W(-e^{r\partial_u}g)\big)\big(-i\phi(g')+\tfrac{1}{2}\langle\, g\,|\, g'\,\rangle\big)dr,$$

since again, due to (4.8), the term containing $\partial_r|_{r=0}W(-e^{r\partial_u}g)$ vanishes. As $\partial_r W(-e^{r\partial_u}g)-\partial_r|_{r=0}W(-e^{r\partial_u}g)=\int_0^r\partial_x^2 W(-e^{x\partial_u}g)dx$ and $\partial_x^2W(-e^{x\partial_u}g)=-\tau_x([A,[A,W(-g)]])$, which is an operator of the form $\tau_x(W(-g)P)$, where P is a polynomial of degree two in field operators ($\phi(g')$ and $\phi(g'')$), we obtain $\|T_2'''\varphi\|\prec t^2\|\bar N^{3/2}\varphi\|$. It follows that the contribution to (4.43) coming from T_2' is $\prec\eta^3\|\bar N^{3/2}\varphi\|\,\|\bar N^{1/2}\psi\|$. Hence

$$\left|\int_{\mathbb{R}}\langle\,\varphi\,|\,T_2\psi\,\rangle d\mu(s)\right|\prec\eta^2\|\bar N^{1/2}\varphi\|\,\|\bar N^{1/2}\psi\|+\eta^3\|\bar N^{3/2}\varphi\|\,\|\bar N^{1/2}\psi\|. \tag{4.46}$$

Up to now, only two derivatives of g are assumed to exist. Finally we estimate the term with T_3,

$$\int_{\mathbb{R}}d\mu(s)\int_0^{\eta s}dt\;\langle\, W(-g)\varphi\,|\,\big(\partial_t\phi(e^{t\partial_u}g')-\partial_t|_{t=0}\phi(e^{t\partial_u}g')\big)\psi\,\rangle \tag{4.47}$$

where we inserted the term containing $\partial_t|_{t=0}\phi(e^{t\partial_u}g')$ for free, once again due to (4.8). Since $\partial_t\phi(e^{t\partial_u}g')-\partial_t|_{t=0}\phi(e^{t\partial_u}g')=\phi(e^{t\partial_u}g''-g'')$ we can apply the estimate (4.38) (with g' replaced by g'' and ηs replaced by t) to obtain

$$\left|\int_{\mathbb{R}}\langle\,\varphi\,|\,T_3\psi\,\rangle d\mu(s)\right|\prec\eta^{\gamma+1}\|\varphi\|\,\|\bar N^{1/2}\psi\|, \tag{4.48}$$

for any $\gamma\in[0,1]$ and provided $\|\,|\partial_u|^{\gamma+2}g\|<\infty$. Combining (4.42), (4.46) and (4.48) yields the bound (4.14). The estimate for $K(\eta)^*$ is obtained in the same way. □

Proof of Lemma 4.6. For $l\geqslant 0$ we have

$$\bar N^{l/2}R_z(\eta)\psi=\bar N^{1/2}R_z(\eta)\bar N^{1/2}\bar N^{(l-2)/2}\psi+\bar N^{1/2}[\bar N^{(l-1)/2},R_z(\eta)]\psi. \tag{4.49}$$

The second term on the right side is

$$
\begin{aligned}
&\bar{N}^{1/2}[\bar{N}^{(l-1)/2}, R_z(\eta)]\psi = \Delta\, \bar{N}^{1/2} R_z(\eta)[I(\eta), \bar{N}^{(l-1)/2}] R_z(\eta)\psi \\
&= \Delta\, \bar{N}^{1/2} R_z(\eta) \bar{N}^{1/2} \big(\bar{N}^{-1/2} I(\eta) \bar{N}^{1/2} - \bar{N}^{(l-2)/2} I(\eta) \bar{N}^{-(l-2)/2}\big) \bar{N}^{(l-2)/2} R_z(\eta)\psi.
\end{aligned} \tag{4.50}
$$

Using that $\|\bar{N}^{1/2} R_z(\eta) \bar{N}^{1/2}\| \leqslant C\eta^{-1}$ (see Lemma 4.3(4)) and that, as we show below,

$$
\sup_{\eta>0} \|\bar{N}^\alpha \bar{I}(\eta) \bar{N}^{-\alpha}\| < \infty, \tag{4.51}
$$

for all $\alpha \in \mathbb{R}$, we obtain from (4.49) and (4.50) that

$$
\|\bar{N}^{l/2} R_z(\eta)\psi\| \prec \eta^{-1} \|\bar{N}^{(l-2)/2} R_z(\eta)\psi\| + \eta^{-1} \|\bar{N}^{(l-2)/2}\psi\|. \tag{4.52}
$$

We now iterate (4.52). For l even, we obtain after $l/2$ iterations

$$
\|\bar{N}^{l/2} R_z(\eta)\psi\| \prec \eta^{-(l+1)/2} \|\psi\|_1 + \sum_{j=1}^{l/2} \eta^{-j} \|\bar{N}^{(l-2j)/2}\psi\| \prec \big(\|\psi\|_1 + \eta^{1/2} \|\bar{N}^{(l-2)/2}\psi\|\big) \eta^{-(l+1)/2}.
$$

We use $\|\bar{N}^{1/2} R_z(\eta)\psi\| \leqslant c\eta^{-1/2} \|\psi\|_1$ (see (4.16)) in the last iteration step. This gives (4.29) for l even. The estimate for l odd is obtained in the same way, iterating (4.52).

It remains to show the bound (4.51), which is equivalent to $\|\bar{N}^\alpha W \bar{N}^{-\alpha}\| < \infty$, where $W = W(2if_\beta)$. Relation (3.5) (with $D = \mathbb{1}$) gives, for any integer $m \geqslant 1$,

$$
\bar{N}^m W \bar{N}^{-m} = \bar{N}^{m-1} W \big(\mathbb{1} + (\phi + c)\bar{N}^{-1}\big) \bar{N}^{-(m-1)} = \bar{N}^{m-1} W \bar{N}^{-(m-1)} B_m, \tag{4.53}
$$

where c is a constant, $\phi = \phi(2if_\beta)$ and $B_m = \bar{N}^{m-1}(\mathbb{1} + (\phi + c)\bar{N}^{-1})\bar{N}^{-(m-1)}$. By using repeatedly the commutation relation $N\phi = \phi N + 2^{-1/2}(a^*(2if_\beta) - a(2if_\beta))$ one sees that B_m is bounded. Next, we show that $\bar{N}^{1/2} B_m \bar{N}^{-1/2}$ is bounded. It suffices to prove that $\bar{N}^{1/2}[B_m, \bar{N}^{-1/2}]$ is bounded. The representation $\bar{N}^{-1/2} = \pi^{-1} \int_0^\infty x^{-1/2} (\bar{N} + x)^{-1} dx$ (see [24], equation (3.53)) gives

$$
\bar{N}^{1/2}[B_m, \bar{N}^{-1/2}] = \pi^{-1} \int_0^\infty x^{-1/2} \bar{N}^{1/2} (\bar{N} + x)^{-1} [\bar{N}, B_m] (\bar{N} + x)^{-1} dx. \tag{4.54}
$$

Next, $\|\bar{N}^{1/2}(\bar{N}+x)^{-1}\| \leqslant (1+x)^{-1/2}$, $\|(\bar{N}+x)^{-1}\| \leqslant (1+x)^{-1}$ and $[\bar{N}, B_m] = \bar{N}^{m-1}[\bar{N}, \phi]\bar{N}^{-m}$, which is easily seen to be bounded. The norm of the integrand in (4.54) is thus bounded above by a constant times $x^{-1/2}(1+x)^{-3/2}$, which is integrable in $x \in [0, \infty)$. Thus the operator (4.54) is bounded.

Iterating (4.53) gives $\bar{N}^m W \bar{N}^{-m} = W B_1 \cdots B_m$. Let $\alpha \geqslant 0$ and set $\alpha = m + \xi$, with $m = 0, 1, \ldots$ and $0 \leqslant \xi < 1$. Then

$$
\bar{N}^\alpha W \bar{N}^{-\alpha} = \bar{N}^\xi \bar{N}^m W \bar{N}^{-m} \bar{N}^{-\xi} = \bar{N}^\xi W B_1 \cdots B_m \bar{N}^{-\xi}. \tag{4.55}
$$

To show boundedness of $\bar{N}^\alpha W \bar{N}^{-\alpha}$ it suffices to show it for $\bar{N}^\xi [W B_1 \cdots B_m, \bar{N}^{-\xi}]$, as $W B_1 \cdots B_m$ is bounded. The representation $\bar{N}^{-\xi} = \pi^{-1} \sin(\pi\xi) \int_0^\infty x^{-\xi} (\bar{N} + x)^{-1} dx$ (see [24], equation (3.53)) gives

$$
\bar{N}^\xi [W B_1 \cdots B_m, \bar{N}^{-\xi}] = \pi^{-1} \sin(\pi\xi) \int_0^\infty x^{-\xi} \bar{N}^\xi (\bar{N} + x)^{-1} [\bar{N}, W B_1 \cdots B_m] (\bar{N} + x)^{-1} dx. \tag{4.56}
$$

Using that $\|\bar{N}^\xi(\bar{N}+x)^{-1}\| \leqslant (1+x)^{-1+\xi}$, $\|\bar{N}^{1/2}(\bar{N}+x)^{-1}\| \leqslant (1+x)^{-1/2}$, and, as we show below,

$$\|[\bar{N}, WB_1\cdots B_m]\bar{N}^{-1/2}\| < \infty, \tag{4.57}$$

we see that the norm of the integrand in (4.56) is bounded from above by a constant times $x^{-\xi}(1+x)^{-3/2+\xi}$, which is integrable. To complete the proof of Lemma 4.6, we show (4.57).

Expanding the commutator gives a sum of terms, each being of the form either $T_1 = [\bar{N}, W]B_1\cdots B_m\bar{N}^{-1/2}$ or $T_2 = WB_1\cdots B_k[\bar{N}, B_{k+1}]B_{k+2}\cdots B_m\bar{N}^{-1/2}$. T_1 is bounded since $[\bar{N}, W]\bar{N}^{-1/2}$ and $\bar{N}^{1/2}B_k\bar{N}^{-1/2}$ $(k=1,\ldots,m)$ are. Finally, T_2 is bounded since $[\bar{N}, B_{k+1}] = \bar{N}^k[\bar{N},\phi]\bar{N}^{-k-1}$ is. This shows (4.51) (for $\alpha \geqslant 0$; for $\alpha \leqslant 0$ the derivation is the same). The proof of Lemma 4.6 is complete. □

4.2. **Proof of Theorem 4.2.** We want to prove that $\mathfrak{F}(x)$, $x \neq 0$, is invertible. For $y > 0$,

$$\mathfrak{F}(x) - iy = \mathfrak{F}(x+iy)\big\{\mathbb{1} + \mathfrak{F}(x+iy)^{-1}\big(\mathfrak{F}(x) - iy - \mathfrak{F}(x+iy)\big)\big\}, \tag{4.58}$$

where $\mathfrak{F}(x+iy)^{-1} = P(\mathcal{L} - x - iy)^{-1}P$. By Theorem 4.1, $z \mapsto PI\bar{P}(\bar{\mathcal{L}} - z)^{-1}\bar{P}IP$ extends to a Hölder continuous map in $z \in \bar{\mathbb{C}}_+$, with exponent one. Hence

$$\|\mathfrak{F}(x) - iy - \mathfrak{F}(x+iy)\| \leqslant \Delta^2\|PI\bar{P}\big((\bar{\mathcal{L}} - x - iy)^{-1} - (\bar{\mathcal{L}} - x)^{-1}\big)\bar{P}IP\| \leqslant C\Delta^2 y, \tag{4.59}$$

uniformly in $x \in \mathbb{R}$. Moreover, $x \neq 0$ is not an eigenvalue of $\mathcal{L}$, so $w - \lim_{y\to 0_+} iy(\mathcal{L} - x - iy)^{-1} = 0$, from which it follows that

$$\lim_{y\to 0_+} iy\mathfrak{F}(x+iy)^{-1} = 0. \tag{4.60}$$

Combining this with (4.59) and (4.58) shows for any $x \neq 0$ there is a y_0 s.t. if $|y| < y_0$, then $\mathfrak{F}(x) - iy$ is invertible, and

$$\|(\mathfrak{F}(x) - iy)^{-1}\| \leqslant 2\|\mathfrak{F}(x+iy)^{-1}\| \leqslant (2y)^{-1}. \tag{4.61}$$

In the last step, we have again used (4.60). This implies that the kernel of $\mathfrak{F}(x)$ is $\{0\}$. Indeed, if $\mathfrak{F}(x)\psi = 0$ for some $\psi \in \mathrm{Ran}P$, $\|\psi\| = 1$, then $\|(\mathfrak{F}(x) + iy)^{-1}\psi\| = 1/y$. But (4.61) gives $\|(\mathfrak{F}(x) + iy)^{-1}\psi\| \leqslant 1/(2y)$, a contradiction.

Since $\mathfrak{F}(x)$ is invertible $(x \neq 0)$ there is a constant c_x s.t. $\|\mathfrak{F}(x)^{-1}\| \leqslant c_x$. We have $\mathfrak{F}(x)^{-1} = \mathfrak{F}(x')^{-1}[\mathbb{1} - (\mathfrak{F}(x) - \mathfrak{F}(x'))\mathfrak{F}(x)^{-1}]$ and for x' close enough to x, $\|(\mathfrak{F}(x) - \mathfrak{F}(x'))\mathfrak{F}(x)^{-1}\| < 1/2$. It follows that $\|\mathfrak{F}(x')^{-1}\| \leqslant 2c_x$. This completes the proof of Theorem 4.2 □

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