Mean field evolution of open quantum systems
An exactly solvable model

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Abstract

We consider quantum particles coupled to local and collective thermal quantum environments. The coupling is energy conserving, and the collective coupling is scaled in the mean field way. There is no direct interaction between the particles. We show that an initially factorized state of the particles remains factorized at all times, in the limit of large particle number. Each single-particle factor evolves according to an explicit, nonlinear, dissipative and time-dependent Hartree-Lindblad equation. The model is exactly solvable, we do not make any weak coupling, or any markovian approximations, and our results are mathematically rigorous.

1 Introduction and Results

Studying effects of noise on quantum systems is of central importance in modern quantum theory. Quantum information processing is based on manipulation of superposition and entanglement of basic quantum bits forming a quantum processor. As any such system is subject to noise, e.g. due to contact with thermal environments, it is crucial to understand the mechanisms of noise effects, and to quantify them mathematically. It is known that interactions with environments generically destroy phase coherence (decoherence), as well as quantum correlations (entanglement). The literature on this subject is huge, we only refer to [12, 13, 21, 22] and references therein. Another (more positive) aspect is that, starting with disentangled states, one can create and control entanglement by coupling quantum systems to a common thermal noise, [2–6, 8–11, 14, 19, 23]. This opens up the possibility of manipulating quantum bits by controlling their surroundings. In the works cited above, the focus is on the entanglement of two subsystems interacting indirectly via a joint quantum thermal noise reservoir. (See, however, [14], for a numerical investigation of arbitrary numbers of subsystems, where it is shown that creation of two-spin entanglement is suppressed if N is large.)

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In the present paper, we consider entanglement creation in complex open quantum systems, i.e., for a large number N of particles interacting indirectly via a common thermal environment. As the number of particles increases, the total energy of the system increases (linearly in N). It is plausible that the interaction to the common heat bath creates an effective interaction between particles. The size of the interaction energy, for instance an induced two-body interaction, will be proportional to the number of pairs of particles, hence of order N^2. Therefore, in order to have a balanced competition between individual energy and interaction energy, one scales the interaction with a negative power of N. As we show, there is exactly one "correct" scaling, having the property that the dynamics is non-trivial in the limit of infinitely many particles. In many-body quantum physics, an N-dependent scaling of particle interactions is called a *mean field* scaling. So far, mean field models of directly interacting particles (coupled via two-body potentials) have been considered. For these systems, it is known that in the mean field picture, each particle feels the same averaged effect of all other particles [20]. Each particle thus evolves independently, albeit according to a more complicated dynamics containing interaction effects. The equation governing the evolution of the state of a single particle turns out to be nonlinear in the state and is called the nonlinear Hartree equation. The emerging picture is that high complexity (large numbers N of particles) favours particle independence, but complicates the individual dynamics.

In the present paper, we use ideas from mean field theory in the context of indirect interactions in complex open quantum systems. Here, a large number N of particles do not interact directly, but via a common thermal quantum noise. One motivation to consider such interactions is the question of whether a common noise can create entanglement in a particle system, the answer to which is shown here to be negative if N is large. We are not aware of any previous work considering this type of interaction. We show that for disentangled initial states, and for all times, the state of the particles is very close to a product state, the difference being of the order $1/N$. We identify the dynamical equation of a single particle. It is a time-dependent effective evolution equation containing nonlinear Hartree terms as well as dissipative Lindblad contributions. The dynamics is not markovian, as is visible in our dynamical equations via the dependence on time, of both the nonlinear and the dissipative contributions. Our model has purely energy-conserving (purely dephasing, non-demolition) particle-reservoir interactions. As a consequence, we can solve the dynamical equations explicitly and obtain exact results, for all times and for all strengths of interaction between particles and reservoirs. This kind of interactions are physically justified for systems (or time spans) in which decoherence effects dominate relaxation effects (see also references above). A rigorous treatment of systems including energy-exchange interactions between particles and reservoirs is technically much more involved. We are going to address this problem elsewhere.

Explanation of main result. We consider N quantum particles interacting with local and collective thermal environments. Each particle is in contact with its own, independent environment, and coupled to another one, common to all particles. There is no direct interaction, but the collective environment induces interaction among

all particles. Our main result is the following: Starting in a factorized (product) initial state for N particles, $\rho_0 \otimes \cdots \otimes \rho_0$, the reduced density matrix for any n particles ($n \leq N$ fixed), at any given moment in time t, approaches a product state as $N \to \infty$. We summarize this as

$$\rho_{n,N}(t) \longrightarrow \rho_t \otimes \cdots \otimes \rho_t, \quad N \to \infty,$$

where the single particle density matrix ρ_t is the solution of the equation

$$\mathrm{i}\dot{\rho}_t = [A, \rho_t] + \mathrm{Tr}_2[W_{\mathrm{eff}}(t), \rho_t \otimes \rho_t] + \mathcal{L}(t)\rho_t.$$

See Theorem 1.1 below for a precise statement. The first commutator term represents the free dynamics, A being the Hamiltonian of a single particle. The trace term (the trace is taken on the second factor of the space of two particles) is due to the indirect interaction of particles via the common environment. It is called the Hartree term and is quadratic in the density matrix. The operator $W_{\mathrm{eff}}(t)$ is an effective two-body interaction. The interaction with the collective thermal quantum noise creates k-body interactions in the particle system, for all $k \geq 2$. The lowest order is a pair interaction, and higher order corrections to the Hartree-Lindblad equation, which are proportional to powers in $1/N$, involve many-body effective interactions (we are going to explain the details of this in a subsequent publication). The last term in the evolution equation is a time-dependent Lindblad term, describing the effect of the local environment. Both the effective two-body potential and the Lindblad operator are given by explicit time-dependent operators.

The convergence to a product state implies that complexity (large N) prevents the particles from ever becoming entangled, even though they interact via a common environment. We give an estimate on the difference between the reduced n-body density matrix and the limit product state in Theorem 1.2,

$$\mathrm{Tr}|\rho_{n,N}(t) - \rho_t \otimes \cdots \otimes \rho_t| \leq \frac{C_n(t)}{N},$$

provided N is large enough, and with an explicit $C_n(t)$. An upper bound is $C_n(t) \leq \varkappa^2(1 + t + \varkappa^2 t)\mathrm{e}^{n[1+n\varkappa^2 t(1+\varkappa^2 t)]}$, where $\varkappa \in \mathbb{R}$ is the particle-reservoir interactions strength (see, however, Theorem 1.2 for a better bound).

Relation to previous work. There are previous works on mean field open quantum systems, see [7] and references therein. However, in all of them, the dynamics of the N particles is considered to be given by a master equation. By using a master equation, one implicitly assumes that a markovian (weak coupling, van Hove) limit of the dynamics has been taken. The mean field limit dynamics, originating from the master equation in the $N \to \infty$ limit, is thus not guaranteed to approximate the true dynamics (which is not markovian). The technical issue is that of an interchange of two limits: that of a large number of particles and that of small coupling. In the present paper, we do not encounter this problem, as we are able to solve the dynamical equations *exactly*. As a consequence, the single-particle dynamics we derive exhibits the correct non-markovian behaviour, expressed through a time-dependence of the

nonlinear and dissipative parts in the equation. This feature of the dynamics cannot be seen in an analysis based on master equations, as the mean field limit of a markovian dynamics will still be markovian.

Another difference between our and previous works is the model itself. In previous works, particles are taken to interact directly (via pair-potentials). In contrast, we consider here particles which do not interact directly, but only *indirectly*, via a coupling to a common noise (collective quantum heat bath reservoir). One of our motivations for considering such an interaction is the question of whether a common noise can produce entanglement in a particle system. We show here that entanglement creation is disabled by complexity (in the mean field limit).

Mathematical description. The Hilbert space of pure states of each particle is $\mathcal{H}$ with

$$d = \dim \mathcal{H} < \infty.$$

The Hamiltonian of particle j is denoted by A_j. To each particle is associated a local environment, and there is another, collective environment. Each of them is given by a spatially infinitely extended free Bose gas initially in thermal equilibrium (a heat bath). Without further effort, one could consider individual temperatures for each reservoir, but we assume they are all at a common temperature $T = 1/\beta > 0$. The free field Hamiltonian of a reservoir is given by

$$K = \int_{\mathbb{R}^3} |k| a^*(k) a(k) \mathrm{d}^3 k,$$

acting on the (symmetric) Fock space over the one-Boson space $L^2(\mathbb{R}^3, \mathrm{d}^3 k)$,

$$\mathcal{F} = \bigoplus_{n \geq 0} L^2_{\mathrm{symm}}(\mathbb{R}^{3n}, \mathrm{d}^{3n} k)$$

(momentum representation, where $|k|$ is the energy of a Boson). In some physics literature, one would write $K = \sum_k |k| a_k^\dagger a_k$ and understand that an infinite-volume (continuous momentum) limit is performed. The creation and annihilation operators satisfy the usual canonical commutation relations $[a(k), a^*(l)] = \delta(k - l)$. The equilibrium state of a reservoir is determined by $\langle a^*(k) a(l) \rangle_\beta = \frac{\delta(k-l)}{\mathrm{e}^{\beta|k|} - 1}$ (plus Wick's rule). We refer to [1] for more detail on the mathematical description of reservoirs. Let $f \in L^2(\mathbb{R}^3, \mathrm{d}^3 k)$ be a "form factor". We define the field operator

$$\varphi(f) = \frac{1}{\sqrt{2}} \left(a^*(f) + a(f) \right),$$

where $a^*(f) = \int_{\mathbb{R}^3} f(k) a^*(k) \mathrm{d}^3 k$, and $a(f)$ is the adjoint of $a^*(f)$. Particle j interacts with its local reservoir through an interaction $\varkappa_j V_j \otimes \varphi_j(f_j)$, and to a collective reservoir via $\varkappa W_j \otimes \varphi(f)$. The operator $\varphi_j(f_j)$ acts on the j-th reservoir as $\varphi(f_j)$. We have introduced coupling strenghts $\varkappa$ and $\varkappa_j$, which are arbitrary real numbers (not necessarily small). The Hilbert space of the total system is

$$\mathcal{H} \otimes \cdots \otimes \mathcal{H} \otimes \mathcal{F} \otimes \mathcal{F} \otimes \cdots \otimes \mathcal{F},$$

with N copies of the single particle space $\mathcal{H}$ and $N+1$ copies (N local plus one collective) of the single reservoir space $\mathcal{F}$ (Fock space). The total Hamiltonian is

$$\begin{aligned} H_N &= \sum_{j=1}^{N} A_j + \sum_{j=1}^{N} K_j + K && (1.1)\\ &\quad + \sum_{j=1}^{N} \varkappa_j V_j \otimes \varphi_j(f_j) && (1.2)\\ &\quad + \frac{\varkappa}{\sqrt{N}} \sum_{j=1}^{N} W_j \otimes \varphi(f). && (1.3) \end{aligned}$$

Each of the K_j is the same operator K, but acting on the space of the j-th reservoir. The collective interaction is homogeneous, in that W_j is a fixed operator W, acting on the space of the jth particle. It is assumed that the interactions are purely dephasing (energy conserving, of non-demolition type), in the sense that they are diagonalized jointly with the free Hamiltonians A_j. This kind of interaction produces phase processes, such as decoherence and entanglement [18]. It does not describe relaxation processes (populations, or the diagonal of the density matrix in the energy basis, are constant).

The time-dependent reduced n-body density matrix is

$$\rho_{n,N}(t) = \mathrm{Tr}_{[n+1,N]} \mathrm{e}^{-\mathrm{i}tH_N} \rho_0 \otimes \cdots \otimes \rho_0 \otimes \rho_{\vec{R}}\ \mathrm{e}^{\mathrm{i}tH_N}. \tag{1.4}$$

Here, ρ_0 is the initial single-particle density matrix, and $\rho_{\vec{R}}$ is the product state of all N local reservoirs and the collective one, each in its own equilibrium (at a fixed temperature $1/\beta$). The symbol $\mathrm{Tr}_{[n+1,N]}$ means that we take the trace over all degrees of freedom of particles $n+1, n+2, \ldots, N$, and over all reservoirs.

The following two functions are key dynamical quantities

$$\begin{aligned} S(t) &= \frac{1}{2} \int_{\mathbb{R}^3} |f(k)|^2 \frac{|k|t - \sin(|k|t)}{|k|^2} \mathrm{d}^3k, && (1.5)\\ \Gamma(t) &= \int_{\mathbb{R}^3} |f(k)|^2 \coth(\beta|k|/2) \frac{\sin^2(|k|t/2)}{|k|^2} \mathrm{d}^3k. && (1.6) \end{aligned}$$

We denote by $S_j(t)$ the quantity (1.5) with f replaced by f_j, and similarly we define $\Gamma_j(t)$.

Theorem 1.1 (Convergence to Hartree-Lindblad dynamics) *For any $t \in \mathbb{R}$ and $n \geq 1$ we have*

$$\lim_{N\to\infty} \mathrm{Tr}\, |\rho_{n,N}(t) - \rho_{1,t} \otimes \cdots \otimes \rho_{n,t}| = 0.$$

The single particle density matrix $\rho_{j,t}$ satisfies the time-dependent Hartree-Lindblad equation

$$\mathrm{i}\dot{\rho}_{j,t} = [A_j, \rho_{j,t}] + \varkappa^2 \mathrm{Tr}_2[W_{\mathrm{eff}}(t), \rho_{j,t} \otimes \rho_{j,t}] + \varkappa_j^2 \mathcal{L}_j(t)\rho_{j,t}, \tag{1.7}$$

where the effective two-particle interaction and the Lindblad operator are

$$\begin{aligned} W_{\text{eff}}(t) &= 2\dot{S}(t) W \otimes W \\ \mathcal{L}_j(t)\rho &= \dot{S}_j(t)[V_j^2, \rho] - \mathrm{i}\dot{\Gamma}_j(t)[V_j, [V_j, \rho]]. \end{aligned}$$

Remarks: **1.** The coupling constants $\varkappa$ and $\varkappa_j$ can take arbitrary values in $\mathbb{R}$ (they do not have to be small). In the outline of our results at the beginning of the introduction, we have taken them to be equal to one, for simplicity of the explanations.

2. Equation (1.7) has three terms on the right side. The first one originates from the free individual dynamics of spin j. The second one describes the effect of all spins on spin j, through an effective interaction operator W_{eff}, and is quadratic in the density matrix. This effective interaction is due entirely through indirect coupling of the spins, mediated by a common heat bath. The third term is due to the local reservoirs. Note that the noise effects are independent, in the sense that each term on the right side of (1.7) can be switched on and off individually by its own coupling constant.

3. Equation (1.7) is time-dependent. This means that the dynamics is not markovian, a well-known property of open quantum systems.

The convergence of the density matrix as $N \to \infty$ is given in the "trace norm", which is the trace of the absolute value of an operator. The absolute value of an operator is defined by $|B| = \sqrt{B^*B}$. The choice of the trace norm is motiated as follows. Suppose that A is an operator acting on n particles, where $n \leq N$ is a fixed number. By Theorem 1.1 we have

$$\begin{aligned} &|\mathrm{Tr}(\rho_{n,N}(t)A) - \mathrm{Tr}(\rho_{1,t} \otimes \cdots \otimes \rho_{n,t}A)| \\ &\quad \leq \|A\| \ \mathrm{Tr}|\rho_{n,N}(t) - \rho_{1,t} \otimes \cdots \otimes \rho_{n,t}| \longrightarrow 0 \end{aligned}$$

as $N \to \infty$. We use here that $|\mathrm{Tr}AB| \leq \|A\| \,\mathrm{Tr}|B|$, where $\|\cdot\|$ is the operator norm, $\|A\| = \sup_{\|x\|=1} \|Ax\|$ (the supremum is taken over all normalized vectors in the Hilbert space on which the operator A acts). This means that for large N, the expectation of the observable A is obtained as an average in a product state. In particular, (any measure of) entanglement between arbitrary n particles vanishes for all times, in the limit of large complexity. The following result estimates how quickly this happens as complexity grows.

Theorem 1.2 (Convergence speed) *Let $\eta = 4\varkappa^2 n\|W\|^2$ and recall that d is the dimension of the single particle Hilbert space. For $t \in \mathbb{R}$ and $1 \leq n \leq N$, such that $N > \eta|S(t)|$, we have*

$$\mathrm{Tr}\,|\rho_{n,N}(t) - \rho_{1,t} \otimes \cdots \otimes \rho_{n,t}| \leq \frac{C_n(t)}{N},$$

where $C_n(t) = \eta d^{2n}\left((2n + 2\eta|S(t)| + 1)\mathrm{e}^{\eta|S(t)|(n+2\eta|S(t)|+1)} + n|\Gamma(t)|\right)$.

For large times, we have $S(t) \to \frac{t}{4}\int_{\mathbb{R}^3} |k|^{-1}|f(k)|^2 \mathrm{d}^3k$ and $\Gamma(t) \to \frac{t}{2\beta}\lim_{\omega\downarrow 0} J(\omega)/\omega$, where $J(\omega) = \pi\omega^2 \int_{S^2} |f(\omega, \Sigma)|\mathrm{d}\Sigma$ is the (collective) reservoir spectral density, see [15, 16].

2 Illustration: N spins $1/2$ (qubits)

We consider (1.1)-(1.3) for N spins, each one having pure state space $\mathcal{H} = \mathbb{C}^2$ and each Hamiltonian A_j being equal to

$$A = \omega S^z = \frac{\omega}{2} \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right],$$

$\omega > 0$ being the frequency of a spin (in units of $\hbar$). The noise interaction operators are the same for each spin, i.e., $V_j = V$ and $W_j = W$, with

$$V = W = S^z.$$

The coupling constants are taken to be homogeneous, $\varkappa_j = \varkappa_{\rm l}$ (local) for all j and the collective coupling constant is denoted $\varkappa = \varkappa_{\rm c}$. We assume the local form factors in (1.2) to be all equal, $f_j = f_{\rm l}$ and denote the collective one in (1.3) by $f = f_{\rm c}$. According to Theorem 1.1, the single spin density matrix evolves according to the Hartree-Lindblad equation (1.7),

$$\mathrm{i}\dot{\rho} = \omega[S^z, \rho_t] + 2\varkappa_{\rm c}^2 \dot{S}_{\rm c}(t) \mathrm{Tr}_2[S^z \otimes S^z, \rho_t \otimes \rho_t] - \mathrm{i}\dot{\Gamma}_{\rm l}(t)\varkappa_{\rm l}^2 [S^z, [S^z, \rho_t]]. \tag{2.1}$$

This equation is the same for each spin j, hence we do not write this index. Here, $S_{\rm c}(t)$ and $\Gamma_{\rm l}(t)$ are the quantities (1.5) and (1.6) with f the collective and local form factors $f_{\rm c}$ and $f_{\rm l}$, respectively. Denote by $[\rho_t]_{m,n}$ the density matrix of ρ_t in the ordered energy basis $\{\varphi_1, \varphi_2\}$, where $S^z\varphi_1 = \varphi_1$, $S^z\varphi_2 = -\varphi_2$. Let ρ_0 be the initial condition $\rho_t|_{t=0}$ and denote its populations by $[\rho_0]_{1,1} = p \in [0,1]$, $[\rho_0]_{2,2} = 1 - p$. The solution to (2.1) is given by (see also (3.11))

$$[\rho_t]_{j,j} = [\rho_0]_{j,j}, \qquad j = 1, 2, \tag{2.2}$$

$$[\rho_t]_{1,2} = \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{e}^{-\frac{\mathrm{i}}{2}\varkappa_{\rm c}^2 (2p-1) S_{\rm c}(t)} \, \mathrm{e}^{-\varkappa_{\rm l}^2 \Gamma_{\rm l}(t)} [\rho_0]_{1,2}. \tag{2.3}$$

The populations are constant since all interactions commute with the single particle Hamiltonian A. The single particle off-diagonal evolution (2.3) has two oscillatory factors and a decaying one. The first oscillatory factor is due to the free evolution of a single spin, the other one is induced by the collective interaction and depends on the initial state. The relaxation, or decay term originates purely from the local coupling. One can view the collection of all spins as some environment for each fixed single spin. Equation (2.3) shows that the effect of such an environment is entirely different from the usual one, induced by thermal noises (an infinite Bose gas initially in equilibrium). Indeed, the 'other spin environment' only contributes with a phase factor $\mathrm{e}^{-\frac{\mathrm{i}}{2}\varkappa_{\rm c}^2(2p-1)S_{\rm c}(t)}$ to the dynamics of the off-diagonal single particle density matrix elements, while the thermal reservoir generates a decaying factor $\mathrm{e}^{-\varkappa_l^2 \Gamma_{\rm l}(t)}$. We phrase this as follows.

In the mean field limit, the cumulative effect of all spins on a fixed single one gives a modulation of the dephasing process, it does not accelerate decoherence. Decoherence is driven entirely by the coupling to local reservoirs.

One can view the spins as magnetic entities. Denote by S^x, S^y and S^z the three spatial components of a single spin (spin operators, proportional to the Pauli matrices). The Hamiltonian A introduced at the beginning of this section is the energy of a spin placed in a magnetic field in the z direction, of magnitude $B_z = \omega/\gamma$, where γ is the gyromagnetic ratio. Define the transversal spin operator by

$$S^- := S^x - \mathrm{i}S^y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The average of S^- at time t is precisely $\mathrm{Tr}(\rho_t S^-) = [\rho_t]_{2,1}$, the complex conjugate of (2.3). It follows that the effect of the collective coupling, in the mean field way, is directly visible in the angular speed of precession of the spins around the z axis. When the spins are coupled merely to their local reservoirs, then this speed is simply ω. However, the effect of the presence of a large number of other spins, coupled indirectly via the collective reservoir, and in the mean field scaling limit, becomes time-dependent and is given by

$$\omega + \varkappa_{\mathrm{c}}(p - 1/2)\frac{S_{\mathrm{c}}(t)}{t}.$$

Note that when $S_{\mathrm{c}}(t)$ approaches its linear limit at, the precession speed becomes $\omega + \varkappa_{\mathrm{c}}(p - 1/2)a$.

3 Proof of Theorems 1.1 and 1.2

3.1 Extraction of the main term

There are rank-one projections $P_j^{(m)}$, acting on the jth particle space, s.t.

$$A_j = \sum_m a_j^{(m)} P_j^{(m)} \tag{3.1}$$

$$V_j = \sum_m v_j^{(m)} P_j^{(m)} \tag{3.2}$$

$$W_j = \sum_m w^{(m)} P_j^{(m)}, \tag{3.3}$$

where $a_j^{(m)}$, $v_j^{(m)}$ and $w^{(m)}$ are the (possibly repeated) eigenvalues of the operators A_j, V_j and W, respectively.

Let $K_{\vec{R}} = \sum_{j=1}^N K_j + K$ and let p_k be the population of level k in ρ_0, i.e., $P_j^{(m_k)} \rho_0 P_j^{(m_k)} = p_k P_j^{(m_k)}$. We have

$$\begin{aligned} \rho_{n,N}(t) = {} & \mathrm{e}^{-\mathrm{i}t(A_1+\cdots+A_n)} \sum_{m_1,\ldots,m_N} \sum_{m'_1,\ldots m'_n} \prod_{j=1}^n \left(P_j^{(m_j)} \rho_0 P_j^{(m'_j)}\right) \prod_{j=n+1}^N p_j \\ & \times \mathrm{Tr}_{\vec{R}} \left\{ \mathrm{e}^{-\mathrm{i}t(K_{\vec{R}}+I_\ell+I_{\mathrm{c}})} \rho_{\vec{R}} \, \mathrm{e}^{\mathrm{i}t(K_{\vec{R}}+I'_\ell+I'_{\mathrm{c}})} \right\} \mathrm{e}^{\mathrm{i}t(A_1+\cdots+A_n)}, \end{aligned} \tag{3.4}$$

where

$$
\begin{aligned}
I_\ell &= \sum_{j=1}^{N} \varkappa_j v_j^{(m_j)} \varphi_j(f_j) \\
I_{\rm c} &= \frac{\varkappa}{\sqrt{N}} \sum_{j=1}^{N} w^{(m_j)} \varphi(f) \\
I'_\ell &= \sum_{j=1}^{n} \varkappa_j v_j^{(m'_j)} \varphi_j(f_j) + \sum_{j=n+1}^{N} \varkappa_j v_j^{(m_j)} \varphi_j(f_j) \\
I'_{\rm c} &= \frac{\varkappa}{\sqrt{N}} \sum_{j=1}^{n} w^{(m'_j)} \varphi(f) + \frac{\varkappa}{\sqrt{N}} \sum_{j=n+1}^{N} w^{(m_j)} \varphi(f).
\end{aligned}
$$

We simplify the trace in (3.4). By independence of the reservoirs, we have

$$
{\rm e}^{-{\rm i}t(K_{\vec R}+I_\ell+I_{\rm c})} = {\rm e}^{-{\rm i}t[K+\frac{\varkappa}{\sqrt{N}}\sum_{j=1}^{N} w^{(m_j)}\varphi(f)]} \prod_{j=1}^{n} {\rm e}^{-{\rm i}t[K_j+\varkappa_j v_j^{(m_j)}\varphi_j(f_j)]},
$$

and similarly for the second exponential in the trace in (3.4). That trace thus becomes the product

$$
\begin{aligned}
&\prod_{j=1}^{n} \left\langle {\rm e}^{{\rm i}t[K_j+\varkappa_j v_j^{(m'_j)}\varphi_j(f_j)]}\ {\rm e}^{-{\rm i}t[K_j+\varkappa_j v_j^{(m_j)}\varphi_j(f_j)]} \right\rangle_\beta \\
&\quad \times \left\langle {\rm e}^{{\rm i}t[K+\frac{\varkappa}{\sqrt{N}}\{\sum_{j=1}^{n} w^{(m'_j)}+\sum_{j=n+1}^{N} w^{(m_j)}\}\varphi(f)]}\ {\rm e}^{-{\rm i}t[K+\frac{\varkappa}{\sqrt{N}}\sum_{j=1}^{N} w^{(m_j)}\varphi(f)]} \right\rangle_\beta, \qquad (3.5)
\end{aligned}
$$

where $\langle X\rangle_\beta$ is the average of X in the Bosonic equilibrium state at temperature $1/\beta$. One has explicitly [17], for $x, y \in \mathbb{R}$,

$$
\left\langle {\rm e}^{{\rm i}t(K+x\varphi(f))}\ {\rm e}^{-{\rm i}t(K+y\varphi(f))} \right\rangle_\beta = {\rm e}^{{\rm i}(x-y)(x+y)S(t)}\ {\rm e}^{-(x-y)^2\Gamma(t)}, \qquad (3.6)
$$

where $S(t)$ and $\Gamma(t)$ are given in (1.5) and (1.6). Using (3.5) and (3.6) we obtain

$$
\begin{aligned}
&{\rm Tr}_{\vec R}\left\{ {\rm e}^{-{\rm i}t(K_{\vec R}+I_\ell+I_{\rm c})} \rho_{\vec R}\ {\rm e}^{{\rm i}t(K_{\vec R}+I'_\ell+I'_{\rm c})} \right\} \\
&\quad = \prod_{j=1}^{n} {\rm e}^{{\rm i}\varkappa_j^2[v_j^{(m'_j)}-v_j^{(m_j)}][v_j^{(m'_j)}+v_j^{(m_j)}]S_j(t)}\ {\rm e}^{-\varkappa_j^2[v_j^{(m'_j)}-v_j^{(m_j)}]^2\Gamma_j(t)} \\
&\qquad \times {\rm e}^{{\rm i}\frac{\varkappa^2}{N}[\sum_{j=1}^{n}(w^{(m'_j)}-w^{(m_j)})][\sum_{j=1}^{n}(w^{(m'_j)}+w^{(m_j)})+2\sum_{j=n+1}^{N} w^{(m_j)}]S(t)} \\
&\qquad \times {\rm e}^{-\frac{\varkappa^2}{N}[\sum_{j=1}^{n}(w^{(m'_j)}-w^{(m_j)})]^2\Gamma(t)}. \qquad (3.7)
\end{aligned}
$$

We need to insert this expression into (3.4). The result is

$$\begin{aligned}\rho_{n,N}(t) &= \mathrm{e}^{-\mathrm{i}t(A_1+\cdots+A_n)} \sum_{m_1,\ldots,m_n} \sum_{m_1',\ldots,m_n'} \prod_{j=1}^{n} P_j^{(m_j)} \rho_0 P_j^{(m_j')} \\ &\times \prod_{j=1}^{n} \mathrm{e}^{\mathrm{i}\varkappa_j^2 [v_j^{(m_j')} - v_j^{(m_j)}][v_j^{(m_j')} + v_j^{(m_j)}] S_j(t)} \ \mathrm{e}^{-\varkappa_j^2 [v_j^{(m_j')} - v_j^{(m_j)}]^2 \Gamma_j(t)} \\ &\times \mathrm{e}^{\mathrm{i}\frac{\varkappa^2}{N} [\sum_{j=1}^n (w^{(m_j')} - w^{(m_j)})][\sum_{j=1}^n (w^{(m_j')} + w^{(m_j)})] S(t)} \\ &\times \mathrm{e}^{-\frac{\varkappa^2}{N} [\sum_{j=1}^n (w^{(m_j')} - w^{(m_j)})]^2 \Gamma(t)} \\ &\times \left[\sum_m p_m \ \mathrm{e}^{2\mathrm{i}\frac{\varkappa^2}{N} w^{(m)} \sum_{j=1}^n (w^{(m_j')} - w^{(m_j)}) S(t)} \right]^{N-n} \mathrm{e}^{\mathrm{i}t(A_1+\cdots+A_n)}. \end{aligned} \tag{3.8}$$

We separate (3.8) into a main term and a remainder, as $N \to \infty$, as follows. Define

$$\begin{aligned}\rho_{n,\infty}(t) &= \mathrm{e}^{-\mathrm{i}t(A_1+\cdots+A_n)} \sum_{m_1,\ldots,m_n} \sum_{m_1',\ldots,m_n'} \prod_{j=1}^{n} P_j^{(m_j)} \rho_0 P_j^{(m_j')} \\ &\times \prod_{j=1}^{n} \mathrm{e}^{\mathrm{i}\varkappa_j^2 [v_j^{(m_j')} - v_j^{(m_j)}][v_j^{(m_j')} + v_j^{(m_j)}] S_j(t)} \ \mathrm{e}^{-\varkappa_j^2 [v_j^{(m_j')} - v_j^{(m_j)}]^2 \Gamma_j(t)} \\ &\times \mathrm{e}^{2\mathrm{i}\varkappa^2 S(t) [\sum_m p_m w^{(m)}] \sum_{j=1}^n (w^{(m_j')} - w^{(m_j)})} \ \mathrm{e}^{\mathrm{i}t(A_1+\cdots+A_n)}. \end{aligned} \tag{3.9}$$

Note that this term is simply (3.8) with "N replaced by ∞". In this limit, of course $\varkappa^2/N \to 0$, and, by using that $a^x = \mathrm{e}^{x \log a}$ and the expansion $\log\left(\sum_m p_m \mathrm{e}^{\mathrm{i}x_m/N}\right) = \frac{\mathrm{i}}{N} \sum_m p_m x_m + O(N^{-2})$, see Proposition 3.1 below for details, we obtain that in the limit $N \to \infty$, the term $[\cdots]^{N-n}$ becomes the second last exponential in (3.9). We thus have, for n and t fixed,

$$\rho_{n,N}(t) = \rho_{n,\infty}(t) + R, \tag{3.10}$$

where $R := \rho_{n,N}(t) - \rho_{n,N}(t)$ is the reminder term, an operator depending on N, n, t. We estimate the size (trace norm) of R Section in 3.2. Here we continue the analysis of the main term $\rho_{n,\infty}(t)$.

We write the seond last eponential in (3.9) as the product

$$\prod_{j=1}^{n} \mathrm{e}^{2\mathrm{i}\varkappa^2 S(t) [\sum_m p_m w^{(m)}](w^{(m_j')} - w^{(m_j)})}$$

and noting that

$$\begin{aligned} &P_j^{(m_j)} \rho_0 P_j^{(m_j')} \mathrm{e}^{2\mathrm{i}\varkappa^2 S(t) [\sum_m p_m w^{(m)}](w^{(m_j')} - w^{(m_j)})} \\ &\quad = P_j^{(m_j)} \mathrm{e}^{-2\mathrm{i}\varkappa^2 S(t) [\sum_m p_m w^{(m)}] W} \rho_0 \ \mathrm{e}^{2\mathrm{i}\varkappa^2 S(t) [\sum_m p_m w^{(m)}] W} P_j^{(m_j')}, \end{aligned}$$

we see that the r.h.s. of (3.9) is of the product form $\rho_{1,t} \otimes \cdots \otimes \rho_{n,t}$, where

$$\begin{aligned}\rho_{j,t} &= \mathrm{e}^{-\mathrm{i}tA_j} \sum_{m,m'} P_j^{(m)} \mathrm{e}^{-2\mathrm{i}\varkappa^2 S(t)[\sum_k p_k w^{(k)}]W} \rho_0 \ \mathrm{e}^{2\mathrm{i}\varkappa^2 S(t)[\sum_k p_k w^{(k)}]W} P_j^{(m')} \\ &\quad \times \mathrm{e}^{\mathrm{i}\varkappa_j^2 [v_j^{(m')} - v_j^{(m)}][v_j^{(m')} + v_j^{(m)}] S_j(t)} \ \mathrm{e}^{-\varkappa_j^2 [v_j^{(m')} - v_j^{(m)}]^2 \Gamma_j(t)} \ \mathrm{e}^{\mathrm{i}tA_j}. \end{aligned} \tag{3.11}$$

We take the derivative w.r.t. time of the last equation to obtain

$$\begin{aligned}\mathrm{i}\dot{\rho}_{j,t} &= [A_j, \rho_{j,t}] + 2\varkappa^2 \dot{S}(t) \sum_m p_m w^{(m)} [W, \rho_{j,t}] \\ &\quad - \varkappa_j^2 \dot{S}_j(t)(\rho_t V^2 - V^2 \rho_t) - \mathrm{i}\varkappa_j^2 \dot{\Gamma}_j(t)(V^2 \rho_t - 2V\rho_t V + \rho_t V^2). \end{aligned} \tag{3.12}$$

The sum over m is

$$\sum_m p_m w^{(m)} = \mathrm{Tr}(\rho_0 W) = \mathrm{Tr}(\rho_{j,t} W),$$

where the last equality holds since $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}(\rho_{j,t}W) = 0$ (use e.g. (3.12) to see this). It follows that

$$\sum_m p_m w^{(m)} [W, \rho_{j,t}] = \mathrm{Tr}_2([W \otimes W, \rho_{j,t} \otimes \rho_{j,t}]). \tag{3.13}$$

The trace is taken over the second space. Combining (3.12) and (3.13) yields the evolution equation in Theorem 1.1.

3.2 Control of convergence speed

We now investigate the speed of convergence of R, as $N \to \infty$. From the definition of R, (3.10), it follows that

$$R = \sum_{m_1,\ldots,m_n} \sum_{m_1',\ldots,m_n'} \mathrm{e}^{\mathrm{i}t\alpha} \{f_1 + f_2\} \prod_{j=1}^n P_j^{(m_j)} \rho_0 P_j^{(m_j')}, \tag{3.14}$$

where $\alpha = \sum_{j=1}^n (a_{m_j'} - a_{m_j})$, and, setting

$$x_m = 2\varkappa^2 w^{(m)} \sum_{j=1}^n (w^{(m_j')} - w^{(m_j)}) S(t), \tag{3.15}$$

$$f_1 = \left[\sum_m p_m \ \mathrm{e}^{\mathrm{i}x_m/N} \right]^{N-n} - \mathrm{e}^{\mathrm{i}\sum_m p_m x_m} \tag{3.16}$$

$$\begin{aligned} f_2 &= \left(\mathrm{e}^{\mathrm{i}\frac{\varkappa^2}{N}[\sum_{j=1}^n (w^{(m_j')} - w^{(m_j)})][\sum_{j=1}^n (w^{(m_j')} + w^{(m_j)})] S(t)} \ \mathrm{e}^{-\frac{\varkappa^2}{N}[\sum_{j=1}^n (w^{(m_j')} - w^{(m_j)})]^2 \Gamma(t)} - 1 \right) \\ &\quad \times \left[\sum_m p_m \ \mathrm{e}^{\mathrm{i}x_m/N} \right]^{N-n} \end{aligned} \tag{3.17}$$

R is an operator on the n-fold tensor product $\mathcal{H}_n := \mathcal{H}\otimes\cdots\otimes\mathcal{H}$, where $\mathcal{H}$ is the Hilbert space of a single particle. Since the space of bounded linear operators on $\mathcal{H}_n$, denoted by $\mathcal{B}(\mathcal{H}_n)$ (and having the usual operator norm $\|\cdot\|$) is the dual space of the Banach space $L_1(\mathcal{H}_n)$ of trace-class operators on $\mathcal{H}_n$ (with norm $\|x\|_1 = \mathrm{Tr}|x|$), we have

$$\|R\|_1 = \sup_{B\in\mathcal{B}(\mathcal{H}_n),\|B\|=1} \big|\mathrm{Tr}RB\big|. \tag{3.18}$$

Let B be a bounded operator on $\mathcal{H}_n$. By cyclicity of the trace, we have

$$\mathrm{Tr}RB = \mathrm{Tr}\left(\rho_0\otimes\cdots\otimes\rho_0 \sum_{m_1,\ldots,m_n}\sum_{m_1',\ldots,m_n'} \mathrm{e}^{\mathrm{i}t\alpha}\{f_1+f_2\}\prod_{j=1}^n P_j^{(m_j')}BP_j^{(m_j)}\right). \tag{3.19}$$

Since $|\mathrm{Tr}XY| \le \|Y\|\mathrm{Tr}|X|$, and since a density matrix has trace one, we obtain from (3.19) the bound

$$\begin{aligned} |\mathrm{Tr}RB| &\le \left\| \sum_{m_1,\ldots,m_n}\sum_{m_1',\ldots,m_n'} \mathrm{e}^{\mathrm{i}t\alpha}\{f_1+f_2\}\prod_{j=1}^n P_j^{(m_j')}BP_j^{(m_j)}\right\| \\ &\le \|B\|\; d^{2n} \sup_{m_1,\ldots,m_n,m_1',\ldots,m_n'} |f_1+f_2|, \end{aligned} \tag{3.20}$$

where $d = \dim\mathcal{H}$.

We estimate the supremum of $|f_2|$. For x real, we have $|\mathrm{e}^{\mathrm{i}x/N}-1| = |\int_0^{x/N}\mathrm{e}^{\mathrm{i}y}\mathrm{d}y| \le |x|/N$. Similarly, for $x\ge 0$, we have $|\mathrm{e}^{-x/N}-1| = |\int_0^{x/N}\mathrm{e}^{-y}\mathrm{d}y| \le x/N$. Furthermore, the last term $[\cdots]^{N-n}$ in the expression (3.17) for f_2 has modulus less than or equal to 1 (since $\sum_m p_m = 1$). We obtain

$$\sup_{m_1,\ldots,m_n,m_1',\ldots,m_n'} |f_2| \le \frac{1}{N}4\varkappa^2n^2\|W\|^2(|S(t)|+|\Gamma(t)|). \tag{3.21}$$

In order to estimate $|f_1|$ we establish the following result.

Proposition 3.1 *Let x_m be as in (3.15) and set $\xi = 4\varkappa^2 n\|W\|^2|S(t)|$. Suppose that $N > 2\xi$. Then we have*

$$\left(\sum_m p_m\mathrm{e}^{\mathrm{i}x_m/N}\right)^{N-n} = \mathrm{e}^{\mathrm{i}\sum_m p_m x_m} + R',$$

where $R'\in\mathbb{C}$ satisfies $|R'| \le \frac{\xi}{N}(n+2\xi+1)\mathrm{e}^{\frac{\xi}{N}(n+2\xi+1)}$.

We give a proof of Proposition 3.1 below. Combining the result of the proposition with (3.21) gives

$$\sup_{m_1,\ldots,m_n,m_1',\ldots,m_n'} |f_1+f_2| \le \frac{\eta}{N}\left((2n+2\eta|S(t)|+1)\mathrm{e}^{\eta|S(t)|(n+2\eta|S(t)|+1)} + n|\Gamma(t)|\right),$$

where $\eta = 4\varkappa^2 n\|W\|^2$. Using this bound in (3.20) we arrive at

$$\|R\|_1 \leq \frac{\eta d^{2n}}{N}\left((2n+2\eta|S(t)|+1)\mathrm{e}^{\eta|S(t)|(n+2\eta|S(t)|+1)}+n|\Gamma(t)|\right). \tag{3.22}$$

This is the bound given in the theorem.

Proof of Proposition 3.1. The left hand side equals $\mathrm{e}^{(N-n)\log\sum_m p_m \mathrm{e}^{\mathrm{i}x_m/N}}$. We have

$$\log\left(\sum_m p_m \mathrm{e}^{\mathrm{i}x_m/N}\right) = \log(1+a), \tag{3.23}$$

where

$$a := \sum_m p_m(\mathrm{e}^{\mathrm{i}x_m/N}-1).$$

Since x_m is real we have (as above) $|\mathrm{e}^{\mathrm{i}x_m/N}-1| \leq \frac{|x_m|}{N}$, so that $|a| \leq \frac{|x_m|}{N} < \frac{1}{2}$ (by the definition of ξ and the condition on N in the proposition). We can thus use the Taylor expansion $\log(1+a) = \sum_{k\geq 1}\frac{(-1)^{k+1}}{k}a^k$, valid for $|a|<1$ in (3.23), $\log(1+a) = a + a^2\sum_{k\geq 0}\frac{(-1)^{k+3}}{k+2}a^k$. The modulus of the last sum is bounded above by the value of the geometric series, $(1-|a|)^{-1} < 2$. It follows that $\log(1+a) = a + a^2 R_1$, with $|R_1| < 2$. Therefore,

$$(N-n)\log(1+a) = (N-n)a + R_2, \tag{3.24}$$

with $|R_2| < 2\frac{\xi^2}{N}$. Next, we have

$$\begin{aligned}(N-n)a &= \mathrm{i}\sum_m p_m x_m \frac{N-n}{N}\left[1+\frac{\mathrm{i}x_m}{N}\sum_{k\geq 0}\left(\frac{\mathrm{i}x_m}{N}\right)^k\frac{1}{(k+2)!}\right]\\ &= \mathrm{i}\sum_m p_m x_m + R_3,\end{aligned} \tag{3.25}$$

where $|R_3| \leq (n+1)\frac{\xi}{N}$. We combine (3.24) and (3.25),

$$(N-n)\log(1+a) = \mathrm{i}\sum_m p_m x_m + R_4, \tag{3.26}$$

with $|R_4| \leq \frac{\xi}{N}(n+2\xi+1)$. If follows from (3.26) that $\mathrm{e}^{(N-n)\log(1+a)} = \mathrm{e}^{\mathrm{i}\sum_m p_m x_m}\mathrm{e}^{R_4}$. Finally, we use the bound $|\mathrm{e}^{R_4}-1| \leq |R_4|\mathrm{e}^{|R_4|}$ (note that R_4 is complex, not real) to conclude that

$$(1+a)^{N-n} = \mathrm{e}^{\mathrm{i}\sum_m p_m x_m} + R_5, \tag{3.27}$$

where $|R_5| \leq \frac{\xi}{N}(n+2\xi+1)\mathrm{e}^{\frac{\xi}{N}(n+2\xi+1)}$. This completes the proof of Proposition 3.1 and hence that of Theorem 1.1. ∎

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