*Journal of Statistical Physics, Vol. 116, Nos. 1/4, August 2004 (© 2004)*

# Ionization of Atoms in a Thermal Field[1]

**J. Fröhlich,[2] M. Merkli,[2, 3] and I. M. Sigal[4]**

*Received June 13, 2003; accepted November 13, 2003*

We study the stationary states of a quantum mechanical system describing an atom coupled to black-body radiation at positive temperature. The stationary states of the non-interacting system are given by product states, where the particle is in a bound state corresponding to an eigenvalue of the particle Hamiltonian, and the field is in its equilibrium state. We show that if Fermi's Golden Rule predicts that a stationary state disintegrates after coupling to the radiation field then it is unstable, provided the coupling constant is sufficiently small (depending on the temperature). The result is proven by analyzing the spectrum of the thermal Hamiltonian (Liouvillian) of the system within the framework of $W^*$-dynamical systems. A key element of our spectral analysis is the positive commutator method.

**KEY WORDS:** Open quantum system; black-body radiation; CCR algebra; positive temperature representation; Fermi golden rule; Liouville operator; virial theorem; positive commutator method.

## 1. INTRODUCTION

In this paper, we study a quantum mechanical model of an atom interacting with black-body radiation. The atom is described by an electron moving in a potential, e.g., the Coulomb potential of a static nucleus.

Our goal is to show that when the atom is coupled to the quantized radiation field in the state of black-body radiation at sufficiently high temperature, it is ionized. We describe this ionization process by showing that stationary states of the system become unstable when the atom is

[1] This paper is dedicated to Elliott H. Lieb, in admiration and friendship.

[2] Institut für Theoretische Physik, ETH Hönggerberg, CH-8093 Zürich, Switzerland: e-mail: juerg@phys.ethz.ch

[3] Present address: Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada; e-mail: merkli@math.mcgill.ca

[4] Department of Mathematics, University of Toronto, Toronto, Ontario, M5S 3G3 Canada.

coupled to the radiation field. Each bound state of the atom leads to a stationary state of the uncoupled system, where the field is in an equilibrium state. We consider a class of small interactions localized in space which couple *finitely* many bound states of the atom to the field. We show that the stationary states for the coupled system correspond to those eigenvalues of the atomic Hamiltonian which are not coupled to the field. In other words, all stationary states arising from atomic bound states which are coupled to the field by the interaction are unstable, provided the coupling constant is small enough.

This instability is explained by the following mechanism. If the electron is in a bound state with energy $E<0$, it will be hit, after some time, by a quantum (photon) of energy $\omega \geqslant -E$, and hence will make a transition to a scattering state of energy $E+\omega$. In other words, the atom is ionized.

The average time, $t_E$, it takes for an atom in a bound state of energy $E$ to be ionized is given, to second order in the perturbation, by the Fermi Golden Rule. Heuristically, for an inverse temperature $\beta$ of the field, it satisfies

$$t_E \propto e^{\beta\,|E|}\,|p(E)|^{-2}, \tag{1.1}$$

where $|p(E)|^2$ is the probability for the electron to make a transition from the bound state to a scattering state by absorbing a photon of energy $>E$. The factor $e^{\beta\,|E|}$ in (1.1) can be explained by Planck's law, which says that the probability density for a photon to have energy $\omega$ is $\frac{1}{e^{\beta\omega}-1}$. At zero temperature, $\beta=\infty$, one finds that $t_E=\infty$, and thermal ionization does not occur. For a given strength of the interaction, as measured by the size of a coupling constant $\lambda\in\mathbb{R}$, we are able to show that thermal ionization takes place, provided $0<|\lambda|<ke^{-2\beta\,|E_0|}$, where $E_0<0$ is the minimal energy of the electron in a bound state coupled to the radiation field, and $k$ is some constant. This restriction is of technical nature; physically, thermal ionization is expected to be observed for arbitrarily small temperatures, provided the coupling constant is small enough *independently* of $\beta$.

Next, we describe the system and our main results in some more detail. The atomic Hamiltonian is a Schrödinger operator $H_p=-\Delta+v$ on the Hilbert space $\mathscr{H}_p=L^2(\mathbb{R}^3,d^3x)$, where $v$ belongs to a certain class of potentials including the Coulomb potential regularized at the origin. The operator $H_p$ generates the Heisenberg dynamics

$$\alpha_t^p(A)=e^{itH_p}Ae^{-itH_p}$$

on the von Neumann algebra $\mathfrak{A}_p=\mathscr{B}(\mathscr{H}_p)$ of bounded operators on $\mathscr{H}_p$.

The field is conveniently described in terms of a $C^*$-algebra $\mathfrak{A}_f$, which can be viewed as a time-averaged Weyl algebra. The dynamics is given by a $*$-automorphism group of $\mathfrak{A}_f$ describing free massless bosons.

The combinded system is described in terms of the algebra

$$\mathfrak{A} = \mathfrak{A}_p \otimes \mathfrak{A}_f,$$

and the uncoupled dynamics is given by the automorphisms

$$\alpha_{t,0} = \alpha_t^p \otimes \alpha_t^f.$$

To define the coupled dynamics, we specify a (regularized) interaction term $\lambda V^{(\epsilon)}$, whose form is motivated by standard models of atoms interacting with the radiation field. The regularization is introduced to guarantee that $V^{(\epsilon)} \in \mathfrak{A}$, for all $\epsilon \neq 0$. The interacting dynamics, $\alpha_{t,\lambda}^{(\epsilon)}$, is then defined as the $*$-automorphism group of $\mathfrak{A}$ obtained by the Schwinger–Dyson series.

At zero temperature, the dynamics of the model is generated by the formal Hamiltonian

$$H = H_p + H_f + \lambda V,$$

where $H_f = d\Gamma(|k|)$ is the free-field Hamiltonian, i.e., the second quantized multiplication operator $|k|$, acting on bosonic Fock space $\mathcal{F}(L^2(\mathbb{R}^3, d^3k))$, and the interaction term $V$ is given by

$$V = \sum_\alpha G_\alpha \otimes (a(g_\alpha) + a^*(g_\alpha)).$$

The sum is over a finite set, $G_\alpha$ are bounded selfadjoint operators on $\mathcal{B}(\mathcal{H}_p)$, and the form factors $g_\alpha$ are in $L^2(\mathbb{R}^3, d^3k)$.

We introduce a reference state

$$\omega^{\text{ref}} = \omega^p \otimes \omega_\beta^f$$

on $\mathfrak{A}$, where $\omega^p$ is given by a (strictly positive) density matrix, and $\omega_\beta^f$ is the $(\beta, \alpha_t^f)$-KMS state of $\mathfrak{A}_f$, i.e., the state of black-body radiation at inverse temperature $\beta$. We are interested in the time evolution of states on $\mathfrak{A}$ which are close to (normal w.r.t.) $\omega^{\text{ref}}$, i.e., which are represented by a density matrix on the GNS Hilbert space $\mathcal{H}$ of $(\mathfrak{A}, \omega^{\text{ref}})$. The GNS representation provides us with a representation map $\pi_\beta\colon \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ and a vector $\Omega^{\text{ref}} \in \mathcal{H}$ s.t. $\omega^{\text{ref}}(A) = \langle \Omega^{\text{ref}}, \pi_\beta(A)\, \Omega^{\text{ref}} \rangle$. There is a selfadjoint operator $L_\lambda^{(\epsilon)}$ on $\mathcal{H}$ generating the coupled time evolution in the representation $\pi_\beta$,

$$\pi_\beta(\alpha_{t,\lambda}^{(\epsilon)}(A)) = e^{itL_\lambda^{(\epsilon)}} \pi_\beta(A)\, e^{-itL_\lambda^{(\epsilon)}},$$

for all $A \in \mathfrak{A}$ and $t \in \mathbb{R}$. We will show that

$$\text{s}-\lim_{\epsilon \to 0} e^{itL_\lambda^{(\epsilon)}} = e^{itL_\lambda}$$

exists, for all $t$, and defines a $*$-automorphism group

$$\sigma_{t,\lambda}(A) = e^{itL_\lambda} A e^{-itL_\lambda}$$

of the von Neumann algebra

$$\mathfrak{M}_\beta = \pi_\beta(\mathfrak{A})'' \subset \mathscr{B}(\mathscr{H}).$$

The pair $(\mathfrak{M}_\beta, \sigma_{t,\lambda})$ is called a $W^*$-dynamical system. Our results concern the structure of the set of normal ($\sigma$-weakly continuous) time-translation ($\sigma_{t,\lambda}$-) invariant states on $\mathfrak{M}_\beta$.

The general theory of von Neumann algebras shows that there is a one-to-one correspondence between normal $\sigma_{t,\lambda}$-invariant states on $\mathfrak{M}_\beta$ and normalized vectors in the set

$$\mathscr{P} \cap \ker L_\lambda, \tag{1.2}$$

where $\mathscr{P}$ is a certain cone in $\mathscr{H}$, the so-called natural cone associated to $(\mathfrak{M}_\beta, \Omega^{\text{ref}})$, provided we choose the thermal Hamiltonian (Liouvillian) $L_\lambda$ in such a way that the unitary one-parameter group $\{e^{itL_\lambda} \mid t \in \mathbb{R}\}$ leaves $\mathscr{P}$ invariant. Let $\mathscr{M}$ be a labelling of the eigenvalues of $H_p$, including multiplicities. An element $m \in \mathscr{M}$ is called a mode and the corresponding eigenvalue of $H_p$ is denoted by $E(m)$. We will see that

$$\mathscr{P} \cap \ker L_0 = \mathscr{P} \cap \text{span}\{\varphi_m \otimes \varphi_n \otimes \Omega \mid m, n \in \mathscr{M}, E(m) = E(n)\}, \tag{1.3}$$

where $\varphi_m$ is the eigenvector of $H_p$ corresponding to the mode $m$, and $\Omega$ is the vector representative of $\omega_\beta^f$. Our main result, Theorem 2.3, shows that (1.2) is the subset of (1.3) with $m, n$ ranging over those modes which are not coupled to the field.

While this result holds for a specific class of potentials (see (2.2)), we prove in Theorem 2.2 a result which holds for a very general class of potentials: For any $\sigma_{t,0}$-invariant normal state $\omega^0$ and any $\sigma_{t,\lambda}$-invariant normal state $\omega^\lambda$ on $\mathfrak{M}_\beta$ we prove that $\|\omega^0 - \omega^\lambda\| \geqslant k > 0$, provided $\lambda \neq 0$ is small enough, for a constant $k$ independent of $\lambda$. Here $\|\cdot\|$ denotes the norm on the space of linear functionals on $\mathfrak{M}_\beta$. Theorem 2.2 is proven for bounded potentials $v$ such that $-\Delta + v$ has only finitely many eigenvalues below the threshold of the continuous spectrum, all of which are coupled to the field. Alternatively, we could relax this finiteness condition but couple only finitely many modes to the field.

## 2. DEFINITION OF THE MODEL AND MAIN RESULTS

In Section 2.1 we introduce the model and show in which way it defines a $W^*$-dynamical system $(\mathfrak{M}_\beta, \sigma_{t,\lambda})$. Our main results are presented in Section 2.2.

### 2.1. Definition of the Model

Starting with an algebra $\mathfrak{A}$ describing the joint system atom-field and a (regularized) dynamics $\alpha_{t,\lambda}^{(\epsilon)}$ on it, we introduce a reference state $\omega^{\mathrm{ref}}$, describing a bound state of the atom and black-body radiation at inverse temperature $\beta$. We then consider the induced (regularized) dynamics $\sigma_{t,\lambda}^{(\epsilon)}$ on $\pi_\beta(\mathfrak{A})$, where $(\mathscr{H}, \pi_\beta, \Omega^{\mathrm{ref}})$ denotes the GNS representation corresponding to $(\mathfrak{A}, \omega^{\mathrm{ref}})$. As $\epsilon \to 0$, $\sigma_{t,\lambda}^{(\epsilon)}$ tends to a $*$-automorphism group, $\sigma_{t,\lambda}$, of the von Neumann algebra $\mathfrak{M}_\beta$, defined as the weak closure of $\pi_\beta(\mathfrak{A})$ in $\mathscr{B}(\mathscr{H})$. We determine the generator, $L_\lambda$, of the unitary group, $e^{itL_\lambda}$, on $\mathscr{H}$ implementing $\sigma_{t,\lambda}$; $L_\lambda$ is called a *Liouvillian*. We explain the relation between eigenvalues of $L_\lambda$ and invariant normal states on $\mathfrak{M}_\beta$.

#### *2.1.1. Kinematical Algebra $\mathfrak{A}$, and Regularized Dynamics $\alpha_{t,\lambda}^{(\epsilon)}$*

We consider a system consisting of a quantum mechanical particle (an electron in the potential of a static nucleus) interacting with a quantized field.

Pure states of the particle system are described by unit vectors in the Hilbert space $\mathscr{H}_p = L^2(\mathbb{R}^3, d^3x)$, their dynamics is determined by the Schrödinger equation with Hamiltonian

$$H_p = -\Delta + v, \tag{2.1}$$

where the potential $v$ is bounded and satisfies one of the following two conditions.

- *Condition $C_A$.* The potential $v$ is s.t. the spectrum of $H_p$ consists of a finite number $d$ of eigenvalues (counting multiplicity) lying below the continuous spectrum which covers $[0, \infty)$. We set $E_0 := \inf \sigma(H_p) < 0$.
- *Condition $C_B$.* The potential $v$ is given by

$$v(x) = -\frac{\rho(|x|)}{|x|^{1+\mu}}, \qquad -1 < \mu \leqslant 1, \tag{2.2}$$

where $\rho(|x|)$ is a smooth, non-negative function that has a zero of order $1+\mu$ at the origin, and increases to a constant value $\rho$ as $|x| \to \infty$, in such a way that $v$ is smooth, and

$$(x \cdot \nabla)^j \, v \tag{2.3}$$

are bounded, for $j=0,\ldots,3$. Notice that the eigenvalues of $H_p$ are all negative and can accumulate only at the threshold 0.

**Remark.** In Condition $C_B$ we admit potentials such that $H_p$ has infinitely many eigenvalues below zero, but couple only finitely many of them to the field, as we explain below.

The field is a scalar massless free bosonic field. (It would be more interesting, physically, to consider the quantized electromagnetic field. Our methods can be applied to the resulting model at the price of slightly more complicated notations.) The scalar free field is conveniently described in terms of a "time-averaged" Weyl algebra, $\mathfrak{A}_f$, which is the $C^*$-algebra (of "observables"), defined as follows. Let $\mathfrak{W}$ be the Weyl algebra over the Hilbert space

$$L_0^2 = L^2(\mathbb{R}^3, d^3k) \cap L^2(\mathbb{R}^3, |k|^{-1}\, d^3k), \tag{2.4}$$

i.e., $\mathfrak{W}$ is the $C^*$-algebra generated by Weyl operators, $W(f)$, $f \in L_0^2$, satisfying the Weyl relations

$$W(f)\, W(g) = e^{-i\, \mathrm{Im}\langle f, g\rangle} W(g)\, W(f). \tag{2.5}$$

The free field dynamics on $\mathfrak{W}$ is given by the $*$-automorphism group

$$W(f) \mapsto \alpha_t^{\mathfrak{W}}(W(f)) = W(e^{i\omega t} f), \tag{2.6}$$

where $\omega(k) = |k|$ is the energy of a single boson. For functions $f \in L_0^2$, the expectation functional

$$f \mapsto e^{-\frac{1}{4}\langle f, (1+\frac{2}{e^{\beta\omega}-1}) f\rangle} \tag{2.7}$$

is well defined and determines a $(\beta, \alpha_t^{\mathfrak{W}})$-KMS state on $\mathfrak{W}$. It is well known that the $*$-automorphism group $\alpha_t^{\mathfrak{W}}$ of $\mathfrak{W}$ is not norm-continuous (i.e., $\mathbb{R} \ni t \mapsto \alpha_t^{\mathfrak{W}}(W(f))$ is not continuous in the norm of $\mathfrak{W}$). The time-averaged $C^*$-algebra $\mathfrak{A}_f$ is generated by elements of the form

$$a(h) = \int_{\mathbb{R}} ds\, h(s)\, \alpha_s^{\mathfrak{W}}(a), \tag{2.8}$$

where $a \in \mathfrak{W}$ and $h\colon \mathbb{R} \to \mathbb{C}$ are functions whose Fourier transforms satisfy $\hat{h} \in C_0^\infty$ (this is a convenient class of functions which allows us to define KMS states on $\mathfrak{A}_f$, see ref. 7). The free field dynamics on $\mathfrak{A}_f$ is defined by

$$\alpha_t^f(a(h)) = \int_{\mathbb{R}} ds\, h(s-t)\, \alpha_s^{\mathfrak{W}}(a) =: a(h_t). \tag{2.9}$$

It is a norm-continuous $*$-automorphism group on $\mathfrak{A}_f$. We refer to ref. 7 for more details on the construction and the properties of $\mathfrak{A}_f$.

The joint system describing the particle and the field is described in terms of the $C^*$-algebra

$$\mathfrak{A} = \mathfrak{A}_p \otimes \mathfrak{A}_f, \tag{2.10}$$

where $\mathfrak{A}_p = \mathscr{B}(\mathscr{H}_p)$ is the von Neumann algebra of all bounded operators on the Hilbert space $\mathscr{H}_p$. The uncoupled dynamics is given by the $*$-automorphism group

$$\alpha_{t,0} = \alpha_t^p \otimes \alpha_t^f \tag{2.11}$$

of $\mathfrak{A}$, where $\alpha_t^p(\cdot) = e^{itH_p} \cdot e^{-itH_p}$. In order to define the dynamics of the interacting system in a representation independent way (i.e., as a $*$-automorphism group on $\mathfrak{A}$), we need to introduce a regularized interaction term. For $\epsilon \neq 0$, this term is given by

$$V_{\#}^{(\epsilon)} = \sum_\alpha G_{\alpha,\#} \otimes \frac{1}{2i\epsilon} \{W(\epsilon g_\alpha)(h_\epsilon) - W(\epsilon g_\alpha)(h_\epsilon)^*\} \in \mathfrak{A}, \tag{2.12}$$

where the sum is over finitely many indices $\alpha$, with $G_{\alpha,\#} = G_{\alpha,\#}^* \in \mathscr{B}(\mathscr{H}_p)$, $g_\alpha \in L_0^2$, for all $\alpha$, and where $h_\epsilon$ is an approximation of the Dirac distribution localized at zero. To be specific we can take $h_\epsilon(t) = \frac{1}{\epsilon} e^{-t^2/\epsilon^2}$. The symbol $G_{\alpha,\#}$ (and similarly $V_{\#}^{(\epsilon)}$) stands for either $G_\alpha$ or $G_{\alpha,J}$, where $J$ is some cutoff determining which modes of the particle are coupled to the field. In order to describe this more precisely, we introduce the following terminology. Let $\mathscr{M}$ be the index set of the discrete "modes" of $H_p$, i.e., a labelling of the eigenvalues of $H_p$ including multiplicity. Given $m \in \mathscr{M}$, $E(m)$ denotes the corresponding eigenvalue of $H_p$. An eigenvalue $E$ of $H_p$ is simple if and only if there is a unique $m \in \mathscr{M}$ s.t. $E = E(m)$. We denote the rank-one projection corresponding to the mode $m \in \mathscr{M}$ by $p_m$.

Let $J_d \subset \mathscr{M}$ be a set of finitely many discrete modes of $H_p$ and let $J_c$ be an open interval in the continuous spectrum $\mathbb{R}_+$ of $H_p$ (we may also take a finite union of disjoint intervals), s.t. $J_c \subset [r, R]$, for some $r, R$ satisfying $0 < r < R < \infty$. The set

$$J := J_d \cup J_c$$

determines the modes of the particle which are coupled to the field, according to the interaction

$$G_{\alpha,J} = (p_{J_d} + \mu(H_p))\, G_\alpha (p_{J_d} + \mu(H_p)), \tag{2.13}$$

where $G_\alpha$ is a bounded, selfadjoint operator on $\mathscr{H}_p$, and

$$p_{J_d} = \sum_{m \in J_d} p_m, \tag{2.14}$$

$\mu \in C_0^\infty(J_c)$ is a smooth version of the indicator function with support in $J_c$, and $\mu(H_p)$ is defined via the Fourier transform

$$\mu(H_p) = \int \hat{\mu}(s)\, e^{isH_p}.$$

Clearly, $G_{\alpha, J}$ tends to $G_\alpha$, in the strong sense as $\mu$ increases to the characteristic function of $\mathbb{R}_+$ (i.e. $J_c \uparrow \mathbb{R}_+$) and $J_d$ increases to the set of all discrete modes of $H_p$. Thus, $V_J^{(\epsilon)}$ can be viewed as an approximation of $V^{(\epsilon)} = V_{J=\mathbb{R}}^{(\epsilon)}$.

The interaction term (2.12) determines a $*$-automorphism group $\alpha_{t,\lambda}^{(\epsilon)}$ of $\mathfrak{A}$, the coupled dynamics, via the norm-convergent Dyson series

$$\begin{aligned} \alpha_{t,\lambda}^{(\epsilon)}(A) &\\ := \alpha_{t,0}(A) &\\ + \sum_{n \geqslant 1} (i\lambda)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n [\alpha_{t_n,0}(V_{\#}^{(\epsilon)}), [\cdots [\alpha_{t_1,0}(V_{\#}^{(\epsilon)}), \alpha_{t,0}(A)] \cdots]], \end{aligned} \tag{2.15}$$

where $A \in \mathfrak{A}$, and $\lambda \in \mathbb{R}$ is the coupling constant. The multiple integral in (2.15) is understood in the product topology coming from the strong topology of $\mathscr{B}(\mathscr{H}_p)$ and the norm topology of $\mathfrak{A}_f$.

One may view $\alpha_{t,\lambda}^{(\epsilon)}$ as a *regularized dynamics*, in the sense that it has a limit, as $\epsilon \to 0$, in suitably chosen representations of $\mathfrak{A}$; (this is shown in ref. 7 and explained below).

The functions $g_\alpha \in L_0^2$ are called *form factors*. Using polar coordinates in $\mathbb{R}^3$, we often write $g_\alpha = g_\alpha(\omega, \Sigma)$, where $(\omega, \Sigma) \in \mathbb{R}_+ \times S^2$.

We now specify two sets of assumptions on the interactions.

*Condition A.* The potential $v$ satisfies condition $C_A$, the interaction is given by $V^{(\epsilon)}$, and the following properties hold.

- Infrared and Ultraviolet Behaviour of the Form Factors. For any fixed $\Sigma$, $g_\alpha(\cdot, \Sigma) \in C^4(\mathbb{R}_+)$, and there are two constants $0 < k_1, k_2 < \infty$, s.t. if $\omega < k_1$, then

$$|\partial_\omega^j g_\alpha(\omega, \Sigma)| < k_2 \omega^{p-j}, \qquad \text{for some} \quad p > 2, \tag{2.16}$$

uniformly in $\alpha$, $j=0,\ldots,4$ and $\Sigma \in S^2$. Similarly, there are two constant $0 < K_1, K_2 < \infty$, s.t. if $\omega > K_1$, then

$$|\partial_\omega^j g_\alpha(\omega, \Sigma)| < K_2 \omega^{-q-j}, \qquad \text{for some} \quad q > 3. \tag{2.17}$$

• Relative Bound on $[G_\alpha, H_p]$. Define the commutator $[G_\alpha, H_p]$ in the weak sense on $C_0^\infty \times C_0^\infty$ by

$$\langle \psi, [G_\alpha, H_p]\, \varphi \rangle = \langle G_\alpha \psi, H_p \varphi \rangle - \langle H_p \psi, G_\alpha \varphi \rangle.$$

Then $[G_\alpha, H_p]$ extends to a relatively $(H_p - E_0 + 1)^{1/2}$-bounded operator, i.e., there is a $k < \infty$ s.t. for any $\psi \in C_0^\infty$,

$$\|[G_\alpha, H_p]\, \psi\| \leqslant k\, \|(H_p - E_0 + 1)^{1/2}\, \psi\|, \tag{2.18}$$

where $E_0 = \inf \sigma(H_p) < 0$.

• The Fermi Golden Rule Condition. We define a family of bounded operators on $\mathscr{H}_p$ by $F(\omega, \Sigma) = \sum_\alpha g_\alpha(\omega, \Sigma)\, G_\alpha$ and let, for arbitrary $\epsilon > 0$,

$$T_\epsilon(\omega, E) = \int_{S^2} d\Sigma\, F(\omega, \Sigma)\, \frac{p_c \epsilon}{(H_p - E - \omega)^2 + \epsilon^2}\, F(\omega, \Sigma)^*, \tag{2.19}$$

where $E$ is an eigenvalue of $H_p$ and $p_c$ is the projection onto the continuous subspace of $H_p$. Let $p(E)$ denote the projection onto the eigenspace corresponding to $E$. We assume that there is an $\epsilon_0 > 0$, s.t. for $0 < \epsilon < \epsilon_0$,

$$\int_{-E}^{\infty} d\omega\, \frac{\omega^2}{e^{\beta\omega} - 1}\, p(E)\, T_\epsilon(\omega, E)\, p(E) \geqslant \gamma_E\, p(E), \tag{2.20}$$

for any $E \in \sigma_p(H_p)$, where $\gamma_E$ is a strictly positive constant. We set

$$\gamma := \min_{E \in \sigma_p(H_p)} \gamma_E > 0. \tag{2.21}$$

**Remarks.** (1) All requirements in Condition A are independent of the regularization of the interaction.

(2) For the physical model of an atom interacting with the radiation field, the value of the constant $p$ in (2.16) is $p = -1/2$ (or $p = 1/2$ in the dipole approximation), see, e.g., ref. 3. Although $p > 2$ is quite far from the physical range, we do not attempt here to optimize condition (2.16). This will be the aim of subsequent work. Suffice it to note that the discrete values $p = -1/2, 1/2$ are also admissible in our analysis.

(3) The operator $T_\epsilon(\omega, E)$ is just a (non-negative) number if $E$ is a simple eigenvalue. For $\epsilon$ small, it represents the probability that the particle makes a transition from the bound state corresponding to the energy $E$ into a scattering state with energy $E+\omega \geqslant 0$ by absorbing a photon of energy $\omega$. The probability density for a photon to have energy $\omega$ is given by Planck's law, i.e., by $(e^{\beta\omega}-1)^{-1}$. Hence $\gamma$ is a perturbative bound on the probability of an ionization process; it depends on the inverse temperature $\beta$ as $\gamma \sim e^{\beta E_0}$, where $E_0 < 0$ is the ground state energy of $H_p$. More precisely, if we assume that, for $0<\epsilon<\epsilon_0$, $\omega \mapsto p(E)\, T_\epsilon(\omega, E)\, p(E)$ is continuous, and that there is a constant $t>0$ s.t. $p(E)\, T_\epsilon(\omega, E)\, p(E) \geqslant t \cdot p(E)$ at $\omega = -E$, then one sees that $k \frac{e^{\beta E}}{1+\beta} \leqslant \gamma_E \leqslant k e^{\beta E}$, for some $k$ which does not depend on $\beta$.

*Condition B.* The potential $v$ satisfies condition $C_B$, the interaction is given by $V_J^{(\epsilon)}$, and the following properties hold.

- The infra-red and ultra-violet behaviour of the form factors is as in (2.16), (2.17).
- Spatial decay of $G_\alpha$. There is a constant $k<\infty$ s.t.

$$\|\langle x\rangle^{n_1} G_\alpha \langle x\rangle^{n_2}\| \leqslant k, \qquad n_1+n_2 = 0,..., 5, \tag{2.22}$$

where we set $\langle x\rangle = (x^2+1)^{1/2}$, for $x \in \mathbb{R}^3$. Notice that this is a condition on $G_\alpha$ not depending on the regularization.

- The Fermi Golden Rule Condition. For all eigenvalues $E$ of $H_p$ s.t. $E = E(m)$ for some $m \in J_d$, let $T_\epsilon(\omega, E)$ be defined as in (2.19), with $p_c$ replaced by $\mu(H_p)^2$, and let $p_{J_d}(E) = \sum_{\substack{m \in J_d \\ E(m)=E}} p_m$. There is an $\epsilon_0 > 0$ s.t., for $0<\epsilon<\epsilon_0$,

$$\int_{-E}^{\infty} d\omega \, \frac{\omega^2}{e^{\beta E}-1}\, p_{J_d}(E)\, T_\epsilon(\omega, E)\, p_{J_d}(E) \geqslant \gamma_E\, p_{J_d}(E), \tag{2.23}$$

for some strictly positive constant $\gamma_E$. We set

$$\gamma := \min\{\gamma_E \mid E \in \sigma_p(H_p) \text{ s.t. } E = E(m) \text{ for some } m \in J_d\} > 0. \tag{2.24}$$

**Remarks.** (1) $\gamma$ is exponentially small in $\beta$, as observed in Remark (3) after (2.21).

(2) The operator $T_\epsilon(\omega, E)$ is a decreasing function of $r$, and an increasing function of $R$. Thus, we may assume without loss of generality that $\gamma$ is independent of $r \leqslant 1$, $R \geqslant 2$.

### *2.1.2. Reference State $\omega^{\mathrm{ref}}$*

The *reference state* of the system is given by the product state

$$\omega^{\mathrm{ref}} = \omega^{p} \otimes \omega^{f}_{\beta}, \tag{2.25}$$

where $\omega^p$ is a state on $\mathscr{B}(\mathscr{H}_p)$, determined by a strictly positive density matrix $\rho_p > 0$, i.e.

$$\omega_p(A) = \mathrm{tr}(\rho_p A), \tag{2.26}$$

for any $A \in \mathscr{B}(\mathscr{H}_p)$. The state $\omega^f_\beta$ is the $\beta$-KMS state of $\mathfrak{A}_f$ w.r.t. the free field dynamics (2.9) determined by the expectation functional (2.7). It describes *black body radiation* of the field at temperature $1/\beta$.

Let $(\mathscr{H}, \pi_\beta, \Omega^{\mathrm{ref}})$ be the GNS representation of $(\mathfrak{A}, \omega^{\mathrm{ref}})$, i.e. $\mathscr{H}$ is a Hilbert space, $\pi_\beta$ is a $*$-morphism $\mathfrak{A} \to \mathscr{B}(\mathscr{H})$, and $\Omega^{\mathrm{ref}}$ is a vector in $\mathscr{H}$ s.t. $\pi_\beta(\mathfrak{A})\, \Omega^{\mathrm{ref}}$ is dense in $\mathscr{H}$, and

$$\omega^{\mathrm{ref}}(A) = \langle \Omega^{\mathrm{ref}}, \pi_\beta(A)\, \Omega^{\mathrm{ref}} \rangle, \qquad A \in \mathfrak{A}.$$

An explicit realization of the GNS representation is well known. It was first constructed by Araki and Woods, ref. 2, and has been used recently by several authors. Here we just recall the explicit formulas that are useful in the present paper and refer to refs. 7 and 9 for a more detailed discussion.

The representation Hilbert space is

$$\mathscr{H} = \mathscr{H}_p \otimes \mathscr{H}_p \otimes \mathscr{F}, \tag{2.27}$$

where $\mathscr{F}$ is a shorthand for the Fock space

$$\mathscr{F} = \mathscr{F}((L^2(\mathbb{R} \times S^2, du \times d\Sigma)), \tag{2.28}$$

$du$ being the Lebesgue measure on $\mathbb{R}$, and $d\Sigma$ the uniform measure on $S^2$. Here $\mathscr{F}(X)$ denotes the Bosonic Fock space over a (normed vector) space $X$,

$$\mathscr{F}(X) := \mathbb{C} \oplus \bigoplus_{n \geqslant 1} (\mathscr{S} X^{\otimes n}), \tag{2.29}$$

where $\mathscr{S}$ is the projection onto the symmetric subspace of the tensor product. We use standard notation, e.g., $\Omega$ is the vacuum vector, $[\psi]_n$ is the $n$-particle component of $\psi \in \mathscr{F}(X)$, $\mathrm{d}\Gamma(A)$ is the second quantization of the operator $A$ on $X$, $N = \mathrm{d}\Gamma(\mathbb{1})$ is the number operator.

The representation map $\pi_\beta : \mathfrak{A} \to \mathscr{B}(\mathscr{H})$ is the product

$$\pi_\beta = \pi_p \otimes \pi_f^\beta,$$

where the $*$-homomorphism $\pi_p$: $\mathfrak{A}_p \to \mathscr{B}(\mathscr{H}_p \otimes \mathscr{H}_p)$ is given by

$$\pi_p(A) = A \otimes \mathbb{1}_p. \tag{2.30}$$

The representation map $\pi_f^\beta$: $\mathfrak{A}_f \to \mathscr{B}(\mathscr{F})$ is determined by

$$\pi_f^\beta(a(h)) = \int_{\mathbb{R}} dt\, h(t)\, \pi_{\mathfrak{W}}^\beta(\alpha_t^{\mathfrak{W}}(a)), \tag{2.31}$$

where $\pi_{\mathfrak{W}}^\beta$: $\mathfrak{W} \to \mathscr{B}(\mathscr{F})$ is a representation of the Weyl algebra given by

$$\pi_{\mathfrak{W}}^\beta = \pi_{\text{Fock}} \circ \mathscr{T}_\beta.$$

Here, $\mathscr{T}_\beta$ is the Bogoliubov transformation, mapping $\mathfrak{W}(L_0^2)$ to $\mathfrak{W}(L^2(\mathbb{R}\times S^2))$ defined by $W(f) \mapsto W(\tau_\beta f)$, with $\tau_\beta$: $L^2(\mathbb{R}_+ \times S^2) \to L^2(\mathbb{R}\times S^2)$ given by

$$(\tau_\beta f)(u, \Sigma) = \sqrt{\frac{u}{1-e^{-\beta u}}} \begin{cases} \sqrt{u} f(u, \Sigma), & u > 0, \\ -\sqrt{-u}\, \bar{f}(-u, \Sigma), & u < 0. \end{cases} \tag{2.32}$$

**Remarks.** (1) It is easily verified that $\text{Im}\langle \tau_\beta f, \tau_\beta g\rangle_{L^2(\mathbb{R}\times S^2)} = \text{Im}\langle f, g\rangle_{L^2(\mathbb{R}_+\times S^2)}$, for all $f, g \in L_0^2$, so the CCR (2.5) are preserved under the map $\tau_\beta$.

(2) In the limit $\beta \to \infty$, the r.h.s. of (2.32) tends to

$$\begin{cases} uf(u, \Sigma), & u > 0, \\ 0, & u < 0. \end{cases} \tag{2.33}$$

Notice that $L^2(\mathbb{R}_+ \times S^2) \oplus L^2(\mathbb{R}_+ \times S^2)$ is isometrically isomorphic to $L^2(\mathbb{R}\times S^2)$ via the map

$$(f, g) \mapsto h,\ h(u, \Sigma) = \begin{cases} uf(u, \Sigma), & u > 0, \\ ug(-u, \Sigma), & u < 0, \end{cases} \tag{2.34}$$

so (2.33) can be identified, via (2.34), with $f \in L_0^2$. Thus, $\mathscr{T}_\beta$ reduces to the identity (an imbedding), $\pi_{\mathfrak{W}}^\beta$ becomes the Fock representation of $\mathfrak{W}(L_0^2)$, as $\beta \to \infty$, and we recover the zero temperature situation.

It is useful to introduce the following notation. We define unitary operators $\hat{W}(f)$ on the Hilbert space (2.27) by

$$\hat{W}(f) = e^{i\varphi(f)}, \qquad f \in L^2(\mathbb{R}\times S^2),$$

where $\varphi(f)$ is the selfadjoint operator on $\mathscr{F}$ given by

$$\varphi(f) = \frac{a^*(f) + a(f)}{\sqrt{2}}, \tag{2.35}$$

and $a^*(f)$, $a(f)$ are the creation- and annihilation operators on $\mathscr{F}$, smeared out with $f$. One easily verifies that

$$\pi^{\beta}_{\mathfrak{W}}(W(f)) = \hat{W}(\tau_\beta f).$$

The cyclic GNS vector is given by

$$\Omega^{\text{ref}} = \Omega_p \otimes \Omega,$$

where $\Omega$ is the vacuum in $\mathscr{F}$, and

$$\Omega_p = \sum_{n\geqslant 0} k_n \varphi_n \otimes \mathscr{C}_p \varphi_n \in \mathscr{H}_p \otimes \mathscr{H}_p. \tag{2.36}$$

Here, $\{k_n^2\}_{n=0}^{\infty}$ is the spectrum of $\rho_p$, $\{\varphi_n\}$ is an orthogonal basis of eigenvectors of $\rho_p$, and $\mathscr{C}_p$ is an antilinear involution on $\mathscr{H}_p$. The operator $\mathscr{C}_p$ comes from the identification of $l^2(\mathscr{H}_p)$ (Hilbert–Schmitt operators on $\mathscr{H}_p$) with $\mathscr{H}_p \otimes \mathscr{H}_p$, via $|\varphi\rangle\langle\psi| \mapsto \varphi \otimes \mathscr{C}_p\psi$. We fix a convenient choice for $\mathscr{C}_p$. It is the antilinear involution on $\mathscr{H}_p$ corresponding to complex conjugation of components of vectors in the basis in which the Hamiltonian $H_p$ is diagonal (i.e. it is the the time reversal operator). Then $\mathscr{C}_p H_p \mathscr{C}_p = H_p$.

### 2.1.3. $W^*$-Dynamical System $(\mathfrak{M}_\beta, \sigma_{t,\lambda})$

Let $\mathfrak{M}_\beta$ be the von Neumann algebra obtained by taking the weak closure of $\pi_\beta(\mathfrak{A})$ in $\mathscr{B}(\mathscr{H})$,

$$\mathfrak{M}_\beta = \mathscr{B}(\mathscr{H}_p) \otimes \mathbb{1}_p \otimes \pi_f^\beta(\mathfrak{A}_f)'' \subset \mathscr{B}(\mathscr{H}). \tag{2.37}$$

Since the density matrix $\rho_p$ is strictly positive, $\Omega_p$ is cyclic and separating for the von Neumann algebra $\pi_p(\mathfrak{A}_p)'' = \mathscr{B}(\mathscr{H}_p) \otimes \mathbb{1}_p$. Similarly, $\Omega$ is cyclic and separating for $\pi_f^\beta(\mathfrak{A}_f)''$, since it is the GNS vector of a KMS state (see, e.g., ref. 5II). Consequently, $\Omega^{\text{ref}}$ is cyclic and separating for $\mathfrak{M}_\beta$. Let $J$ be the modular conjugation operator associated to $(\mathfrak{M}_\beta, \Omega^{\text{ref}})$. It is given by

$$J = J_p \otimes J_f, \tag{2.38}$$

where, for $\varphi, \psi \in \mathscr{H}_p$, $J_p(\varphi \otimes \mathscr{C}_p \psi) = \psi \otimes \mathscr{C}_p \varphi$, and, for $\psi = \{[\psi]_n\}_{n \geqslant 0} \in \mathscr{F}$,

$$[J_f \psi]_n \, (u_1,\dots, u_n) = \overline{[\psi]_n \, (-u_1,\dots, -u_n)}, \qquad \text{for} \quad n \geqslant 1,$$

and $[J_f \psi]_0 = \overline{[J_f \psi]_0} \in \mathbb{C}$. Clearly, $J\Omega^{\text{ref}} = \Omega^{\text{ref}}$, and one verifies that

$$J_p \pi_p(A) \, J_p = \mathbb{1}_p \otimes \mathscr{C}_p A \mathscr{C}_p, \tag{2.39}$$

$$J_f \pi^{\beta}_{\mathfrak{W}}(W(f)) \, J_f = \hat{W}(-e^{-\beta u/2} \tau_\beta(f)) = \hat{W}(e^{-\beta u/2} \tau_\beta(f))^*. \tag{2.40}$$

It is not difficult to see (ref. 7) that

$$\sigma_{t,0}(\pi_\beta(A)) := \pi_\beta(\alpha_{t,0}(A)) = e^{itL_0} \pi_\beta(A) \, e^{-itL_0}, \tag{2.41}$$

for all $A \in \mathfrak{A}$, where $L_0$ is the selfadjoint operator on $\mathscr{H}$, given by

$$L_0 = H_p \otimes \mathbb{1}_p - \mathbb{1}_p \otimes H_p + \mathrm{d}\Gamma(u), \tag{2.42}$$

commonly called the (non-interacting, standard) Liouvillian. One easily sees that $\alpha^{(\epsilon)}_{t,\lambda}$ is unitarily implemented in the representation $\pi_\beta$ as

$$\pi_\beta(\alpha^{(\epsilon)}_{t,\lambda}(A)) = e^{itL^{(\epsilon)}_\lambda} \pi_\beta(A) \, e^{-itL^{(\epsilon)}_\lambda} =: \sigma^{(\epsilon)}_{t,\lambda}(\pi_\beta(A)),$$

where the *regularized Liouvillian* $L^{(\epsilon)}_\lambda$ is given by

$$L^{(\epsilon)}_\lambda = L_0 + \lambda \pi_\beta(V^{(\epsilon)}_{\#}) - \lambda J \pi_\beta(V^{(\epsilon)}_{\#}) \, J.$$

An application of the Glimm–Jaffe–Nelson Theorem (Theorem 3.1) shows that $L^{(\epsilon)}_\lambda$ is essentially selfadjoint on

$$\mathscr{D} = C^\infty_0 \otimes C^\infty_0 \otimes (\mathscr{F}(C^\infty_0(\mathbb{R} \times S^2)) \cap \mathscr{F}_0) \subset \mathscr{H}, \tag{2.43}$$

where $\mathscr{F}_0$ is the finite-particle subspace (see ref. 7). Moreover, from the theorem on invariance of domains, Theorem A.1, and the Duhamel formula, one easily sees that

$$\lim_{\epsilon \to 0} e^{itL^{(\epsilon)}_\lambda} = e^{itL_\lambda}, \tag{2.44}$$

in the strong sense on $\mathscr{H}$, where the Liouvillian $L_\lambda$ is given by

$$L_\lambda = L_0 + \lambda I, \tag{2.45}$$

$$I = \sum_\alpha G_{\alpha,\#} \otimes \mathbb{1}_p \otimes \varphi(\tau_\beta(g_\alpha)) - \mathbb{1}_p \otimes \mathscr{C}_p G_{\alpha,\#} \mathscr{C}_p \otimes \varphi(e^{-\beta u/2} \tau_\beta(g_\alpha)).$$

The operator $L_\lambda$ is essentially selfadjoint on the domain $\mathscr{D}$ defined in (2.43) and defines a $*$-automorphism group on $\mathfrak{M}_\beta$ given by

$$\sigma_{t,\lambda}(A) = e^{itL_\lambda} A e^{-itL_\lambda}, \qquad A \in \mathfrak{M}_\beta. \tag{2.46}$$

An important property of $L_\lambda$ is that

$$e^{itL_\lambda} J = J e^{itL_\lambda}, \qquad \text{for all} \quad \lambda \in \mathbb{R}. \tag{2.47}$$

### 2.1.4. Characterization of the $\sigma_{t,\lambda}$-Invariant Normal States on $\mathfrak{M}_\beta$

A state $\omega$ on a von Neumann algebra $\mathfrak{M} \subset \mathscr{B}(\mathscr{H})$ is called normal iff it is given by a density matrix $\rho \in \mathscr{B}(\mathscr{H})$, i.e., $\omega(A) = \operatorname{tr} \rho A$, $A \in \mathfrak{M}$. If $\tau_t$ is a group of homomorphisms of $\mathfrak{M}$ the state is called $\tau_t$-invariant iff $\omega \circ \tau_t = \omega$ for all $t \in \mathbb{R}$. In order to characterize the $\sigma_{t,\lambda}$-invariant normal states on $\mathfrak{M}_\beta$ it is useful to introduce the natural cone $\mathscr{P}$ associated to $(\mathfrak{M}_\beta, \Omega^{\mathrm{ref}})$, which is defined by

$$\mathscr{P} = \overline{\{AJA\Omega^{\mathrm{ref}} \mid A \in \mathfrak{M}_\beta\}} \subset \mathscr{H}, \tag{2.48}$$

where the bar denotes the closure in the norm of $\mathscr{H}$. The following properties of the natural cone are the contents of the *Araki–Connes–Haagerup* theorem, a deep result in the theory of von Neumann algebras (see, e.g., ref. 5).

Given any normal state $\omega$ on $\mathfrak{M}_\beta$, there is a unique vector $\xi \in \mathscr{P}$ s.t. $\omega(A) = \langle \xi, A\xi \rangle$, for all $A \in \mathfrak{M}_\beta$. Moreover, if $\omega_1$ and $\omega_2$ are normal states on $\mathfrak{M}_\beta$ with corresponding vectors $\xi_1$, $\xi_2$ in $\mathscr{P}$ then

$$\|\xi_1 - \xi_2\|^2 \leqslant \|\omega_1 - \omega_2\| \leqslant \|\xi_1 - \xi_2\| \, \|\xi_1 + \xi_2\|. \tag{2.49}$$

The norm of a state $\omega$ of $\mathfrak{M}_\beta$ is given by $\|\omega\| = \sup_{A \in \mathfrak{M}_\beta} |\omega(A)| / \|A\|$.

It is not difficult to see that (2.47) implies that $e^{itL_\lambda} \mathscr{P} = \mathscr{P}$, for all $\lambda \in \mathbb{R}$ and $t \in \mathbb{R}$. From the uniqueness of the vector representative in the natural cone it follows that the $\sigma_{t,\lambda}$-invariant normal states are in one-to-one correspondence with the unit vectors in the set $\mathscr{P} \cap \ker L_\lambda$, which, for $\lambda = 0$, is given by

$$\mathscr{P} \cap \ker L_0 = \mathscr{P} \cap \operatorname{span}\{\varphi_m \otimes \varphi_n \otimes \Omega \mid m, n \in \mathscr{M}, E(m) = E(n)\}. \tag{2.50}$$

We will show in Theorem 2.3 that, for $\lambda \neq 0$, the $\sigma_{t,\lambda}$-invariant normal states are given by the subset of (2.50) determined by the modes $m, n \in \mathscr{M} \setminus J_d$ that do not interact with the field.

### 2.1.5. A Quick-Reference List

For the convenience of the reader and for future reference, we collect the definitions of some important operators in a list. Generally, if $\pi$ is a projection then we set $\bar{\pi} = \mathbb{1} - \pi$.

| | |
|---|---|
| $p_d$ | projection onto the discrete subspace of $H_p$ |
| $p_c$ | projection onto the continuous subspace of $H_p$ |
| $p_m$ | one-dimensional projection onto the mode $m \in \mathscr{M}$ |
| $p_{J_d}$ | $= \sum_{m \in J_d} p_m$ |
| $p_{J_c}$ | spectral projection of $H_p$ onto the interval $J_c$ |
| $p$ | $= p_{J_d} + p_{J_c}$ |
| $P$ | $= p \otimes p \otimes \mathbb{1}_f$ is a projection on $\mathscr{H}_p \otimes \mathscr{H}_p \otimes \mathscr{F}$ |
| $P^l$ | $= p \otimes \bar{p} \otimes \mathbb{1}_f$ |
| $P^r$ | $= \bar{p} \otimes p \otimes \mathbb{1}_f$ |
| $P^0$ | $= \bar{p} \otimes \bar{p} \otimes \mathbb{1}_f$ |
| $P_0$ | projection onto $\ker L_p$ |
| $\Pi$ | $= P_0 \otimes P_\Omega$ is the projection onto $\ker L_0$ |

## 2.2. Main Results

Our main results concern the dynamical system $(\mathfrak{M}_\beta, \sigma_{t,\lambda})$, where we have defined the von Neumann algebra $\mathfrak{M}_\beta$ in (2.37), and where $\sigma_{t,\lambda}$ is the $*$-automorphism group (2.46) of $\mathfrak{M}_\beta$ generated by the Liouvillian (2.45).

The following theorem describes some properties of eigenvectors of $L_\lambda$ which, as we have seen in Section 2.1.4, play an important role in the characterization of invariant normal states.

**Theorem 2.1 (Bounds on Eigenvectors).**

(1) Assume that either Condition A or Condition B of Section 2.1.1 holds. Let $N = d\Gamma(\mathbb{1})$ denote the number operator on Fock space $\mathscr{F}(L^2(\mathbb{R} \times S^2))$. Any eigenvector $\psi$ of $L_\lambda$ satisfies $\psi \in \mathscr{D}(N^{1/2})$, for any $\lambda \in \mathbb{R}$, and there is a constant $k < \infty$ s.t.

$$\|N^{1/2}\psi\| \leqslant k\, |\lambda|\, \|\psi\|. \tag{2.51}$$

The constant $k$ satisfies $k < k'(1 + 1/\beta)$, where $k'$ depends on the interaction, but not on $\beta$.

(2) Assume Condition A. Given any $0 < \beta < \infty$, there are constants $\lambda_0(\beta) > 0$, $k(\beta) > 0$, s.t. if $0 < |\lambda| < \lambda_0(\beta)$, and if $\psi$ is an eigenvector of $L_\lambda$, then

$$\|\bar{P}_0 \otimes P_\Omega \psi\| \geqslant k(\beta)\, \|\psi\|, \tag{2.52}$$

where $P_0$ is the projection onto the zero eigenspace of $L_p$, $\bar{P}_0 = \mathbb{1} - P_0$, and $P_\Omega$ is the projection onto the vacuum sector in $\mathscr{F}$. We have $\lambda_0(\beta) \geqslant k\gamma$ (see (2.21) and remark (2) thereafter) and $k(\beta) \geqslant k\gamma^2$, for some $k$ independent of $\beta$ and $\lambda$ (i.e., both constants decay exponentially in $\beta$, for large $\beta$).

The proof of Theorem 2.1 is given in Section 4. Here we show that the bounds (2.51) and (2.52) imply that bifurcations of stationary states for the interacting dynamics generated by $L_\lambda$ from any stationary states for $\lambda = 0$ cannot occur.

Let $\Pi = P_0 \otimes P_\Omega$ denote the projection onto the zero eigenspace of $L_0$ and set $\bar{\Pi} := \mathbb{1} - \Pi$. Assume that $\psi$ is an eigenvector of $L_\lambda$, for some $0 < |\lambda| < \lambda_0$. Using the decomposition $\bar{\Pi} = \bar{P}_0 \otimes P_\Omega + \bar{P}_\Omega$ and (2.51), (2.52), we have that

$$\|\bar{\Pi}\psi\| \geqslant \|\bar{P}_0 \otimes P_\Omega \psi\| - \|\bar{P}_\Omega \psi\| \geqslant (k(\beta) - k\,|\lambda|)\, \|\psi\|.$$

Let $\psi_0$ be an arbitrary element of $\ker L_0$. Then

$$\|\psi_0 - \psi\| \geqslant \|\bar{\Pi}\psi\| - \|\psi_0 - \Pi\psi\| \geqslant \|\bar{\Pi}\psi\| - \|\psi_0 - \psi\|,$$

so

$$\|\psi_0 - \psi\| \geqslant \tfrac{1}{2}\, (k(\beta) - k\,|\lambda|)\, \|\psi\|. \tag{2.53}$$

This shows that for $0 < |\lambda| < \min(\lambda_0, \frac{1}{2}\frac{k(\beta)}{k})$, the distance between any eigenvector of $L_\lambda$ and any eigenvector of $L_0$ is greater than $k(\beta)/4$. Combining (2.53) with (2.49) yields the following result.

**Theorem 2.2 (No Bifurcation).** Assume that the Condition A of Section 2.1.1 holds, and that $0 < |\lambda| < \min(\lambda_0, \frac{1}{2}\frac{k(\beta)}{k})$, where $\lambda_0$, $k(\beta)$, $k$ are the constants in Theorem 2.1, (2). For any normal $\sigma_{t,0}$-invariant state $\omega^0$ on $\mathfrak{M}_\beta$ and any normal $\sigma_{t,\lambda}$-invariant state $\omega^\lambda$ on $\mathfrak{M}_\beta$,

$$\|\omega^0 - \omega^\lambda\| \geqslant k(\beta)^2/16. \tag{2.54}$$

Our next result shows that the modes of the particle which are coupled to the field do not give rise to invariant states; (compare with Section 2.1.4).

**Theorem 2.3 (Instability of Normal Invariant States).** Assume that Condition B of Section 2.1.1 holds. Given any $0<\beta<\infty$ and any $r>0$, there is a $\lambda_0(\beta,r)>0$ s.t. for $0<|\lambda|\leqslant\lambda_0(\beta,r)$, the $\sigma_{t,\lambda}$-invariant normal states on $\mathfrak{M}_\beta$ are in one-to-one correspondence with the unit vectors in the set

$$\mathscr{P}\cap\operatorname{span}\{\varphi_m\otimes\varphi_n\otimes\Omega\mid m,n\in\mathscr{M}\setminus J_d,\ E(m)=E(n)\}. \tag{2.55}$$

We have $\lambda_0(\beta,r)\geqslant k\gamma^2 r$, for some $k$ which is independent of $\beta$, $r$ and where $\gamma$ is given in (2.24).

## 3. VIRIAL THEOREMS AND THE POSITIVE COMMUTATOR METHOD

Our proofs of Theorems 2.1 and 2.3 are based on the positive commutator method, which we explain in Section 3.2. In the next section we describe an essential ingredient of this method, the virial theorem.

### 3.1. Two Abstract Virial Theorems

Let $\mathscr{H}$ be a Hilbert space, $\mathscr{D}\subset\mathscr{H}$ a core for a selfadjoint operator $Y\geqslant\mathbb{1}$, and $X$ a symmetric operator on $\mathscr{D}$. We say the triple $(X,Y,\mathscr{D})$ satisfies the *GJN (Glimm–Jaffe–Nelson) Condition*, or that $(X,Y,\mathscr{D})$ is a *GJN-triple*, if there is a constant $k<\infty$, s.t. for all $\psi\in\mathscr{D}$:

$$\|X\psi\|\leqslant k\,\|Y\psi\| \tag{3.1}$$

$$\pm i\{\langle X\psi,Y\psi\rangle-\langle Y\psi,X\psi\rangle\}\leqslant k\langle\psi,Y\psi\rangle. \tag{3.2}$$

Notice that if $(X_1,Y,\mathscr{D})$ and $(X_2,Y,\mathscr{D})$ are GJN triples, then so is $(X_1+X_2,Y,\mathscr{D})$. Since $Y\geqslant\mathbb{1}$, inequality (3.1) is equivalent to

$$\|X\psi\|\leqslant k_1\,\|Y\psi\|+k_2\,\|\psi\|,$$

for some $k_1,k_2<\infty$. For a more detailed exposition of the following results (and proofs of Theorems 3.2 and 3.3) we refer to ref. 7.

**Theorem 3.1 (GJN Commutator Theorem).** If $(X,Y,\mathscr{D})$ satisfies the GJN Condition, then $X$ determines a selfadjoint operator (again denoted by $X$), s.t. $\mathscr{D}(X)\supset\mathscr{D}(Y)$. Moreover, $X$ is essentially selfadjoint on any core for $Y$, and (3.1) is valid for all $\psi\in\mathscr{D}(Y)$.

Based on the GJN commutator theorem, we next describe the setting for a general *virial theorem*. Suppose one is given a selfadjoint operator

$\Lambda \geqslant \mathbb{1}$ with core $\mathscr{D} \subset \mathscr{H}$, and operators $L, A, N, D, C_n$, $n = 0, 1, 2, 3$, all symmetric on $\mathscr{D}$, and satisfying

$$\langle \varphi, D\psi \rangle = i\{\langle L\varphi, N\psi \rangle - \langle N\varphi, L\psi \rangle\} \tag{3.3}$$

$$C_0 = L$$

$$\langle \varphi, C_n\psi \rangle = i\{\langle C_{n-1}\varphi, A\psi \rangle - \langle A\varphi, C_{n-1}\psi \rangle\}, \quad n = 1, 2, 3, \tag{3.4}$$

where $\varphi, \psi \in \mathscr{D}$. We assume that

- $(X, \Lambda, \mathscr{D})$ satisfies the GJN Condition, for $X = L, N, D, C_n$. Consequently, all these operators determine selfadjoint operators, which we denote by the same letters.
- $A$ is selfadjoint, $\mathscr{D} \subset \mathscr{D}(A)$, and $e^{itA}$ leaves $\mathscr{D}(\Lambda)$ invariant.

**Remarks.** (1) From the invariance condition $e^{itA}\mathscr{D}(\Lambda) \subset \mathscr{D}(\Lambda)$, it follows that for some $0 \leqslant k, k' < \infty$, and all $\psi \in \mathscr{D}(\Lambda)$,

$$\|\Lambda e^{itA}\psi\| \leqslant k e^{k'|t|} \|\Lambda\psi\|. \tag{3.5}$$

A proof of this can be found in ref. 1, Propositions 3.2.2 and 3.2.5.

(2) Condition (3.1) is phrased equivalently as "$X \leqslant kY$, in the sense of Kato on $\mathscr{D}$."

(3) One can show that if $(A, \Lambda, \mathscr{D})$ satisfies conditions (3.1), (3.2), then the above assumption on $A$ holds; see Theorem A.1.

**Theorem 3.2 (1st Virial Theorem).** Assume $N$ and $e^{itA}$ commute, for all $t \in \mathbb{R}$, in the strong sense on $\mathscr{D}$, and that

$$D \leqslant kN^{1/2}, \tag{3.6}$$

$$C_1 \leqslant kN^p, \qquad \text{for some} \quad 0 \leqslant p < \infty, \tag{3.7}$$

$$C_3 \leqslant kN^{1/2} \tag{3.8}$$

in the sense of Kato on $\mathscr{D}$, for some $k < \infty$. Then, if $\psi \in \mathscr{D}(L)$ is an eigenvector of $L$, there is a family of approximating eigenvectors $\{\psi_\alpha\} \subset \mathscr{D}(L) \cap \mathscr{D}(C_1)$, $\alpha > 0$, such that $\psi_\alpha \to \psi$ (in $\mathscr{H}$) as $\alpha \to 0$, and

$$\lim_{\alpha \to 0} \langle \psi_\alpha, C_1\psi_\alpha \rangle = 0. \tag{3.9}$$

**Remarks.** (1) It is not necessary that $N$ and $e^{itA}$ commute for the result to hold, but this will be the case in all our applications (see, ref. 7 for the case where $N$ and $A$ do not commute).

(2) In a heuristic way, we understand $C_1$ as the commutator $i[L, A] = i(LA - AL)$, and (3.9) as $\langle \psi, i[L, A] \psi \rangle = 0$, which is the usual statement of the virial theorem; see, e.g., refs. 1 and 8 for a comparison (and correction) of virial theorems encountered in the literature.

The Virial Theorem is still valid if we add to the operator $A$ a bounded perturbation $A_0$ leaving the domain of $L$ invariant.

**Theorem 3.3 (2nd Virial Theorem).** Suppose that we are in the situation of Theorem 3.2, and that $A_0$ is a bounded operator on $\mathscr{H}$, s.t. $\operatorname{Ran} A_0 \subseteq \mathscr{D}(L) \cap \operatorname{Ran} P(N \leqslant n_0)$, for some $n_0 < \infty$. The commutator $i[L, A_0] = i(LA_0 - A_0 L)$ is well defined in the strong sense on $\mathscr{D}(L)$. For the same family of approximating eigenvectors as in the previous theorem, we have that

$$\lim_{\alpha \to 0} \langle \psi_\alpha, (C_1 + i[L, A_0]) \psi_\alpha \rangle = 0. \tag{3.10}$$

## 3.2. Outline of the Proofs of Theorems 2.1 and 2.3; the Positive Commutator Method

The positive commutator method gives a conceptually easy proof of the absence of point spectrum of $L$. We outline a version that is adapted to the proofs of Theorems 2.1 and 2.3. The full proofs are given in Sections 4 and 5. In the present section we use the notation of Section 3.1 and write $i[L_\lambda, A]$ for $C_1$.

*Outline of the Proof of Theorem 2.3.* According to the discussion of Section 2.1.4 we have to show that $\mathscr{P} \cap \ker L_\lambda$ is given by the set (2.55). The Liouvillian $L_\lambda$ is reduced by the decomposition

$$\mathscr{H} = \operatorname{Ran} P^0 \oplus \operatorname{Ran} P \oplus \operatorname{Ran} P^l \oplus \operatorname{Ran} P^r,$$

where the various projections are defined in Section 2.1.5. It is easy to see that $\mathscr{P} \cap \operatorname{Ran} P^l = \mathscr{P} \cap \operatorname{Ran} P^r = \{0\}$ and that $L_\lambda \upharpoonright \operatorname{Ran} P^0 = L_0 \upharpoonright \operatorname{Ran} P^0$. Consequently, $\mathscr{P} \cap \ker L_\lambda = \mathscr{P} \cap (\ker L_0 \upharpoonright \operatorname{Ran} P^0 \cup \ker L_\lambda \upharpoonright \operatorname{Ran} P)$, and to prove the theorem, it is enough to show that

$$\ker L_\lambda \upharpoonright \operatorname{Ran} P = \{0\}. \tag{3.11}$$

We construct selfadjoint operators $\Lambda, A, A_0$ such that the conditions of Section 3.1 are fulfilled, with $L = L_\lambda$ and $N = d\Gamma(\mathbb{1})$. The operators $A$ and $A_0$ have the properties that

$$i[L_\lambda, A] + i[L, A_0] \geqslant PM_0P + M_1,$$

where the bounded operators $M_0$ and $M_1$ satisfy

$$\begin{aligned} PM_1P &= 0, \\ \langle \psi, M_0 \psi \rangle &\geqslant \delta \, \|\psi\|^2, \end{aligned} \tag{3.12}$$

for any $\psi \in \ker L_\lambda \restriction \operatorname{Ran} P$, and where $\delta$ is strictly positive. From Theorem 3.3 we obtain

$$0 = \lim_\alpha \, \langle \psi_\alpha, (i[L_\lambda, A] + i[L_\lambda, A_0]) \, \psi_\alpha \rangle \geqslant \delta \, \|\psi\|^2.$$

Because $\delta > 0$, we have that $\ker L_\lambda \restriction \operatorname{Ran} P = \{0\}$.

We now explain how to arrive at the key inequality (3.12). The form of the non-interacting Liouvillian, $L_0$, given in (2.42) suggests to consider

$$A = \chi A_p \chi \otimes \mathbb{1}_p \otimes \mathbb{1}_f - \mathbb{1}_p \otimes \chi A_p \chi \otimes \mathbb{1}_f + \mathbb{1}_p \otimes \mathbb{1}_p \otimes \mathrm{d}\Gamma(i\partial_u),$$

where $A_p$ is the dilation generator, see (3.40), and $\mathrm{d}\Gamma(i\partial_u)$ is the second quantization of the translation generator in the radial variable $u$ of $L^2(\mathbb{R} \times S^2, du \times d\Sigma)$, see (2.28). Here, $\chi$ is a function of $H_p$ with support in an interval in $\mathbb{R}_+$, containing $J_c$ but not $\{0\}$, and such that $\chi|_{J_c} = 1$. Then we have

$$i[L_\lambda, A] = \chi^2 H_p \otimes \mathbb{1}_p \otimes \mathbb{1}_f + \mathbb{1}_p \otimes \chi^2 H_p \otimes \mathbb{1}_f + N + U + \lambda I_1,$$

where $N$ is the number operator, $U$ is a non-negative operator, because of the choice of the parameter $\mu$ in the potential $v$ of Eq. (2.2), and where $I_1 = i[I, A]$ is infinitesimally small w.r.t. $N$,

$$\pm \lambda I_1 \leqslant cN + \frac{\lambda^2}{c} k, \tag{3.13}$$

for any $c > 0$ and some $k < \infty$. The role of $\chi$ is to project out the discrete modes in $J_d$. Using that $P = p \otimes p \otimes \mathbb{1}_f$, $p\chi^2 = p_{J_c}$, and that $p_{J_c} H_p \geqslant r p_{J_c}$, since the interval $J_c$ is away from the origin by a distance of at least $r$, we obtain

$$\begin{aligned} &Pi[L_\lambda, A] \, P \\ &\quad \geqslant P \, (p_{J_c} H_p \otimes \mathbb{1}_p \otimes P_\Omega + \mathbb{1}_p \otimes p_{J_c} H_p \otimes P_\Omega + \tfrac{1}{2} \, \bar{P}_\Omega) \, P - k\lambda^2 P \\ &\quad \geqslant \min(r, 1/2) \, P \, (p_{J_c} \otimes \mathbb{1}_p \otimes P_\Omega + \mathbb{1}_p \otimes p_{J_c} \otimes P_\Omega + \tfrac{1}{2} \, \bar{P}_\Omega) \, P - k\lambda^2 P, \end{aligned}$$

where we choose $c=1/2$ in (3.13). The first term on the r.h.s. is strictly positive except on the subspace Ran $p_{J_d}\otimes p_{J_d}\otimes P_\Omega\subset$ Ran $P$. We decompose

$$p_{J_d}\otimes p_{J_d}\otimes P_\Omega = P\Pi + \sum_{\substack{m,n\in J_d\\ E(m)\neq E(n)}} p_m\otimes p_n\otimes P_\Omega, \tag{3.14}$$

where $\Pi$ is the projection onto the kernel of $L_0$. Note that $P\Pi$ is finite-dimensional. On the range of the second projection on the r.h.s. of (3.14), the free Liouvillian satisfies

$$|L_0|\geqslant \min\{|E(m)-E(n)|\,|\,m,n\in J_d,\,E(m)\neq E(n)\}>0. \tag{3.15}$$

Let $\Delta$ be an interval around zero whose size $|\Delta|$ is smaller than the r.h.s. of (3.15), and let $E^0_\Delta$ be the spectral projection of $L_0$ onto $\Delta$. Then we have that $E^0_\Delta p_{J_d}\otimes p_{J_d}\otimes P_\Omega = E^0_\Delta P\Pi = P\Pi$. Consider the decomposition

$$\text{Ran } PE^0_\Delta = \text{Ran } PE^0_\Delta\bar{P}\oplus \text{Ran } P\Pi. \tag{3.16}$$

From the above discussion it is apparent that on the block Ran $PE^0_\Delta\bar{P}$, $i[L_\lambda, A]$ is bigger than $r-k\lambda^2\geqslant r/2$ (we require $|\lambda|\leqslant k\sqrt{r}$), while the commutator is zero on the block Ran $P\Pi$. This is where we introduce the operator $A_0$.

One can choose $A_0$ s.t. $P\Pi i[L_\lambda, A_0]\,\Pi P$ is strictly positive provided the interaction satisfies the Fermi Golden Rule Condition. Moreover, on Ran $PE^0_\Delta\bar{\Pi}$, $i[L_\lambda, A_0]$ is small relative to $r$. The construction of $A_0$ has been given, in the context of zero temperature systems, in ref. 4, and has been modified for positive temperature systems in ref. 10.

The above discussion shows that the operator $i[L_\lambda, A]+i[L_\lambda, A_0]$ has strictly positive diagonal blocks in the decomposition (3.16). An application of the Feshbach method then shows that

$$E^0_\Delta P(i[L_\lambda, A]+i[L_\lambda, A_0])\,PE^0_\Delta =: E^0_\Delta PM_0PE^0_\Delta \tag{3.17}$$

is strictly positive. Since $\|E^0_\Delta\psi-\psi\|\leqslant k\,|\lambda|\,\|\psi\|$, for any $\psi\in\ker L_\lambda$, we can pass from (3.17) to estimate (3.12).

In this proof, the coupling constant cannot be chosen idependently of the inverse temperature $\beta$. This is due to the fact that the constant $\delta$ in (3.12) is proportional to $\gamma$, see (2.24), which in turn decays exponentially in $\beta$. We have to require that certain error terms which depend on $\lambda$ are small w.r.t. $\gamma$, hence the $\beta$-dependent smallness condition on $\lambda$.

*Outline of the Proof of Theorem 2.1.* Part (1) is an easy consequence of the virial theorem combined with the bound

$$i[L_\lambda, A] \geqslant \tfrac{1}{2} N - k\lambda^2,$$

for $A = \mathrm{d}\Gamma(i\partial_u)$. The proof of part (2) proceeds as follows. We construct $A_0$ (the same as for the proof of Theorem 2.1) s.t.

$$i[L_\lambda, A] + i[L_\lambda, A_0] \geqslant \kappa_1 \Pi - \kappa_2 \bar{P}_0 \otimes P_\Omega, \tag{3.18}$$

for some $\kappa_1, \kappa_2 > 0$, and where $\Pi$ is the projection onto the kernel of $L_0$. If $\psi$ is an eigenvector of $L_\lambda$ then by part (1) we have that $\|\Pi\psi\| \geqslant (1 - k\,|\lambda|)\,\|\psi\| - \|\bar{P}_0 \otimes P_\Omega \psi\|$. Inserting this bound into (3.18) and using the virial theorem yields the bound (2.52).

This outline indicates that the proofs of Theorems 2.1 and 2.3 consist of two steps. First we verify that the virial theorems are applicable and second establish a positive commutator estimate in the above sense. The latter task is carried out in Sections 4 and 5.

### 3.3. Applications of the Virial Theorems

Corresponding to the different hypotheses of Theorems 2.1 and 2.3, we introduce two sets of operators $\Lambda$, $L$, $A$, $A_0$, $N$, and verify, in each case, that the virial theorems are applicable. The following objects appear in both applications: the Hilbert space is the GNS space given in (2.27); the dense domain $\mathscr{D}$ is chosen to be

$$\mathscr{D} = C_0^\infty(\mathbb{R}^3) \otimes C_0^\infty(\mathbb{R}^3) \otimes \mathscr{D}_f, \tag{3.19}$$

where

$$\mathscr{D}_f = \mathscr{F}(C_0^\infty(\mathbb{R} \times S^2)) \cap \mathscr{F}_0,$$

where the Fock space $\mathscr{F}$ has been defined in (2.29), and $\mathscr{F}_0$ denotes the finite-particle subspace. The operator $L$ is the interacting Liouvillian introduced in (2.45), and $N = \mathrm{d}\Gamma(\mathbb{1})$ is the particle number operator in $\mathscr{F} \equiv \mathscr{F}(L^2(\mathbb{R} \times S^2))$. Clearly, $X = L, N$ are symmetric operators on $\mathscr{D}$. The operator $D$, defined in (3.3), is given by

$$\begin{aligned} D = i\lambda \sum_\alpha \{ &G_{\alpha,\#} \otimes \mathbb{1}_p \otimes (-a^*(\tau_\beta(g_\alpha)) + a(\tau_\beta(g_\alpha))) \\ &- \mathbb{1}_p \otimes \mathscr{C}_p G_{\alpha,\#} \mathscr{C}_p \otimes (-a^*(e^{-\beta u/2}\tau_\beta(g_\alpha)) + a(e^{-\beta u/2}\tau_\beta(g_\alpha)))\}. \end{aligned} \tag{3.20}$$

We define a bounded, selfadjoint operator $A_0$ on $\mathscr{H}$ by

$$A_0 = i\theta\lambda(\Pi I R_\epsilon^2 \bar{\Pi} - \bar{\Pi} R_\epsilon^2 I \Pi), \tag{3.21}$$

$$R_\epsilon^2 = (L_0^2 + \epsilon^2)^{-1}. \tag{3.22}$$

Here, $\theta$ and $\epsilon$ are positive parameters, and $\Pi$ is the projection

$$\Pi = P_0 \otimes P_\Omega, \tag{3.23}$$

$$P_0 = P(L_p = 0), \tag{3.24}$$

$$\bar{\Pi} = \mathbb{1} - \Pi. \tag{3.25}$$

We also introduce the notation

$$\bar{R}_\epsilon = \bar{\Pi} R_\epsilon.$$

Notice that the operator $A_0$ satisfies the conditions given in Theorem 3.3 with $n_0 = 1$. Moreover, $[L, A_0] = LA_0 - A_0 L$ extends to a bounded operator on the entire Hilbert space, and

$$\|[L, A_0]\| \leqslant k\left(\frac{\theta\,|\lambda|}{\epsilon} + \frac{\theta\lambda^2}{\epsilon^2}\right). \tag{3.26}$$

This choice for the operator $A_0$ was initially introduced in ref. 4 for the spectral analysis of Pauli–Fierz Hamiltonians (zero temperature systems), and was adopted in ref. 10 to show return to equilibrium (positive temperature systems). The key feature of $A_0$ is that

$$i\Pi[L, A_0]\,\Pi = 2\theta\lambda^2 \Pi I\, \bar{R}_\epsilon^2 I \Pi$$

is a non-negative operator. The Fermi Golden Rule Condition, (2.20) (or (2.23)), says that it is a *strictly positive operator* on Ran $\Pi$.

**Proposition 3.2.** Assume (2.20) and let $0 < \epsilon < \epsilon_0$. Then

$$\Pi I\, \bar{R}_\epsilon^2 I \Pi \geqslant \tfrac{1}{\epsilon}\,\gamma \Pi, \tag{3.27}$$

where $\gamma$ is given by (2.21). Assuming condition (2.23) instead of (2.20), the same lower bound holds (with $\gamma$ given in (2.24)) if we replace $\Pi$ by $\Pi P$ ($P = p \otimes p$, $p = p_{J_d} + p_{J_c}$) and $I$ by the regularized interaction; see (2.13).

The proof of Proposition 3.2 is given in Section A.4.

Next, we define the operators $\Lambda$ and $A$ and verify the hypotheses used in Section 3.1.

### *3.3.1. Setting for Theorem 2.1*

We define

$$\Lambda = \Lambda_p \otimes \mathbb{1}_p \otimes \mathbb{1}_f + \mathbb{1}_p \otimes \Lambda_p \otimes \mathbb{1}_f + \mathbb{1}_p \otimes \mathbb{1}_p \otimes \Lambda_f, \tag{3.28}$$

$$\Lambda_p = H_p - E_0 + 1, \tag{3.29}$$

$$\Lambda_f = \mathrm{d}\Gamma(u^2+1), \tag{3.30}$$

where, we recall, $E_0 = \inf \sigma(H_p) < 0$. Clearly, $\Lambda$ is essentially selfadjoint on the domain $\mathscr{D}$ defined in (3.19), and $\Lambda_p \geqslant \mathbb{1}$, $\Lambda_f \geqslant 0$. In what follows we shall often use the standard fact that if $f \in L^2(\mathbb{R} \times S^2, du \times d\Sigma)$, then $a^{\#}(f)$ is relatively $N^{1/2}$ bounded in the sense of Kato. This implies immediately that $a^{\#}(f)$ is relatively $\Lambda_f^{1/2}$ bounded.

We verify that $(L, \Lambda, \mathscr{D})$ is a GJN triple. The bound (3.1) is trivial by the above observation, and the fact that $\tau_\beta(g_\alpha) \in L^2(\mathbb{R} \times S^2)$. Next, the only contribution to the commutator of $L$ with $\Lambda$ comes from the interaction, and a typical term to estimate is of the form $[G_\alpha, H_p] \otimes \mathbb{1} \otimes \varphi(\tau_\beta(g_\alpha)) + G_\alpha \otimes \mathbb{1}_p \otimes [\varphi(\tau_\beta(g_\alpha)), \Lambda_f]$. Using the bound (2.18), we obtain for the first term

$$\begin{aligned} &|\langle \psi, [G_\alpha, H_p] \otimes \mathbb{1}_p \otimes \varphi(\tau_\beta(g_\alpha))\, \psi \rangle| \\ &\quad \leqslant k\, \|[G_\alpha, H_p](H_p - E_0 + 1)^{-1/2}\|\, \|\Lambda_p^{1/2} \otimes \mathbb{1}_p \psi\|\, \|\Lambda_f^{1/2} \psi\| \\ &\quad \leqslant k \langle \psi, \Lambda \psi \rangle. \end{aligned} \tag{3.31}$$

Next,

$$\begin{aligned} |\langle \psi, G_\alpha \otimes \mathbb{1}_p \otimes [\varphi(\tau_\beta(g_\alpha)), \Lambda_f]\, \psi \rangle| &\leqslant k\, \|(u^2+1)^{1/2}\, \tau_\beta(g_\alpha)\|_{L^2(\mathbb{R} \times S^2)}\, \|\Lambda^{1/2} \psi\|^2 \\ &\leqslant k \langle \psi, \Lambda \psi \rangle, \end{aligned} \tag{3.32}$$

where we have used that

$$[a^*(\tau_\beta(g_\alpha)), \Lambda_f] = a^*((u^2+1)\, \tau_\beta(g_\alpha)),$$

$$[a(\tau_\beta(g_\alpha)), \Lambda_f] = -a((u^2+1)\, \tau_\beta(g_\alpha)),$$

so that $[\varphi(\tau_\beta(g_\alpha)), \Lambda_f]$ is still $N^{1/2}$ bounded, since $\tau_\beta(g_\alpha)$ has the decay property (2.17). The form bound (3.2) follows from these observations. In a similar way, one shows that $(D, \Lambda, \mathscr{D})$ is a GJN triple.

Next, we define the operator $A \equiv A_f$ to be the selfadjoint generator of the translation group acting on the radial variable of elements in $\mathscr{F}$ by

$$[e^{itA_f} \psi]_n\, (u_1, \Sigma_1, \ldots, u_n, \Sigma_n) = [\psi]_n\, (u_1 - t, \Sigma_1, \ldots, u_n - t, \Sigma_n), \quad t \in \mathbb{R}.$$

In what follows, we will often not display the angular variables $\Sigma_1, \ldots, \Sigma_n$. We set $e^{itA_f}\Omega := \Omega$. Clearly, $\mathscr{D} \subset \mathscr{D}(A_f)$, $\mathscr{D}$ is invariant under $e^{itA_f}$, hence a core for $A_f$, and $A_f$ acts on $\mathscr{D}$ as

$$A_f = \mathrm{d}\Gamma(i\partial_u). \tag{3.33}$$

An easy calculation shows that, on $\mathscr{D}$,

$$\Lambda e^{itA_f} = e^{itA_f}(\Lambda + \mathrm{d}\Gamma(2ut - t^2)), \tag{3.34}$$

so estimate (3.5), with $k' = 0$, is satisfied for all $\psi \in \mathscr{D}$, hence for all $\psi \in \mathscr{D}(\Lambda)$. For $A := A_f$, we find that

$$C_1 = N + \lambda I_1, \tag{3.35}$$

$$C_2 = \lambda I_2, \tag{3.36}$$

$$C_3 = \lambda I_3, \tag{3.37}$$

where

$$I_n = i^n \sum_\alpha \{G_\alpha \otimes \mathbb{1}_p \otimes \varphi((-i\partial_u)^n \tau_\beta(g_\alpha)) - \mathbb{1}_p \otimes \mathscr{C}_p G_\alpha \mathscr{C}_p \otimes \varphi((-i\partial_u)^n e^{-\beta u/2} \tau_\beta(g_\alpha))\}. \tag{3.38}$$

We now show that $(C_n, \Lambda, \mathscr{D})$ are GJN triples, for $n = 1, 2, 3$. The operators $I_n$ are $N^{1/2}$-bounded, since $\tau_\beta(g_\alpha)$ and $e^{-\beta u/2}\tau_\beta(g_\alpha)$ are in in the domain of the operators $(i\partial_u)^n$, $n = 1, 2, 3$ (see also (2.16)), hence (3.1) holds. Note that this also yields (3.7). Next, we need to calculate the commutators of $C_n$ with $\Lambda$. The estimates on the commutators of $I_n$ with $\Lambda$, for $n = 2, 3$, are similar to the ones for $n = 1$. The latter has been outlined above; it requires that $(u^2+1)(-i\partial_u)^n \tau_\beta(g_\alpha) \in L^2(\mathbb{R} \times S^2)$, which is guaranteed by conditions (2.16) and (2.17).

This discussion shows that we are in the situation described in Section 3.1, and Theorems 3.2 and 3.3 apply.

### *3.3.2. Setting for Theorem 2.3*

We define the operator $\Lambda$ as in (3.28), but where $\Lambda_p$ is now given by

$$\Lambda_p = -\Delta + x^2. \tag{3.39}$$

$\Lambda$ is essentially selfadjoint on $\mathscr{D}$ (see (3.19)), $\Lambda_p \geqslant \mathbb{1}$, $\Lambda_f \geqslant 0$.

Verifying that $(L_J, \Lambda, \mathscr{D})$ is a GJN triple is done as in Section 3.3.1, using that $[G_{\alpha,J}, \Lambda_p]$ is bounded; see Lemma A.5. It is also easy to check that $(D, \Lambda, \mathscr{D})$ is a GJN triple.

Next, we define an operator $A$ differing substantially from the choice $A = A_f$ (see (3.33)) in Section 3.3.1: We add a (regularized) dilatation on the particle space to $A_f$.

Let $\chi \in C_0^\infty(\mathbb{R}_+)$ be a smooth characteristic function of the set $J_c$ (with the property $\chi \mid_{J_c} = 1$), which has compact support not containing zero. We define $\chi(H_p) = \int \hat{\chi}(s)\, e^{isH_p}$, where $\hat{\chi}$ is the Fourier transform of $\chi$, and we abbreviate $\chi(H_p)$ by $\chi$. Let $A_p$ be the symmetric operator on $C_0^\infty(\mathbb{R}^3)$ given by

$$A_p = -\frac{i}{4}(x \cdot \nabla + \nabla \cdot x). \tag{3.40}$$

Notice that $(A_p, \Lambda_p, C_0^\infty(\mathbb{R}^3))$ is a GJN triple, so $A_p$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)$. We denote the selfadjoint closure again by $A_p$.

**Remark.** To show that $A_p$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)$, we can also use the fact that the dense set $C_0^\infty(\mathbb{R}^3)$ is invariant under the group of dilatations on $L^2(\mathbb{R}^3, d^3x)$, hence a core for the selfadjoint generator of this group. The generator acts on $C_0^\infty$ as in (3.40).

**Proposition 3.3.** $(\chi A_p \chi, \Lambda_p, C_0^\infty(\mathbb{R}^3))$ is a GJN triple. In particular, $\chi A_p \chi$ is well defined and symmetric on $C_0^\infty(\mathbb{R}^3)$, and it is essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)$. We denote the selfadjoint closure again by $\chi A_p \chi$.

We give the proof in Section A.2. Let us now define the operator

$$A = \chi A_p \chi \otimes \mathbb{1}_p - \mathbb{1}_p \otimes \chi A_p \chi + A_f, \tag{3.41}$$

which is essentially selfadjoint on $\mathscr{D}$. It follows immediately from Proposition 3.2, Theorem 1.1, and relation (3.34), that $e^{itA}$ leaves $\mathscr{D}(\Lambda)$ invariant, and that the estimate (3.5) holds true. We calculate explicitly

$$C_n = \delta_{n,1} N + \mathrm{ad}^{(n)}_{\chi A_p \chi}(H_p) \otimes \mathbb{1}_p + (-1)^n\, \mathbb{1}_p \otimes \mathrm{ad}^{(n)}_{\chi A_p \chi}(H_p) + \lambda I_n, \tag{3.42}$$

for $n = 1, 2, 3$, where we define the multiple commutators $\mathrm{ad}^{(0)}_Y(X) = X$, and for $n \geqslant 1$, $\mathrm{ad}^{(n)}_Y(X) = i[\mathrm{ad}^{(n-1)}_Y(X), Y]$, in the weak sense on $C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$. For $n = 1, 2, 3$, we have defined

$$\begin{aligned} I_n = \sum_{k=0}^{n} \binom{n}{k} 2^{-(n-k)} \sum_\alpha \{&\mathrm{ad}^{(n-k)}_{\chi A_p \chi}(G_\alpha) \otimes \mathbb{1}_p \otimes \varphi((-i\partial_u)^k\, \tau_\beta(g_\alpha)) \\ &+ (-1)^{n-k}\, \mathbb{1}_p \otimes \mathrm{ad}^{(n-k)}_{\chi A_p \chi}(G_\alpha) \otimes \varphi((-i\partial_u)^k\, e^{\beta u/2} \tau_\beta(g_\alpha))\}. \end{aligned} \tag{3.43}$$

Note that

$$\mathrm{ad}^{(1)}_{\chi A_p \chi}(H_p) = \chi(H_p + W)\,\chi, \tag{3.44}$$

with

$$W = -\frac{1}{2}\,(x\cdot\nabla v + 2v) = \frac{1}{2}\left(\frac{\rho'(|x|)}{|x|^{\mu}} + (1-\mu)\,\frac{\rho(|x|)}{|x|^{1+\mu}}\right), \tag{3.45}$$

and the choice of $\mu$, $\rho$ given in (2.2) implies that

$$W \geqslant 0. \tag{3.46}$$

In Appendix A.3 we prove the following proposition.

**Proposition 3.4.** $(C_n, A, \mathscr{D})$ are GJN triples, for $n = 1, 2, 3$, and the estimates (3.6)–(3.8) are satisfied.

This shows that with the choice of operators introduced in this section, Theorems 3.2 and 3.3 apply.

## 4. PROOF OF THEOREM 2.1

(1) Set $\tilde{I}_1 = i[I, A_f]$, where $A_f$ and $I$ are given in (3.33), (2.45). For $\psi \in \mathscr{D}(N^{1/2})$, we have

$$|\langle \psi, \tilde{I}_1 \psi\rangle| \leqslant |\langle \psi, \tilde{I}_1 \bar{P}_\Omega \psi\rangle| + |\langle \psi, \bar{P}_\Omega \tilde{I}_1 P_\Omega \psi\rangle| \leqslant 2\,\|\tilde{I}_1 N^{-1/2} \bar{P}_\Omega\|\,\|\psi\|\,\|N^{1/2}\psi\|.$$

This shows that in the sense of quadratic forms on $\mathscr{D}(N)$, $\lambda \tilde{I}_1 \geqslant -cN - \frac{\lambda^2 k}{c}$, for any $c > 0$, where $k = \|\tilde{I}_1 N^{-1/2} \bar{P}_\Omega\|^2 \leqslant k' \sum_\alpha \|\partial_u \tau_\beta(g_\alpha)\|^2_{L^2} \leqslant k'(1+1/\beta)$. Let $\tilde{C}_1 = i[L_\lambda, A_f] = N + \lambda \tilde{I}_1$ and choose $c = 1/2$. Then we find $\tilde{C}_1 \geqslant \frac{1}{2} N - k\lambda^2$, in the sense of forms on $\mathscr{D}(N) \subset \mathscr{D}(\tilde{C}_1)$, and from Theorem 3.2

$$0 = \lim_{\alpha \to 0}\,\langle \psi_\alpha, \tilde{C}_1 \psi_\alpha\rangle \geqslant \tfrac{1}{2} \lim_{\alpha \to 0}\,\|N^{1/2}\psi_\alpha\|^2 - k\lambda^2\,\|\psi\|^2,$$

where $\psi$ is an eigenvector of $L_\lambda$, and $\psi_\alpha$ its regularization. It follows that

$$\lim_{\alpha \to 0}\,\|N^{1/2}\psi_\alpha\|^2 \leqslant k\lambda^2\,\|\psi\|^2,$$

which tells us that $\psi \in \mathscr{D}(N^{1/2})$, and that $\|N^{1/2}\psi\| \leqslant k\,|\lambda|\,\|\psi\|$.

(2) In what follows, the constants $k, k_1, \lambda_1, \lambda_2$ are independent of $\beta \geqslant \beta_0$, where $\beta_0 > 0$ is arbitrary but fixed. In the course of the proof we

will impose several conditions on the parameters $\epsilon$, $\lambda$, $\theta$ which are collected in (4.14). We adopt the notation of Section 3.3.1.

On $\mathscr{D}(N) \subset \mathscr{D}(C_1)$, we define the operator

$$B = C_1 + i[L_\lambda, A_0].$$

Recall that $A_0$ is defined in (3.21), that we write $\bar{R}_\epsilon = \bar{\Pi} R_\epsilon$, and that $P_\Omega I P_\Omega = P_\Omega I_1 P_\Omega = 0$. Using that $P_\Omega = \Pi + \bar{P}_0 \otimes P_\Omega$, one finds that

$$\begin{aligned} P_\Omega B P_\Omega &= \Pi B \Pi + \bar{P}_0 \otimes P_\Omega B \Pi + \Pi B \bar{P}_0 \otimes P_\Omega + \bar{P}_0 \otimes P_\Omega B \bar{P}_0 \otimes P_\Omega \\ &= 2\theta\lambda^2 \Pi I \bar{R}_\epsilon^2 I \Pi + \theta\lambda^2 (\bar{P}_0 \otimes P_\Omega I \bar{R}_\epsilon^2 I \Pi + \Pi I \bar{R}_\epsilon^2 I \bar{P}_0 \otimes P_\Omega), \qquad (4.1) \\ P_\Omega B \bar{P}_\Omega &= \lambda P_\Omega I_1 \bar{P}_\Omega + \theta\lambda \Pi I \bar{R}_\epsilon^2 L_0 \bar{P}_\Omega + \theta\lambda^2 \Pi I \bar{R}_\epsilon^2 I \bar{P}_\Omega, \qquad (4.2) \\ \bar{P}_\Omega B P_\Omega &= \lambda \bar{P}_\Omega I_1 P_\Omega + \theta\lambda \bar{P}_\Omega L_0\, \bar{R}_\epsilon^2 I \Pi + \theta\lambda^2 \bar{P}_\Omega I \bar{R}_\epsilon^2 I \Pi, \qquad (4.3) \\ \bar{P}_\Omega B \bar{P}_\Omega &= \bar{P}_\Omega N + \bar{P}_\Omega (\lambda I_1 - \theta\lambda^2 (I \Pi I \bar{R}_\epsilon^2 + \bar{R}_\epsilon^2 I \Pi I))\, \bar{P}_\Omega. \end{aligned}$$

From the estimates $\|\bar{P}_\Omega N^{-1/2} I_1 N^{-1/2} \bar{P}_\Omega\| \leqslant k$, $\|I \Pi I\| \leqslant k$, $\|\bar{R}_\epsilon^2\| \leqslant \epsilon^{-2}$, we see that there is some constant $\lambda_1 < \infty$ (independent of $\lambda$, $\epsilon$, $\theta$), s.t.

$$\bar{P}_\Omega B \bar{P}_\Omega \geqslant \tfrac{1}{2}\, \bar{P}_\Omega, \qquad (4.4)$$

provided

$$|\lambda|, \qquad \frac{\theta\lambda^2}{\epsilon^2} < \lambda_1, \qquad (4.5)$$

see also (4.14). Using the estimates

$$\|I \bar{R}_\epsilon^2 I \Pi\|, \|\Pi I \bar{R}_\epsilon^2 I\| \leqslant \epsilon^{-2} k \qquad \text{and} \qquad \|P_\Omega I_1\|, \|I_1 P_\Omega\| \leqslant k,$$

where $k$ is independent of the parameters $\lambda$, $\theta$, $\epsilon$, we arrive at the following lower bound. For any $\phi \in \mathscr{D}(N)$ and some $k_1 < \infty$

$$\begin{aligned} \langle B \rangle_\phi \geqslant{}& 2\theta\lambda^2 \langle \Pi I \bar{R}_\epsilon^2 I \Pi \rangle_\phi + \frac{1}{2} \|\bar{P}_\Omega \phi\|^2 \\ & - k_1 \frac{\theta\lambda^2}{\epsilon^2} \|\Pi\phi\|\, \|\bar{P}_0 \otimes P_\Omega \phi\| - k_1\, |\lambda|\, \|P_\Omega \phi\|\, \|\bar{P}_\Omega \phi\| \\ & - \left(2\theta\, |\lambda| + 2k_1 \frac{\theta\lambda^2}{\epsilon}\right) \|\bar{R}_\epsilon I \Pi \phi\|\, \|\bar{P}_\Omega \phi\|. \qquad (4.6) \end{aligned}$$

Clearly, $\| \bar{R}_\epsilon I\Pi\phi\| \, \|\bar{P}_\Omega\phi\| \leqslant \delta\langle \Pi I\bar{R}_\epsilon^2 I\Pi\rangle_\phi + \delta^{-1} \|\bar{P}_\Omega\phi\|^2$, for any $\delta > 0$. Choosing appropriate values of $\delta$, we bound the last line in (4.6) from below by

$$-\theta\lambda^2\langle \Pi I\bar{R}_\epsilon^2 I\Pi\rangle_\phi - 4\left(\theta + k_1^2\,\frac{\theta\lambda^2}{\epsilon^2}\right)\|\bar{P}_\Omega\phi\|^2,$$

and it follows that

$$\begin{aligned}\langle B\rangle_\phi \geqslant{}& \frac{\theta\lambda^2}{\epsilon}\,\gamma\,\|\Pi\phi\|^2 + \left(\frac{1}{2} - 4\theta - 4k_1^2\,\frac{\theta\lambda^2}{\epsilon^2}\right)\|\bar{P}_\Omega\phi\|^2\\ &- k_1\,\frac{\theta\lambda^2}{\epsilon^2}\,\|\Pi\phi\|\,\|\bar{P}_0\otimes P_\Omega\phi\| - k_1\,|\lambda|\,\|P_\Omega\phi\|\,\|\bar{P}_\Omega\phi\|,\end{aligned}\tag{4.7}$$

where we have used (3.27) and hence assumed that $0 < \epsilon < \epsilon_0$. Using that $\|P_\Omega\phi\| \leqslant \|\Pi\phi\| + \|\bar{P}_0\otimes P_\Omega\phi\|$, we estimate the two terms in the last line on the r.h.s. of (4.7) as

$$-k_1\,|\lambda|\,\|P_\Omega\phi\|\,\|\bar{P}_\Omega\phi\| \geqslant -\frac{1}{4}\,\|\bar{P}_\Omega\phi\|^2 - 8\lambda^2k_1^2(\|\Pi\phi\|^2 + \|\bar{P}_0\otimes P_\Omega\phi\|^2),$$

$$-k_1\,\frac{\theta\lambda^2}{\epsilon^2}\,\|\Pi\phi\|\,\|\bar{P}_0\otimes P_\Omega\phi\| \geqslant -\frac{1}{2}\,\frac{\theta\lambda^2}{\epsilon}\,\gamma\,\|\Pi\phi\|^2 - 2k_1^2\,\frac{\theta\lambda^2}{\epsilon^3}\,\gamma\,\|\bar{P}_0\otimes P_\Omega\phi\|^2.$$

Using these two estimates in (4.7), we arrive at

$$\langle B\rangle_\phi \geqslant \lambda^2 = \left(\frac{1}{2}\,\frac{\theta}{\epsilon}\,\gamma - 8k_1^2\right)\|\Pi\phi\|^2 - 2\lambda^2k_1^2\left(4 + \frac{\theta}{\epsilon^3\gamma}\right)\|\bar{P}_0\otimes P_\Omega\phi\|^2,\tag{4.8}$$

where we require the condition

$$\frac{1}{4} - 4\theta - 4k_1^2\,\frac{\theta\lambda^2}{\epsilon^2} \geqslant 0,\tag{4.9}$$

which guarantees that the contribution of the term in (4.7) which is proportional to $\|\bar{P}_\Omega\phi\|^2$ is non-negative, and can hence be dropped. (4.9) is satisfied if (4.14) holds. Let $\phi = \psi_\alpha \in \mathscr{D}(N)$ be the regularization of the eigenvector $\psi$ as defined in Theorem 3.2. Then it follows from (4.8) that

$$0 = \lim_{\alpha\to 0}\,\langle B\rangle_{\psi_\alpha} \geqslant \kappa_1\,\|\Pi\phi\|^2 - \kappa_2\,\|\bar{P}_0\otimes P_\Omega\phi\|^2,\tag{4.10}$$

where

$$\kappa_1 = \frac{1}{2}\frac{\theta}{\epsilon}\gamma - 8k_1^2 > k_1^2, \tag{4.11}$$

$$\kappa_2 = 2k_1^2\left(4 + \frac{\theta}{\gamma\epsilon^3}\right) > 0. \tag{4.12}$$

The lower bound (4.11) is a consequence of $\epsilon < \frac{\theta\gamma}{18k_1^2}$, see (4.14).

From (2.51), we find that

$$\|\Pi\psi\| \geqslant \|\psi\| - \|\bar{P}_\Omega\psi\| - \|\bar{P}_0 \otimes P_\Omega\psi\| \geqslant (1 - k\,|\lambda|)\,\|\psi\| - \|\bar{P}_0 \otimes P_\Omega\psi\|.$$

Thus there is a positive constant $\lambda_2$ (independent of $\epsilon, \lambda, \theta$) s.t. if $0 < |\lambda| < \lambda_2$ then $\|\Pi\psi\| \geqslant \frac{1}{2}\|\psi\| - \|\bar{P}_0 \otimes P_\Omega\psi\|$. Thus we get from (4.10)

$$\|\bar{P}_0 \otimes P_\Omega\psi\| \geqslant \frac{1}{2}\frac{1}{1 + (\kappa_2/\kappa_1)^{1/2}}\|\psi\|. \tag{4.13}$$

Consequently, under the conditions that

$$0 < |\lambda| < \min\left\{\lambda_1, \lambda_2, \frac{\epsilon}{\sqrt{\theta}}\left(\sqrt{\lambda_1} + \frac{1}{4\sqrt{2}\,k_1}\right)\right\}, \theta < \frac{1}{32}, \epsilon < \min\left\{\frac{\theta\gamma}{18k_1^2}, \epsilon_0\right\}, \tag{4.14}$$

we obtain

$$\|\bar{P}_0 \otimes P_\Omega\psi\| \geqslant \frac{1}{2}\frac{1}{1 + \sqrt{2(4 + \frac{\theta}{\gamma\epsilon^3})}} \tag{4.15}$$

Choose for instance $\theta = 1/100$, $\epsilon = \min\{\frac{\gamma}{2000k_1^2}, \epsilon_0\}$. Then (4.14) holds provided $0 < |\lambda| < k\gamma$, for some $k$ independent of $\beta$ provided that $\beta \geqslant \beta_0$, with $\beta_0 > 0$ arbitrary but fixed. For large $\beta$, the r.h.s. of (4.15) behaves like $\gamma^2$. ∎

## 5. PROOF OF THEOREM 2.3

The $\sigma_{t,\lambda}$-invariant normal states on $\mathfrak{M}_\beta$ are in one-to-one correspondence with the normalized vectors in the span of $\mathscr{P} \cap \ker L_\lambda$ (see Section 2.1.4). Our task is to show that $\mathscr{P} \cap \ker L_\lambda$ equals the set (2.55).

In this section, we will always deal with the cutoff interaction (determined by $G_{\alpha,J}$) but we shall drop the subscript $_J$ in the notation.

## 5.1. Reduction of the Liouvillian

We define the projection $p = p_{J_d} + p_{J_c}$, where $p_{J_d}, p_{J_c}$ are the projections corresponding to the discrete and continuous modes in $J_d$ and $J_c$, respectively; (see also (2.14)). Setting $\bar{p} = \mathbb{1}_p - p$, $P = p \otimes p \otimes \mathbb{1}_f$, we decompose $\bar{P} = \mathbb{1} - P$ as $\bar{P} = P^l + P^r + P^0$, where

$$P^l = p \otimes \bar{p} \otimes \mathbb{1}_f \qquad P^r = \bar{p} \otimes p \otimes \mathbb{1}_f, \qquad P^0 = \bar{p} \otimes \bar{p} \otimes \mathbb{1}_f. \tag{5.1}$$

It is easy to verify that the (regularized) Liouvillian $L_\lambda$, defined in (2.45), is reduced by the decomposition

$$\mathscr{H} = \text{Ran } P \oplus \text{Ran } P^l \oplus \text{Ran } P^r \oplus \text{Ran } P^0,$$

and that

$$L_\lambda \upharpoonright \text{Ran } P^0 = L_0 \upharpoonright \text{Ran } P^0. \tag{5.2}$$

From the definition, (2.38), of the modular conjugation $J$, it follows that

$$JP^l = P^r J. \tag{5.3}$$

Because every $\psi \in \mathscr{P}$ satisfies $J\psi = \psi$, (5.3) implies that $\mathscr{P} \cap \text{Ran } P^l = \mathscr{P} \cap \text{Ran } P^r = \{0\}$, and, consequently, we have that

$$\mathscr{P} \cap \ker L_\lambda = \mathscr{P} \cap (\ker L_\lambda \upharpoonright \text{Ran } P^0 \cup \ker L_\lambda \upharpoonright \text{Ran } P). \tag{5.4}$$

We prove in the next section that $\ker L_\lambda \upharpoonright \text{Ran } P = \{0\}$, which, together with (5.4) and (5.2), shows that $\mathscr{P} \cap \ker L_\lambda$ is given by the subspace defined in (2.55).

## 5.2. The Kernel of $L_\lambda \upharpoonright$ Ran $P$

**Theorem 5.1.** Given any $0 < \beta < \infty$ and any $r > 0$ there is a $\lambda_0(\beta, r) > 0$ s.t. if $0 < |\lambda| < \lambda_0(\beta, r)$ then $\ker L_\lambda \upharpoonright \text{Ran } P = \{0\}$. Here, $\lambda_0(\beta, r) \geqslant k\gamma^2 r$, for some $k$ independent of $\beta, r$. The constant $\gamma$ is given in (2.24).

*Proof.* We use the notation of Section 3.3.2 and write $L$ for $L_\lambda$. In the spirit of the positive commutator method outlined in Section 3.2, we want to establish a lower bound on the expectation value $\langle \psi, (C_1 + i[L, A_0])\, \psi \rangle$ (see also (3.42)), where $\psi$ is a (hypothetical) eigenvector of $L$ in Ran $P$.

Using the relative bound

$$\pm \lambda I_1 \geqslant -cN - \frac{\lambda^2}{c}, \qquad \forall c > 0, \tag{5.5}$$

with $c = 1/10$, and (3.46), we obtain a lower bound (always in the sense of quadratic forms on $\mathscr{D}$)

$$\begin{aligned} &C_1 + i[L, A_0] \\ &\geqslant \chi^2 H_p \otimes \mathbb{1}_p + \mathbb{1}_p \otimes \chi^2 H_p + \frac{9}{10} N + i[L, A_0] - \frac{\lambda^2}{10} \\ &\geqslant P\left(\chi^2 H_p \otimes \mathbb{1}_p \otimes P_\Omega + \mathbb{1}_p \otimes \chi^2 H_p \otimes P_\Omega + \frac{9}{10} \bar{P}_\Omega + i[L, A_0] - \frac{\lambda^2}{10}\right) P \\ &\quad + \bar{P} i[L, A_0]\, P + P i[L, A_0]\, \bar{P} + \bar{P} i[L, A_0]\, \bar{P} - \frac{\lambda^2}{10} \bar{P} \\ &=: P M_0 P + M_1, \end{aligned} \tag{5.6}$$

where the bounded operators $M_0$ and $M_1$ are given by

$$\begin{aligned} M_0 &:= p_{J_c} H_p \otimes \mathbb{1}_p \otimes P_\Omega + \mathbb{1}_p \otimes p_{J_c} H_p \otimes P_\Omega + \frac{9}{10} \bar{P}_\Omega \\ &\quad + i[L, A_0] - \frac{\lambda^2}{10}, \end{aligned} \tag{5.7}$$

$$M_1 := \bar{P} i[L, A_0]\, P + P i[L, A_0] \bar{P} + \bar{P} i[L, A_0]\, \bar{P} - \frac{\lambda^2}{10} \bar{P}. \tag{5.8}$$

The difficult part of the proof of Theorem 2.3 is contained in the following two propositions.

**Proposition 5.1.** Suppose Proposition 3.2 holds and that the parameters satisfy

$$0 < |\lambda| < \min\left(1, \sqrt{r}, \frac{\epsilon}{3\sqrt{\theta}\, k}, \frac{\epsilon}{\sqrt{k}}\right), 0 < \epsilon < \min(5\theta\gamma, \epsilon_0), \quad 0 < \theta < r/32, \tag{5.9}$$

where $k$ is a constant depending on the interaction, but not on any of the parameters $\epsilon$, $\lambda$, $\theta$, nor on $\beta$ (for $\beta \geqslant \beta_0$, with $\beta_0 > 0$ fixed). Then there is an interval $\Delta$ around zero such that

$$PE_\Delta^0 M_0 E_\Delta^0 P \geqslant \frac{\theta\lambda^2}{\epsilon} \gamma E_\Delta^0 P, \tag{5.10}$$

where $E_\Delta^0 = E_\Delta(L_0)$ is the spectral projection of $L_0$ onto $\Delta$.

**Proposition 5.2.** Assume that the conditions of Proposition 5.1 are satisfied and that

$$|\lambda| < \frac{\gamma}{k} \min(1, \epsilon, \theta/\epsilon). \tag{5.11}$$

If $\psi \in \operatorname{Ran} P$ is an eigenvector of $L_\lambda$ then

$$\langle \psi, M_0 \psi \rangle \geqslant \frac{1}{2} \frac{\theta\lambda^2}{\epsilon} \gamma \, \|\psi\|^2. \tag{5.12}$$

We may choose the parameters as $\lambda = \tilde{\lambda}\gamma^2$, $\epsilon = \tilde{\epsilon}\gamma$, with $\tilde{\lambda}$, $\tilde{\epsilon}$, and $\theta$ independent of the inverse temperature $\beta$. Conditions (5.9) and (5.11) are satisfied provided $0 < |\tilde{\lambda}| < kr$, for some $k$ independent of $\beta$ and $r$.

Theorem 5.1 is now proven as follows. Assume that $\psi \in \operatorname{Ran} P$ is an eigenvector of $L_\lambda$, and let $\psi_\alpha$ be the family of approximate eigenvectors given in Theorem 3.3. From (5.6), (5.12), and using that $\langle \psi, M_1 \psi \rangle = 0$, we obtain

$$0 = \lim_\alpha \, \langle \psi_\alpha, (C_1 + i[L, A_0]) \, \psi_\alpha \rangle \geqslant \frac{1}{2} \frac{\theta\lambda^2}{\epsilon} \gamma \, \|\psi\|^2,$$

which is a contradiction, because the r.h.s. is strictly positive.

*Proof of Proposition 5.1.* Pick $\Delta$ such that

$$|\Delta| < \tfrac{1}{2} \min\{|E(m) - E(n)| \mid m, n \in J_d, E(m) \neq E(n)\}. \tag{5.13}$$

The Hilbert space $\operatorname{Ran} E_\Delta^0 P$ has the decomposition

$$\operatorname{Ran} E_\Delta^0 P = \operatorname{Ran} P\Pi \oplus \operatorname{Ran} E_\Delta^0 P\bar{\Pi}, \tag{5.14}$$

where, we recall, $\Pi = P_0 \otimes P_\Omega$ is the projection onto the kernel of $L_0$. We analyze the spectrum of the operator $PE_\Delta^0 M_0 E_\Delta^0 P$ on $\operatorname{Ran} E_\Delta^0 P$ using the Feshbach method. For details on the Feshbach method, we refer to ref. 3.

Let $m$ be a number in the resolvent set of $\bar{\Pi} P E_\Delta^0 M_0 E_\Delta^0 P \bar{\Pi}$ (viewed as an operator on Ran $E_\Delta^0 P \bar{\Pi}$). The Feshbach map $F_{\Pi,m}$ applied to the operator $PE_\Delta^0 M_0 E_\Delta^0 P$ is defined as

$$F_{\Pi,m}(PE_\Delta^0 M_0 E_\Delta^0 P) \\ = \Pi(PM_0P - PM_0\bar{\Pi}PE_\Delta^0(\bar{\Pi}PE_\Delta^0 M_0 E_\Delta^0 P\bar{\Pi} - m)^{-1} E_\Delta^0 P\bar{\Pi} M_0 P)\,\Pi, \tag{5.15}$$

and has the following property of *isospectrality*: Let $\sigma$ and $\rho$ denote the spectrum and the resolvent set of an operator. Then

$$z \in \sigma(PE_\Delta^0 M_0 E_\Delta^0 P) \cap \rho(\bar{\Pi}PE_\Delta^0 M_0 E_\Delta^0 P\bar{\Pi}) \\ \Leftrightarrow z \in \sigma(F_{\Pi,m}(PE_\Delta^0 M_0 E_\Delta^0 P)) \cap \rho(\bar{\Pi}PE_\Delta^0 M_0 E_\Delta^0 P\bar{\Pi}). \tag{5.16}$$

The point of the Feshbach method is that it can be easier to analyze the spectrum of the operator (5.15) than the one of $PE_\Delta^0 M_0 E_\Delta^0 P$, because the operator (5.15) acts on the smaller space Ran $P\Pi$.

Let us examine the diagonal blocks of $PE_\Delta^0 M_0 E_\Delta^0 P$ in the decompositon (5.14). It is readily verified that

$$\Pi PM_0 P\Pi = 2\theta\lambda^2 \Pi PI\bar{R}_\epsilon^2 IP\Pi - \frac{\lambda^2}{10} P\Pi. \tag{5.17}$$

Because of (5.13), $E_\Delta^0 \bar{P}_0 p_{J_d} \otimes p_{J_d} \otimes P_\Omega = 0$. Hence

$$E_\Delta^0 P\bar{\Pi} = E_\Delta^0 P\bar{P}_\Omega + E_\Delta^0 P(p_{J_d} \otimes p_{J_c} + p_{J_c} \otimes p_{J_d} + p_{J_c} \otimes p_{J_c}) \otimes P_\Omega.$$

Set $Q_1 = E_\Delta^0 P\bar{P}_\Omega$ and let $Q_2$ be the other projection on the right side. $PE_\Delta^0 M_0 E_\Delta^0 P$ is diagonal in this decomposition of Ran $E_\Delta^0 P\bar{\Pi}$. We have the estimates

$$Q_1 M_0 Q_1 \geqslant \left(\frac{9}{10} - \frac{\lambda^2}{10} - k\frac{\theta\lambda^2}{\epsilon^2}\right) Q_1, \qquad Q_2 M_0 Q_2 \geqslant \left(r - \frac{\lambda^2}{10}\right) Q_2,$$

where $k = 2\,\|I\Pi I\|$ and we used that $p_{J_c} H_p \geqslant r p_{J_c}$. Consequently, we obtain the lower bound

$$P\bar{\Pi} E_\Delta^0 M_0 E_\Delta^0 \bar{\Pi} P \geqslant \min\left(\frac{9}{10} - \frac{\lambda^2}{10} - k\frac{\theta\lambda^2}{\epsilon^2}, r - \frac{\lambda^2}{10}\right) E_\Delta^0 \bar{\Pi} P \geqslant \frac{r}{2} E_\Delta^0 \bar{\Pi} P, \tag{5.18}$$

due to (5.9). It follows that any $m < r/4$ is in the resolvent set of the operator $P\bar{\Pi}E^0_\Delta M_0 E^0_\Delta \bar{\Pi}P$ and

$$\|(P\bar{\Pi}E^0_\Delta M_0 E^0_\Delta \bar{\Pi}P - m)^{-1}\| \leqslant 4/r. \tag{5.19}$$

We show now that

$$F_{\Pi,m}(PE^0_\Delta M_0 E^0_\Delta P) \geqslant \frac{\theta\lambda^2}{\epsilon}\gamma E^0_\Delta P, \tag{5.20}$$

uniformly in $m < r/4$, provided (5.9) is satisfied, and where the Feshbach map was introduced in (5.15). The bound (5.20) and the isospectrality property (5.16) of the Feshbach map imply (5.10).

We complete the proof of the proposition by showing (5.20). From (5.19) it follows that for any $\psi$ and $m < r/4$

$$\begin{aligned}
&\langle\psi, \Pi PM_0\bar{\Pi}PE^0_\Delta(\bar{\Pi}PE^0_\Delta M_0 E^0_\Delta P\bar{\Pi} - m)^{-1}\, E^0_\Delta P\bar{\Pi}M_0 P\Pi\psi\rangle \\
&\quad \leqslant \frac{4}{r}\, \|\bar{\Pi}PE^0_\Delta M_0 P\Pi\psi\|^2 \\
&\quad \leqslant \frac{8\theta^2\lambda^2}{r}\,(1 + k\lambda^2/\epsilon^2)\langle\psi, P\Pi I\bar{R}^2_\epsilon I\Pi P\psi\rangle,
\end{aligned} \tag{5.21}$$

where we estimate $\bar{\Pi}PE^0_\Delta M_0\Pi P = \theta\lambda\bar{\Pi}PE^0_\Delta L\,\bar{R}^2_\epsilon I\Pi P$ as

$$\|\bar{\Pi}PE^0_\Delta M_0\Pi P\psi\| \leqslant \theta\,|\lambda|\,(1 + k\,|\lambda|/\epsilon)\,\|\bar{R}_\epsilon I\Pi P\psi\|,$$

with $k = \|IP(N \leqslant 2)\|$. Taking into account (5.17) and Proposition 3.1, we obtain the estimate

$$F_{\Pi,m}(PE^0_\Delta M_0 E^0_\Delta P) \geqslant 2\,\frac{\theta\lambda^2}{\epsilon}\,\gamma\left(1 - \frac{4\theta}{r}\,(1 + k\lambda^2/\epsilon^2)\right)\Pi P - \frac{\lambda^2}{10}\,\Pi P. \tag{5.22}$$

The bound (5.20) follows from (5.22) and conditions (5.9). This finishes the proof of Proposition 5.1.

*Proof of Proposition 5.2.* Let $0 \leqslant g \leqslant 1$ be a smooth function with support in the interval $\Delta$, s.t. $g = 1$ on the interval $(-|\Delta|/4, |\Delta|/4)$, and set $g_0 = g(L_0)$. Since the interaction $I$ is relatively $N^{1/2}$-bounded, it follows in a standard way (by using, e.g., the functional calculus presented in Appendix A) that

$$\|(1 - g_0)\,\psi\| \leqslant k\,|\lambda|\,\|(N+1)^{1/2}\,\psi\| \leqslant k\,|\lambda|\,\|\psi\|, \tag{5.23}$$

$$\|(1 - g_0)\,\bar{P}_\Omega\psi\| \leqslant k\,|\lambda|\,\|N^{1/2}\bar{P}_\Omega\psi\| \leqslant k\,|\lambda|^2\,\|\psi\|, \tag{5.24}$$

where we used (2.51) in the last line. Notice also that $\Pi(1-g_0)=0$ and that on Ran $(1-g_0)$ we have $|L_0| \geqslant |\Delta|/4$ hence $\|(1-g_0)\, R_\epsilon\| \leqslant 4/|\Delta|$. These estimates are used below without explicit mention. We decompose

$$PM_0P = g_0PM_0Pg_0$$

$$+2\,\mathrm{Re}(1-g_0)\, PM_0Pg_0 \tag{5.25}$$

$$+(1-g_0)\, PM_0P(1-g_0). \tag{5.26}$$

Proposition 5.1 yields the bound

$$\langle \psi, g_0PM_0Pg_0\psi\rangle \geqslant \frac{\theta\lambda^2}{\epsilon}\,\gamma\, \|g_0\psi\|^2 = \frac{\theta\lambda^2}{\epsilon}\,\gamma(1-k\,|\lambda|)^2\, \|\psi\|^2. \tag{5.27}$$

We estimate

$$2\,\mathrm{Re}(1-g_0)\, PM_0Pg_0 \geqslant 2\,\mathrm{Re}(1-g_0)\, Pi[L, A_0]\, Pg_0 - \lambda^2(1-g_0)\, g_0P, \tag{5.28}$$

and since

$$(1-g_0)\, i[L, A_0]\, g_0 = -\theta\lambda(1-g_0)(\lambda I\Pi I\bar{R}_\epsilon^2 - L\bar{R}_\epsilon^2 I\Pi - \lambda\bar{R}_\epsilon^2 I\Pi I)\, g_0$$

we conclude that

$$2\,\mathrm{Re}\langle\psi, (1-g_0)\, PM_0Pg_0\psi\rangle \geqslant -k\,\frac{\theta\lambda^2}{\epsilon}\left(\frac{|\lambda|}{\epsilon}+\frac{|\lambda|\,\epsilon}{\theta}\right)\|\psi\|^2. \tag{5.29}$$

Next, we have that

$$(1-g_0)\, M_0(1-g_0) \geqslant -\theta\lambda^2(1-g_0)(I\Pi I\bar{R}_\epsilon^2+\bar{R}_\epsilon^2 I\Pi I)(1-g_0) - \frac{\lambda^2}{10}(1-g_0)^2,$$

from which it follows that

$$\langle\psi, (1-g_0)\, PM_0P(1-g_0)\,\psi\rangle \geqslant -k\,\frac{\theta\lambda^2}{\epsilon}\left(\lambda^4\epsilon+\frac{\lambda^2\epsilon}{\theta}\right)\|\psi\|^2. \tag{5.30}$$

Collecting the bounds (5.27), (5.29), and (5.30) we obtain

$$\langle\psi, PM_0P\psi\rangle \geqslant \frac{\theta\lambda^2}{\epsilon}\,\gamma\left(1-k\,|\lambda| - \frac{k}{\gamma}\left(\frac{|\lambda|}{\epsilon}+\frac{|\lambda|\,\epsilon}{\theta}\right)\right)\|\psi\|^2,$$

and (5.12) follows from the conditions (5.11). This completes the proof of Proposition 5.2 and of Theorem 5.1. ∎

## APPENDIX A

### A.1. Invariance of Domains, Commutator Expansion

The following two theorems are useful in our analysis.

**Theorem A.1 (Invariance of Domain, ref. 6).** Suppose $(X, Y, \mathscr{D})$ satisfies the GJN Condition, (3.1), (3.2). Then the unitary group generated by the selfadjoint $X$, $e^{itX}$, leaves $\mathscr{D}(Y)$ invariant, and we have the estimate

$$\|Ye^{itX}\psi\| \leqslant e^{k\,|t|}\,\|Y\psi\|, \tag{A.1}$$

for some $k \geqslant 0$, and all $\psi \in \mathscr{D}(Y)$.

**Theorem A.2 (Commutator Expansion, ref. 6).** Suppose $\mathscr{D}$ is a core for the selfadjoint $Y \geqslant \mathbb{1}$. Let $X, Z$ be two symmetric operators on $\mathscr{D}$, and define the symmetric operators $\mathrm{ad}_X^{(n)}(Z)$ on $\mathscr{D}$ by

$$\mathrm{ad}_X^{(0)}(Z) = Z,$$
$$\langle \psi, \mathrm{ad}_X^{(n)}(Z)\,\psi\rangle = i\{\langle \mathrm{ad}_X^{(n-1)}(Z)\,\psi, X\psi\rangle - \langle X\psi, \mathrm{ad}_X^{(n-1)}(Z)\,\psi\rangle\},$$

for all $\psi \in \mathscr{D}$, $n = 1,..., M$. We suppose that the triples $(\mathrm{ad}_X^{(n)}(Z), Y, \mathscr{D})$, $n = 0, 1,..., M$, satisfy the GJN Condition (3.1), (3.2), and that $X$ is selfadjoint, $\mathscr{D} \subset \mathscr{D}(X)$, $e^{itX}$ leaves $\mathscr{D}(Y)$ invariant, and that (A.1) holds. The we have on $\mathscr{D}(Y)$

$$e^{itX}Ze^{-itX} = Z - \sum_{n=1}^{M-1} \frac{t^n}{n!}\,\mathrm{ad}_X^{(n)}(Z)$$
$$-\int_0^t dt_1 \cdots \int_0^{t_{M-1}} dt_M\, e^{it_M X}\,\mathrm{ad}_X^{(M)}(Z)\,e^{-it_M X}. \tag{A.2}$$

The following lemma is a consequence of the above two theorems.

**Lemma A.1.** Suppose $(X, Y, \mathscr{D})$ and $(\mathrm{ad}_X^{(n)}(Y), Y, \mathscr{D})$ are GJN triples, for $n = 1,..., M$, some $M \geqslant 1$. Moreover, assume that in the sense of Kato on $\mathscr{D}(Y)$: $\pm\mathrm{ad}_X^{(M)}(Y) \leqslant kX$, for some $k \geqslant 0$. For $\chi \in \mathscr{S}(\mathbb{R})$, a smooth function of rapid decrease, define $\chi(X) = \int \hat{\chi}(s)\, e^{isX}$, where $\hat{\chi}$ is the Fourier transform of $\chi$. Then $\chi(X)$ leaves $\mathscr{D}(Y)$ invariant.

*Proof.* For $R>0$, set $\chi_R(X)=\int_{-R}^{R}\hat{\chi}(s)\,e^{isX}$, then $\chi_R(X)\to\chi(X)$ in operator norm, as $R\to\infty$. From the invariance of domain theorem, we see that $\chi_R(X)$ leaves $\mathscr{D}(Y)$ invariant. Let $\psi\in\mathscr{D}(Y)$, then using the commutator expansion theorem above, we have

$$\begin{aligned} Y\chi_R(X)\,\psi &= \chi_R(X)\,Y\psi+\int_{-R}^{R}\hat{\chi}(s)\,e^{isX}(e^{-isX}Ye^{isX}-Y)\,\psi \\ &= \chi_R(X)\,Y\psi-\int_{-R}^{R}\hat{\chi}(s)\,e^{isX}\left(\sum_{n=1}^{M-1}\frac{(-s)^n}{n!}\,\mathrm{ad}_X^{(n)}(Y)\right. \\ &\quad \left.+(-1)^M\int_0^s ds_1\cdots\int_0^{s_{M-1}}ds_M e^{-is_MX}\,\mathrm{ad}_X^{(M)}(Y)\,e^{is_MX}\right)\psi. \end{aligned} \tag{A.3}$$

The integrand of the $s$-integration in (A.3) is bounded in norm by

$$k(|s|^M+1)(\|Y\psi\|+\|X\psi\|)\leqslant k(|s|^M+1)\,\|Y\psi\|,$$

where we used that $\|\mathrm{ad}_X^{(M)}(Y)\,e^{is_MX}\psi\|\leqslant k\,\|Xe^{is_MX}\psi\|=k\,\|X\psi\|$. Since $\hat{\chi}$ is of rapid decrease, it can be integrated against any power of $|s|$, and we conclude that the r.h.s. of (A.3) has a limit as $R\to\infty$. Since $Y$ is a closed operator it follows that $\chi(X)\,\psi\in\mathscr{D}(Y)$. ∎

**Lemma A.2.** Let $\chi\in C_0^\infty(\mathbb{R}^3)$, $\chi=F^2\geqslant 0$. Suppose $(X, Y, \mathscr{D})$ satisfies the GJN condition and define $F(X)$, $\chi(X)$ via the Fourier transform as in Lemma A.1. Suppose $F(X)$ leaves $\mathscr{D}(Y)$ invariant. Let $Z$ be a symmetric operator on $\mathscr{D}$, s.t. for some $M\geqslant 1$, and $n=0, 1,..., M$, the triples $(\mathrm{ad}_X^{(n)}(Z), Y, \mathscr{D})$ satisfy the GJN condition. Moreover, assume that the multi-commutators for $n=1,..., M-1$ are bounded, and the $M$th multi-commutator is relatively $X$-bounded in the sense of Kato on $\mathscr{D}$: there is a $k<\infty$ s.t. $\forall\psi\in\mathscr{D}$:

$$\begin{aligned} &\|\mathrm{ad}_X^{(n)}(Z)\,\psi\|\leqslant k\,\|\psi\|, \qquad n=1,..., M-1, \\ &\|\mathrm{ad}_X^{(M)}(Z)\,\psi\|\leqslant k\,\|X\psi\|. \end{aligned}$$

Then the commutator $[\chi(X), Z]=\chi(X)\,Z-Z\chi(X)$ is well defined on $\mathscr{D}$ and bounded: there is a $k<\infty$ s.t. $\|[\chi(X), Z]\,\psi\|\leqslant k\,\|\psi\|$, $\forall\psi\in\mathscr{D}$.

*Proof.* We write $F, \chi$ instead of $F(X), \chi(X)$. Since $F$ leaves $\mathscr{D}(Y)$ invariant we have in the strong sense on $\mathscr{D}(Y)$:

$$[\chi, Z]=F[F, Z]+[F, Z]\,F.$$

We expand the commutator

$$[F, Z] = \int \hat{F}(s)\, e^{isX}(Z - e^{-isX} Z e^{isX})$$

$$= \int \hat{F}(s)\, e^{isX} \left\{ \sum_{n=1}^{M-1} \frac{s^n}{n!} \operatorname{ad}_X^{(n)}(Z) \right.$$

$$\left. + \int_0^s ds_1 \cdots \int_0^{s_{M-1}} ds_M\, e^{-is_M X} \operatorname{ad}_X^{(M)}(Z)\, e^{is_M X} \right\}.$$

Multiplying this from the left with $F$ (and noticing that $F$ commutes with $e^{is_M X}$), we see immediately that $[F, Z]\, F$ is bounded. Next, for any $\phi, \psi$ in the dense set $\mathscr{D}$, we have the estimate $|\langle \phi, F[F, Z]\, \psi \rangle| = |\langle [F, Z]\, F\phi, \psi \rangle| \leqslant k\, \|\phi\|\, \|\psi\|$, hence $\|F[F, Z]\, \psi\| = \sup_{0 \neq \phi \in \mathscr{D}} \|\phi\|^{-1}\, |\langle \phi, F[F, Z]\, \psi \rangle| \leqslant k\, \|\psi\|$. ∎

## A.2. Proof of Proposition 3.3

Before proving Proposition 3.3, we show certain triples satisfy the GJN conditions.

**Lemma A.3.** The following triples satisfy the GJN Condition (3.1), (3.2):

$$(H_p, \Lambda_p, C_0^\infty(\mathbb{R}^3)) \tag{A.4}$$

$$(\Lambda_p, \Lambda_p, C_0^\infty(\mathbb{R}^3)) \tag{A.5}$$

$$(\operatorname{ad}_{H_p}^{(n)}(\Lambda_p), \Lambda_p, C_0^\infty(\mathbb{R}^3)), \qquad n = 1, 2. \tag{A.6}$$

***Proof.*** For $\psi \in C_0^\infty(\mathbb{R}^3)$ we have $\|H_p \psi\|^2 \leqslant 2\, \|(-\Delta)\, \psi\|^2 + 2\, \|v\|_\infty^2\, \|\psi\|^2$. The first term on the r.h.s. is bounded from above by $2\, \|\Lambda_p \psi\|^2 + 4\, \mathrm{Re}\langle \psi, \Delta x^2 \psi \rangle$ and we have the estimate $\mathrm{Re}\langle \psi, \Delta x^2 \psi \rangle = \sum_{j=1}^3 (\langle \psi, x_j \Delta x_j \psi \rangle + \langle \psi, \partial_j x_j \psi \rangle) \leqslant \sum_{j=1}^3 \|\partial_j \psi\|\, \|x_j \psi\| \leqslant \langle \psi, (-\Delta + x^2)\, \psi \rangle$. Therefore, since $\Lambda_p \geqslant \mathbb{1}$, it follows that that $\|H_p \psi\|^2 \leqslant k\, \|\Lambda_p \psi\|^2$. In a similar way it is simple to verify that $\pm \operatorname{ad}_{\Lambda_p}^{(1)}(H_p) \leqslant k\, \Lambda_p$. This shows that (A.4) satisfies the GJN conditions. The proof for (A.5) is similarly easy.

From the above calculations, and $\operatorname{ad}_{H_p}^{(1)}(\Lambda_p) = -\operatorname{ad}_{\Lambda_p}^{(1)}(H_p)$, we see that $\|\operatorname{ad}_{H_p}^{(1)}(\Lambda_p)\, \psi\| \leqslant k\, \|\Lambda_p^{1/2} \psi\|$. Moreover, we calculate

$$\operatorname{ad}_{\Lambda_p}^{(1)}(\operatorname{ad}_{H_p}^{(1)}(\Lambda_p)) = -(2x \cdot \nabla v + 2\partial_m v_{mn} \partial_n + \partial_n v_{nmm} + 4\Delta + 4x^2) + \text{h.c.},$$

where $v_{klm} = \partial_k \partial_l \partial_m v$, etc. Since all the derivatives of $v$ involved are bounded, the r.h.s. is again relatively $\Lambda_p$-bounded, in the form sense on $C_0^\infty(\mathbb{R}^3)$. This shows (A.6) for $n=1$. For $n=2$, we calculate

$$\mathrm{ad}_{H_p}^{(2)}(\Lambda_p) = -(\nabla v \cdot \nabla v + 2x \cdot \nabla v + 2\,\partial_m v_{nm}\,\partial_n + \partial_m v_{mnn} + 4\Delta) + \mathrm{h.c.} \tag{A.7}$$

Notice that $\mathrm{ad}_{H_p}^{(2)}(\Lambda_p)$ is relatively $H_p$-bounded, hence relatively $\Lambda_p$-bounded, in the sense of Kato on $C_0^\infty(\mathbb{R}^3)$, because $\nabla v \cdot \nabla v$ and $x \cdot \nabla v$ are bounded. Moreover, one can calculate $\mathrm{ad}_{\Lambda_p}^{(1)}(\mathrm{ad}_{H_p}^{(2)}(\Lambda_p))$, and see that it is $\Lambda_p$ form bounded on $C_0^\infty(\mathbb{R}^3)$. This shows (A.6) for $n=2$. ▮

*Proof of Proposition 3.3.* We start by showing that $\chi$ leaves $\mathscr{D}(\Lambda_p)$ invariant. This follows from Lemma A.1 and the following two facts: firstly, $(H_p, \Lambda_p, C_0^\infty(\mathbb{R}^3))$ and $(\mathrm{ad}_{H_p}^{(n)}, \Lambda_p, C_0^\infty(\mathbb{R}^3))$ satisfy the GJN Condition, for $n = 1, 2$ (see Lemma A.3), and secondly, $\pm\mathrm{ad}_{H_p}^{(2)}(\Lambda_p) \leqslant kH_p$, in the sense of Kato on $\mathscr{D}(\Lambda_p)$ (see (A.7)). This shows that $\chi\colon \mathscr{D}(\Lambda_p) \to \mathscr{D}(\Lambda_p)$, and consequently, $\chi\Lambda_p\chi$ is well defined on $\mathscr{D}(\Lambda_p)$.

We have in the strong sense on $\mathscr{D}(\Lambda_p)$

$$\chi\Lambda_p\chi = \chi^2\Lambda_p + \chi[\Lambda_p, \chi]. \tag{A.8}$$

Let us estimate each term on the r.h.s. separately. For $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned}\|\chi^2\Lambda_p\psi\| &= \|\chi^2(\nabla\cdot x + 3/2)\,\psi\| \leqslant \sum_n \|\chi^2\partial_n\|\,\|x_n\psi\| + \tfrac{3}{2}\|\psi\| \\ &\leqslant k\sum_n \langle\psi, x_n^2\psi\rangle^{1/2} + \tfrac{3}{2}\|\psi\| \leqslant k\langle\psi, (-\Delta + x^2)\,\psi\rangle^{1/2},\end{aligned}$$

where we used that $\|\chi^2\partial_n\| < \infty$. Consequently,

$$\|\chi^2\Lambda_p\psi\| \leqslant k\,\|\Lambda_p^{1/2}\psi\|. \tag{A.9}$$

Next we have on $\mathscr{D}(\Lambda_p)$

$$\chi[\Lambda_p, \chi] = -\chi\int ds\,\hat{\chi}(s)\int_0^s ds_1\,e^{is_1H_p}\,\mathrm{ad}_{\Lambda_p}^{(1)}(H_p)\,e^{-is_1H_p},$$

where $\mathrm{ad}_{\Lambda_p}^{(1)}(H_p) = H_p + W$, see (3.44), and since $\chi$ commutes with $e^{is_1H_p}$, we obtain the estimate

$$\|\chi[\Lambda_p, \chi]\| \leqslant \int ds\,|\hat{\chi}(s)|\,s(\|\chi H_p\| + \|W\|_\infty) < \infty. \tag{A.10}$$

It follows, together with with (A.8) and (A.9), that

$$\|\chi A_p \chi\psi\| \leqslant k(\|\Lambda_p^{1/2}\psi\| + \|\psi\|) \leqslant 2k\,\|\Lambda_p^{1/2}\psi\|.$$

This shows that the first GJN condition, (3.1), is satisfied for our triple. Next, we write

$$\langle \chi A_p \chi\psi, \Lambda_p \psi\rangle = \langle A_p \chi\psi, \Lambda_p \chi\psi\rangle + R_1, \tag{A.11}$$

where

$$R_1 = \langle A_p \chi\psi, [\chi, \Lambda_p]\,\psi\rangle. \tag{A.12}$$

Since $\chi\psi \in \mathscr{D}(\Lambda_p)$, and $\Lambda_p$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^3)$, there exists a sequence $\{\varphi_n\} \subset C_0^\infty(\mathbb{R}^3)$ s.t. $\varphi_n \to \chi\psi$ and $\Lambda_p \varphi_n \to \Lambda_p \chi\psi$. Moreover $\Lambda_p$ leaves $C_0^\infty(\mathbb{R}^3)$ invariant, so we have

$$\begin{aligned}\langle A_p \chi\psi, \Lambda_p \chi\psi\rangle &= \lim_n \langle A_p \chi\psi, \Lambda_p \varphi_n\rangle \\ &= \lim_n \{\langle \Lambda_p \chi\psi, A_p \varphi_n\rangle + \langle \chi\psi, [A_p, \Lambda_p]\,\varphi_n\rangle\},\end{aligned} \tag{A.13}$$

and we calculate (strongly on $C_0^\infty(\mathbb{R}^3)$): $i\,[A_p, \Lambda_p] = H_p + W - x\cdot\nabla v - 2x^2$. Since $\chi(H_p + W - x\cdot\nabla v)$ is bounded, and $\chi\psi \in \mathscr{D}(x^2)$ (because it is in $\mathscr{D}(\Lambda_p)$), we get

$$(\mathrm{A.13}) = \lim_n \langle \Lambda_p \chi\psi, A_p \varphi_n\rangle + R_2, \tag{A.14}$$

where we defined

$$R_2 = \langle \psi, \chi(H_p + W - x\cdot\nabla v - 2x^2)\,\chi\psi\rangle. \tag{A.15}$$

Next, it is not difficult to see that if $\psi \in C_0^\infty$ then $\chi\psi \in \mathscr{D}(\Lambda_p^2)$. Consequently, we can move $A_p$ in (A.14) to the left factor in the scalar product (recall that $\mathscr{D}(A_p) \supset \mathscr{D}(\Lambda_p)$), perform the limit, and move $A_p$ back to the right factor. We then obtain

$$\langle A_p \chi\psi, \Lambda_p \chi\psi\rangle = \langle \Lambda_p \chi\psi, A_p \chi\psi\rangle + R_2 = \langle \Lambda_p \psi, \chi A_p \chi\psi\rangle + R_2 - \bar{R}_1, \tag{A.16}$$

where the bar denotes complex conjugate. Together with (A.11) this gives

$$\langle \chi A_p \chi\psi, \Lambda_p \psi\rangle - \langle \Lambda_p \psi, \chi A_p \chi\psi\rangle = R_1 + R_2 - \bar{R}_1. \tag{A.17}$$

Let us first consider $R_1$. We estimate

$$|R_1| \leqslant \|A_p \chi \psi\| \, \|[\chi, \Lambda_p] \psi\| \leqslant (\|A_p \psi\| + \|[A_p, \chi] \psi\|) \, \|[\chi, \Lambda_p] \psi\|.$$

It is clear that $\|A_p \psi\| \leqslant k \, \|\Lambda_p^{1/2} \psi\|$, and by the same argument as the one leading to (A.10), that $\|[A_p, \chi] \psi\| \leqslant k \, \|\psi\|$ (use that $\chi = F^2 \geqslant 0$, as in the proof of Lemma A.2), so that

$$|R_1| \leqslant k \, \|\Lambda_p^{1/2} \psi\| \, \|[\chi, \Lambda_p] \psi\| \leqslant k \, \|\Lambda_p^{1/2} \psi\| \, (\|\psi\| + \|[\chi, x^2] \psi\|), \qquad \text{(A.18)}$$

where we used in the second step that $[\chi, -\Delta]$ is bounded. Next, Lemma A.2 (with $X = H_p$, $Z = x_n$, $Y = \Lambda_p$, $M = 1$) shows that $[\chi, x_n]$ is bounded, hence $\|[\chi, x^2] \psi\| \leqslant \sum_n (k \, \|x_n \psi\| + \|x_n [\chi, x_n] \psi\|) \leqslant k \, \|\Lambda_p^{1/2} \psi\| + \sum_n \|x_n [\chi, x_n] \psi\|$. We write $x_n [\chi, x_n] \psi = [\chi, x_n] \, x_n \psi + [x_n, [\chi, x_n]] \, \psi$. As above, $\|[\chi, x_n] \, x_n \psi\| \leqslant k \, \|\Lambda_p^{1/2} \psi\|$, and using $\chi^2 = F$, we write

$$[x_n, [\chi, x_n]] = -2[F, x_n]^2 + F[x_n [F, x_n]] + [x_n, [F, x_n]] \, F. \qquad \text{(A.19)}$$

As above, $[F, x_n]$ is bounded, and a by now standard commutator expansion shows that so are the other two terms in (A.19). We conclude that $\|[\chi, x^2] \psi\| \leqslant k(\|\Lambda_p^{1/2} \psi\| + \|\psi\|) \leqslant k \, \|\Lambda_p^{1/2} \psi\|$. Combining this with (A.18) yields

$$|R_1| \leqslant k \, \|\Lambda_p^{1/2} \psi\|^2. \qquad \text{(A.20)}$$

Finally, let us obtain the same upper bound for $R_2$. Since $\chi(H_p + W - x \cdot \nabla v)$ is bounded we only need to show that $|\langle \chi \psi, x^2 \chi \psi \rangle| \leqslant k \, \|\Lambda_p^{1/2} \psi\|^2$. Using that $[\chi, x_n]$ is bounded, we arrive at

$$|\langle \chi \psi, x^2 \chi \psi \rangle| = \sum_n \|x_n \chi \psi\|^2 \leqslant k \, \|\psi\|^2 + k \sum_n \|x_n \psi\|^2 \leqslant k \, \|\Lambda_p^{1/2} \psi\|^2,$$

which shows that

$$|R_2| \leqslant k \, \|\Lambda_p^{1/2} \psi\|^2. \qquad \text{(A.21)}$$

Combining (A.17), (A.20), and (A.21) shows that $(\chi A_p \chi, \Lambda_p, C_0^\infty(\mathbb{R}^3))$ satisfies the second GJN condition, (3.2). ▮

## A.3. Proof of Proposition 3.4

We give the following lemma without a proof, which is not difficult to find, e.g., by using the results of Appendix A.1.

**Lemma A.4.** Let $v \in C^{p-1}(\mathbb{R}^d)$ be s.t. $x^\alpha \partial^\alpha v$ is bounded, for any multi-index $|\alpha| \leqslant p-1$. Let $H = -\Delta + v$, which is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$. Given a function $\mu \in C_0^\infty(\mathbb{R})$, define $\mu(H) = \int \hat{\mu}(s)\, e^{isH}$. Then we have

$$\|\langle x\rangle^{\mp n}\, \mu(H) \langle x\rangle^{\pm n}\| \leqslant K(p, \mu), \qquad n = 0, 1,..., p, \tag{A.22}$$

where $K(p, \mu)$ is some finite constant.

**Lemma A.5.** The regularized $G_{\alpha, J} = (p_{J_d} + \mu)\, G_\alpha (p_{J_d} + \mu)$ satisfies the same bounds (2.22) as $G_\alpha$. Moreover, $\mathrm{ad}^{(n)}_{\chi A_p \chi}(G_{\alpha, J})$ is bounded, $n = 0, 1, 2, 3$.

*Proof.* The first assertion follows easily from the fact that $\langle x\rangle^m\, p_{J_d} \langle x\rangle^n$ is bounded, for all $m, n$, and from Lemma A.4 (use $\langle x\rangle^n\, \mu = \langle x\rangle^n\, \mu \langle x\rangle^{-n}\, \langle x\rangle^n$).

In order to show boundedness of the multi-commutators, we treat a typical term appearing in $\mathrm{ad}^{(3)}_{\chi A_p \chi}(G_{\alpha, J})$:

$$\begin{aligned} &\chi A_p \chi G_{\alpha, J} \chi A_p \chi \chi A_p \chi \\ &\quad = \chi A_p \langle x\rangle^{-1}\, \langle x\rangle\, \chi \langle x\rangle^{-1}\, \langle x\rangle\, G_{\alpha, J} \langle x\rangle^2\, \langle x\rangle^{-2}\, \chi A_p \chi \langle x\rangle \langle x\rangle^{-1}\, A_p \chi. \end{aligned}$$

Since $\chi A_p \langle x\rangle^{-1}$ and $\langle x\rangle^{-1} A_p \chi$ are bounded, we see from Lemmas A.4 and the bound (2.22) (for $G_{\alpha, J}$), that the r.h.s. is bounded, provided $\|\langle x\rangle^{-2}\, \chi A_p \chi \langle x\rangle\| < \infty$. To obtain the latter bound, it is enough (due to due to Lemma A.4) to show $\|\langle x\rangle^{-2}\, \chi A_p \langle x\rangle\| < \infty$, which in turn is proved by writing

$$\langle x\rangle^{-2}\, \chi A_p \langle x\rangle = \frac{i}{2} \langle x\rangle^{-2}\, \chi \sum_n\, (x_n \langle x\rangle\, \partial_n + 1/2),$$

and proceeding as in the proof of Lemma (A.4), by commuting $x_n \langle x\rangle$ through $\chi$ to the left. ∎

*Proof of Proposition 3.4.* The operator $\chi(H_p + W)\, \chi$ is bounded, hence relatively $\Lambda_p$-bounded. We will use below the fact that $[\chi, A_p]$ is bounded, which follows from Lemma A.2, with $X = H_p$, $Z = A_p$, $Y = \Lambda_p$, $M = 1$. We have, in the strong sense on $\mathscr{D}(\Lambda_p)$,

$$\mathrm{ad}^{(2)}_{\chi A_p \chi}(H_p) = \chi[\chi^2 H_p, A_p]\, \chi + [\chi W \chi, \chi A_p \chi],$$

where the commutator in the first term is bounded, and the second term equals $\chi(W \chi^2 A_p - A_p \chi^2 W)\, \chi$, which is easily seen to be bounded, too (use

Lemma A.4 together with the fact that $\chi A_p \langle x \rangle^{-1}$ is bounded, and so is $W \langle x \rangle$). Next, we show that

$$\mathrm{ad}^{(3)}_{\chi A_p \chi}(H_p) = [\chi[\chi^2 H_p, A_p]\,\chi, \chi A_p \chi] + [[\chi W \chi, \chi A_p \chi], \chi A_p \chi] \quad (A.23)$$

is bounded. The first term on the r.h.s. is the sum of two bounded operators plus $\chi[\,[\chi^2 H_p, A_p], \chi A_p \chi]\,\chi$, which can be written (by commuting $\chi$ through $A_p$) as a bounded operator plus

$$\chi[\,[\chi^2 H_p, A_p], A_p \chi^2]\,\chi. \quad (A.24)$$

Setting $\chi_1 = \chi H_p$, we expand $[\chi_1, A_p] = \chi_1' H_p + \int \hat{\chi}_1(s) \int_0^s e^{i(s-s_1) H_p} W e^{is_1 H_p}$, and (A.24) splits into two terms, the first one, $\chi[\chi_1' H_p, A_p]\,\chi^4$, is bounded. To see boundedness of the second term, write it as

$$\int \hat{\chi}_1(s) \int_0^s \chi\{[e^{i(s-s_1) H_p} W e^{is_1 H_p}, A_p]\,\chi^2 + A_p e^{i(s-s_1) H_p}[W, \chi^2]\, e^{is_1 H_p}\}\,\chi,$$

and use that $\langle x \rangle W$ is bounded. The second term in (A23) can be written as a bounded operator plus $\chi[W\chi^2 A_p - A_p \chi^2 W, \chi^2 A_p]\,\chi$. The latter term equals again a bounded operator plus $2\,\mathrm{Re}[W\chi^2 A_p, \chi^2 A_p]$, which is easily seen to be bounded, by again noticing that $\langle x \rangle W$, and derivatives of $W$ multiplied by $\langle x \rangle^2$ are bounded.

We have thus shown that the multi-commutators appearing in (3.42) are bounded (hence $\Lambda_p$-bounded in the sense of Kato). Clearly, $N$ is $\Lambda_f$ bounded, and Lemma A.5, together with the fact that $\varphi((-i\partial_u)^k\, \tau_\beta(g_\alpha)))$ is relatively $N^{1/2}$-bounded, shows that $I_n$ is $\Lambda_f$-bounded in the sense of Kato. Consequently, condition (3.1) is satisfied for $X = C_n$.

Next, we verify that (3.2) is satisfied. Let us start with the commutator of $C_1$ with $A$. We need to show relative boundedness of $[\chi(H+W)\,\chi, A_p]$ and $[I_1, A]$ (relative to $\Lambda_p$ and $\Lambda_f$, respectively, in the sense of quadratic forms). Noticing that $\Lambda_p = H_p - v + x^2$, we write $[\chi(H+W)\,\chi, \Lambda_p]$ as a sum of a bounded operator plus

$$[\chi^2 H_p, x^2] + [\chi W \chi, x^2]. \quad (A.25)$$

Now setting $\chi_1 = \chi^2 H_p$, the first term in (A.25) equals $\sum_n (x_n[\chi_1, x_n] + [\chi_1, x_n]\,x_n)$, so for any $\psi \in C_0^\infty$,

$$|\langle \psi, [\chi_1, x^2]\,\psi \rangle \leqslant k \sum_n \|x_n \psi\|\, \|\psi\| \leqslant k\langle \psi, \Lambda_p \psi \rangle. \quad (A.26)$$

Next,

$$|\langle\psi, [\chi W\chi, x^2]\,\psi\rangle|$$
$$\leqslant 2\,|\langle\psi, \chi W[\chi, x^2]\,\psi\rangle| \leqslant 2\sum_n |\langle\psi, (\chi W x_n[\chi, x_n]+\chi W[\chi, x_n]\,x_n)\,\psi\rangle|. \tag{A.27}$$

Commuting $x_n$ in the first term in the sum through $\chi$ to the left, one sees that $|\langle\psi, \chi W x_n[\chi, x_n]\,\psi\rangle| \leqslant k(\|\psi\|^2+\|x_n\psi\|\,\|\psi\|)$, which is bounded from above by $k\langle\psi, \Lambda_p\psi\rangle$ (proceed as in (A.26)). The second term in the sum in (A.27) is estimated in the same way. This shows that (A.25) is $\Lambda_p$-form-bounded.

Next, in order to show the relative bound on $[I_n, \Lambda]$, it is enough to show that $[\mathrm{ad}^{(n)}_{\chi A_p\chi}(G_\alpha), \Lambda_p]$ is relatively $\Lambda_p$-form-bounded, and that

$$[\varphi((-i\partial_u)^n\,\tau_\beta(g_\alpha), \Lambda_f]$$

is relatively $\Lambda_f$-form-bounded. The former bound is easily obtained from (2.22), and the latter has been treated in Section 3.3.1. This shows that $I_n$ are relatively $\Lambda$-form-bounded, hence also completing the proof that $C_1$ satisfies condition (3.2).

Next, we consider the commutator of $C_2$ with $\Lambda$. The only thing to check is that $[\mathrm{ad}_{\chi A_p\chi}(H_p), \Lambda_p]$ is $\Lambda_p$-form-bounded. This commutator can be written as a bounded operator plus $[\mathrm{ad}^{(2)}_{\chi A_p\chi}(H_p), x^2]$, hence it suffices to show that $x_n[\mathrm{ad}^{(2)}_{\chi A_p\chi}(H_p), x_n]$ is $\Lambda_p$-form-bounded ($n=1, 2, 3$). One shows that $[\mathrm{ad}^{(2)}_{\chi A_p\chi}(H_p), x_n]$ is bounded, by simple estimates as above. Relative boundedness of $x_n[\mathrm{ad}^{(2)}_{\chi A_p\chi}(H_p), x_n]$ then follows easily (proceeding as in (A.26)). Consequently, (3.2) is satisfied for $C_2$.

We now consider the commutator of $C_3$ with $\Lambda$, and it is enough to show that $[\mathrm{ad}^{(3)}_{\chi A_p\chi}(H_p), x^2]$ is relatively $\Lambda_p$-form-bounded. We write this commutator as $2\,\mathrm{Re}[\mathrm{ad}^{(2)}_{\chi A_p\chi}(H_p)\,\chi A_p\chi, x^2]$. Now we have

$$[\mathrm{ad}^{(2)}_{\chi A_p\chi}(H_p)\,\chi A_p\chi, x_n]$$
$$=[\mathrm{ad}^{(2)}_{\chi A_p\chi}(H_p), x_n]\,\chi A_p\chi+\mathrm{ad}^{(2)}_{\chi A_p\chi}(H_p)[\chi A_p\chi, x_n],$$

and it is clear that $[\mathrm{ad}^{(2)}_{\chi A_p\chi}(H_p)\,\chi A_p\chi, x_n]\langle x\rangle^{-1}$ is bounded. Consequently,

$$|\langle\psi, [\mathrm{ad}^{(3)}_{\chi A_p\chi}(H_p), x^2]\,\psi\rangle| \leqslant k(\|\psi\|^2+\langle\psi, x^2\psi\rangle) \leqslant k\,\|\Lambda_p^{1/2}\psi\|^2.$$

Hence $C_3$ satisfies (3.2) and the proof of Proposition 3.4 is complete. ∎

## A.4. Proof of Proposition 3.2

We consider first the case when (2.20) holds. From $\Pi I \Pi = 0$ we have $\Pi I \bar{R}_\epsilon^2 I \Pi = \Pi I R_\epsilon^2 I \Pi$. We recall that $P_0$ is the projection onto the kernel of $L_p$ and $\bar{P}_0 = \mathbb{1} - P_0$ is given by

$$\bar{P}_0 = \sum_{\substack{m,n \in \mathcal{M} \\ E(m) \neq E(n)}} p_m \otimes p_n + p_d \otimes p_c + p_c \otimes p_d + p_c \otimes p_c, \tag{A.28}$$

where $p_d$ and $p_c$ are the projections onto the discrete and continuous subspaces corresponding to $H_p$. One can see that

$$\epsilon \Pi I R_\epsilon^2 P_0 I \Pi \to 0, \qquad \epsilon \Pi I R_\epsilon^2 \sum_{\substack{m,n \in \mathcal{M} \\ E(m) = E(n)}} (p_m \otimes p_n) \, I \Pi \to 0,$$

as $\epsilon \to 0$, that $\Pi I R_\epsilon^2 (p_c \otimes p_c) \, I \Pi = 0$, and that $\Pi I R_\epsilon^2 (p_c \otimes p_d) \, I \Pi = \Pi I R_\epsilon^2 (p_d \otimes p_c) \, I \Pi$. From formula (2.45) for the interaction, we obtain the bound

$$\begin{aligned} \Pi I \bar{R}_\epsilon^2 I \Pi &\geqslant \Pi I R_\epsilon^2 (p_c \otimes p_d) \, I \Pi \\ &= \sum_{\substack{m,n,m' \in \mathcal{M} \\ E(m) = E(n) = E(m')}} \sum_{\alpha, \alpha'} (p_m \otimes p_n \otimes P_\Omega) \{ G_\alpha \otimes \mathbb{1}_p \otimes a(\tau_\beta(g_\alpha)) \} \\ &\quad \times \frac{p_c \otimes p_d \otimes \mathbb{1}_f}{L_0^2 + \epsilon^2} \{ G_{\alpha'} \otimes \mathbb{1}_p \otimes a^*(\tau_\beta(g_{\alpha'})) \} (p_{m'} \otimes p_n \otimes P_\Omega). \end{aligned}$$

We write $a(\tau_\beta(g_\alpha)) = \int_{\mathbb{R} \times S^2} \overline{\tau_\beta(g_\alpha)}(u, \Sigma) \, a(u, \Sigma)$ and use the pull through formula $a(u, \Sigma) \, L_0 = (L_0 + u) \, a(u, \Sigma)$ and obtain

$$\begin{aligned} \Pi I \bar{R}_\epsilon^2 I \Pi \geqslant & \sum_{\substack{m,n,m' \in \mathcal{M} \\ E(m) = E(n) = E(m')}} \sum_{\alpha, \alpha'} \int_{-\infty}^{E(m)} du \int_{S^2} d\Sigma \, \frac{u^2}{e^{-\beta u} - 1} \, g_\alpha(-u, \Sigma) \, \overline{g_{\alpha'}}(-u, \Sigma) \\ & \times \left( p_m G_\alpha \frac{p_c}{(H_p - E(m) + u)^2 + \epsilon^2} G_{\alpha'} \, p_{m'} \right) \otimes P_n \otimes P_\Omega, \end{aligned} \tag{A.29}$$

where we recall that $E(m)$ is the eigenvalue of $H_p$ corresponding to the mode $m$. We have dropped the integration over the values $u \geqslant E(m)$ because $\epsilon((H_p - E(m) + u)^2 + \epsilon^2)^{-1} \to \delta(H_p - E(m) + u)$ as $\epsilon \to 0$, hence

$u=-H_p+E(m)\leqslant E(m)$. Recalling the definition of $F$, see before 2.19, and making the change of variable $u\mapsto -u$ in the integral, we arrive at

$$\Pi I\bar{R}_\epsilon^2 I\Pi \geqslant \sum_{\substack{m,n,m'\in\mathcal{M}\\ E(m)=E(n)=E(m')}} \int_{-E(m)}^{\infty} du \int_{S^2} d\Sigma\, \frac{u^2}{e^{\beta u}-1} \times\left(p_m F(u,\Sigma)\frac{p_c}{(H_p-E(m)-u)^2+\epsilon^2}F(u,\Sigma)^*\, p_{m'}\right)\otimes p_n\otimes P_\Omega. \tag{A.30}$$

The projection $p(E)$ onto the eigenspace corresponding to an eigenvalue $E$ of $H_p$ is given by $\sum_{m\in\mathcal{M}\,:\,E(m)=E} p_m$ and we use

$$\sum_{\substack{m,n,m'\in\mathcal{M}\\ E(m)=E(n)=E(m')}} = \sum_{E\in\sigma_p(H_p)}\ \sum_{\substack{m\in\mathcal{M}\\ E(m)=E}}\ \sum_{\substack{n\in\mathcal{M}\\ E(n)=E}}\ \sum_{\substack{m'\in\mathcal{M}\\ E(m')=E}}$$

to arrive at

$$\Pi I\bar{R}_\epsilon^2 I\Pi \geqslant \sum_{E\in\sigma_p(H_p)} \int_{-E}^{\infty} du \int_{S^2} d\Sigma\, \frac{u^2}{e^{\beta u}-1} \times\left(p(E)\, F(u,\Sigma)\frac{p_c}{(H_p-E-u)^2+\epsilon^2}F(u,\Sigma)^*\, p(E)\right) \otimes p(E)\otimes P_\Omega. \tag{A.31}$$

The desired bound (3.27) now follows from (2.20) and (2.21).

The case when (2.23) holds and $\gamma$ is given by (2.24) is done similarly. ▮

## ACKNOWLEDGMENTS

I.M.S. was supported by NSERC under Grant NA7901. M.M. was supported by an NSERC postdoctoral fellowship and by ETH Zürich.

## REFERENCES

1. W. Amrein, A. Boutet de Monvel, and V. Georgescu, *$C_0$-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians* (Birkhäuser, Basel–Boston–Berlin, 1996).
2. H. Araki and E. Woods, Representations of the canonical commutation relations describing a non-relativistic infinite free bose gas, *J. Math. Phys.* **4**:637–662 (1963).
3. V. Bach, J. Fröhlich, and I. M. Sigal, Quantum electrodynamics of confined nonrelativistic particles, *Adv. Math.* **137**:299–395 (1995).
4. V. Bach, J. Fröhlich, I. M. Sigal, and A. Soffer, Positive commutators and the spectrum of Pauli–Fierz hamiltonians of atoms and molecules, *Comm. Math. Phys.* **207**:557–587 (1999).

5. O. Bratteli and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics I, II*, 2nd ed., Texts and Monographs in Physics (Springer-Verlag, Berlin, 1987).
6. J. Fröhlich, Application of commutator theorems to the integration of representations of lie algebras and commutation relations, *Comm. Math. Phys.* **54**:135–150 (1977).
7. J. Fröhlich and M. Merkli, *Thermal Ionization*, to appear in *Math. Phys., Anal. Geom.* (2004).
8. V. Georgescu and C. Gérard, On the virial theorem in quantum mechanics, *Comm. Math. Phys.* **208**:275–281 (1999).
9. V. Jakšić and C.-A. Pillet, On a model for quantum friction III. Ergodic properties of the Spin–Boson system, *Comm. Math. Phys.* **178**:627–651 (1996).
10. M. Merkli, Positive commutators in non-equilibrium quantum statistical mechanics, *Comm. Math. Phys.* **223**:327–362 (2001).