Suppression of Decoherence by Periodic Forcing

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Dedicated to Herbert Spohn,
for his scientific contributions and his friendship.

Abstract

We consider a finite-dimensional quantum system coupled to a thermal reservoir and subject to a time-periodic, energy conserving forcing. We show that, if a certain *dynamical decoupling condition* is fulfilled, then the periodic forcing counteracts the decoherence induced by the reservoir: for small system-reservoir coupling λ and small forcing period T, the system dynamics is approximated by an energy conserving and non-dissipative dynamics, which preserves coherences. For times up to order $(\lambda T)^{-1}$, the difference between the true and approximated dynamics is of size $\lambda + T$. Our approach is rigorous and combines Floquet and spectral deformation theory. We illustrate our results on the spin-fermion model and recover previously known, heuristically obtained results.

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1   Introduction and main results

The phenomenon of decoherence – the destruction of quantum coherence – which leads to the transition from quantum to classical behaviour, is one of the central phenomena in quantum physics. Decoherence is crucial for processing quantum information and it presents the main obstacle to building quantum computers.

The most relevant aspect for quantum information processing is decoherence due to the interaction with the environment [1, 2], and its mathematical understanding is at the heart of possible schemes to prevent it, or to slow it down. The present work investigates the control of environment-induced decoherence by subjecting the system to a periodic external force. Periodic forcing as a means (among others) to dampen decoherence has been proposed in the recent theoretical physics literature [3, 4]. Our contribution to the topic in the present paper is a mathematical analysis of the evolution of periodically driven open quantum systems and a rigorous proof that decoherence is suppressed by periodic forcing, provided the latter fulfills the *Dynamical Decoupling Condition* (12), which we adopt from [3, 4].

Let $\rho_{\mathrm{s}}(t)$ be the *reduced density matrix* of a quantum system, not subject to external forcing, but in contact with a thermal reservoir at temperature $1/\beta > 0$. Let H_{s} be the Hamiltonian of the system, with eigenvalues E_i and orthonormalized eigenvectors φ_i. The (energy basis) matrix elements of $\rho_{\mathrm{s}}(t)$ are given by $[\rho_{\mathrm{s}}(t)]_{i,j} = \langle \varphi_i, \rho_{\mathrm{s}}(t)\varphi_j \rangle$. If the system does not interact with the reservoir then its dynamics is unitary and simply given by $[\rho_{\mathrm{s}}(t)]_{i,j} = \mathrm{e}^{it(E_i-E_j)}[\rho_{\mathrm{s}}(0)]_{i,j}$. When the interaction is turned on, however, the system dynamics becomes irreversible, as energy transferred to the reservoir is dissipated into the vastness of its (infinite) volume. In generic situations where energy exchange takes place, the system plus its environment evolve into their *joint* equilibrium state at temperature $1/\beta$, regardless of the initial state of the system alone. This dynamical process is called thermalization. The reduction of the final equilibrium state to the system alone (obtained by tracing out the reservoir degrees of freedom) is, to lowest approximation in the interaction, the Gibbs equilibrium state $\rho_{\mathrm{Gibbs}} = \mathrm{e}^{-\beta H_{\mathrm{s}}}/\mathrm{Tr}\,\mathrm{e}^{-\beta H_{\mathrm{s}}}$. This means that

$$\lim_{t\to\infty} \rho_{\mathrm{s}}(t) \;=\; \rho_{\mathrm{Gibbs}} + R,$$

where R is a remainder of the order of the system-reservoir interaction strength. The approximate final state ρ_{Gibbs} is *diagonal in the energy basis*. Since off-diagonal density matrix elements represent quantum coherences [1], this means

that thermalization causes decoherence. Open systems with conserved energy H_{s} are also widely studied (they are often explicitly solvable), and one can show that typically, off-diagonal reduced density matrix elements decay to zero, while the diagonal elements are time-independent (here, thermalization does not occur) [2]. The latter process is called "pure dephasing" and is a form of decoherence. It thus appears that a generic effect of noise is to drive off-diagonal system density matrix elements to zero (modulo some small error). In [5, 6, 7], thermalization and decoherence are analyzed rigorously. It is shown there that the decay of off-diagonals is exponentially quick in time, with decay rates given by imaginary parts of complex system energies, associated to quantum resonances.

What happens if we superpose, to the noise coupling, a *structured* (time-periodic) external force $H_{\mathrm{c}}(t)$ acting on the system? Let T be the period of the forcing, and let λ be the system-environment coupling strength. Both T and λ are taken to be small. We show in Corollary 1.2 below that for times up to order $(\lambda T)^{-1}$, the dynamics is given by

$$[\rho_{\mathrm{s}}(t)]_{i,j} \;=\; \mathrm{e}^{i\Phi_{i,j}(t)}[\rho_{\mathrm{s}}(0)]_{i,j} + \mathcal{O}(\lambda + T),$$

where $\Phi_{i,j}(t)$ is a real phase. The off-diagonal density matrix elements do *not* decay, so the forcing has counteracted the decoherence effect of the reservoir. Comparing to the usual time asymptotics for matrix elements which generically decay on the scale $t \sim \lambda^{-2}$, the forcing has a noticeable effect provided $T \ll |\lambda|$ (see Remark 1.3).This result is a consequence of Theorem 1.1 below, which is our main result. In this theorem, we prove that if λ and T are small enough, then the system dynamics is close to the one generated by the system Hamiltonian *plus the forcing term*, but *without the interaction with the heat bath*. The approximating effective dynamics has thus the coherence-preserving property mentioned above, since it is assumed that the forcing operator commutes with the system Hamiltonian.

It is not difficult to see that, in general, if the interaction of the system with the reservoir commutes with H_{c}, then reservoir-induced, pure-phase decoherence cannot be suppressed by the additional periodic force on the system. We must therefore impose a condition of "effective" coupling of the external force to the system-reservoir interaction, called the *dynamical decoupling condition* (12), which has been identified in the literature before [3, 4]. To arrive at our results, we link the reduced dynamics to the spectrum of an effective propagator and employ spectral deformation techniques to analyze the latter.

Model and main results. We consider a small system, $\mathcal{S}$, interacting with the environment, also called the *reservoir*, $\mathcal{R}$, described by a free quantum field.

The main application we have in mind is a small system, such as one or a few qubits, which is coupled to the quantized radiation field or to the phonon field - both being boson fields.

Although free Bose fields are natural for modelling environments, in order to keep the exposition technically as simple as possible we assume that the reservoir $\mathcal{R}$ consists of free fermions. In the fermionic case, the interaction is a bounded operator and is easier to deal with. We expect, however, that, using the results and methods of [8, 9, 6, 7], where such models with bosonic reservoirs were analyzed rigorously, our treatment can be extended to the bosonic case without altering the essence of our conclusions and of our approach (a further discussion is given in Remark 1.9).

The forced dynamics of the system is generated by a time-dependent Hamiltonian

$$H_{\mathrm{s}} + H_{\mathrm{c}}(t)$$

acting on $\mathbb{C}^d$, the pure state Hilbert space of the system $\mathcal{S}$. H_{s} is the intrinsic Hamiltonian of $\mathcal{S}$. $H_{\mathrm{c}}(t)$ is an external forcing, the "control term". We assume that *the control term has period T and commutes with the system Hamiltonian at all times $t \geq 0$ (see Remark 1.10)*:

$$H_{\mathrm{c}}(t + T) = H_{\mathrm{c}}(t), \tag{1}$$

$$[H_{\mathrm{s}}, H_{\mathrm{c}}(t)] = 0. \tag{2}$$

The commutator $\delta_{\mathrm{s,c}} = \delta_{\mathrm{s}} + \delta_{\mathrm{c}}(t) = i[H_{\mathrm{s}}, \cdot] + i[H_{\mathrm{c}}(t), \cdot]$ is a symmetric derivation on the bounded linear operators $\mathcal{B}(\mathbb{C}^d)$. It determines the time-dependent Heisenberg evolution $\tau_{t,0}^{\mathrm{s,c}}$ of the forced system,

$$\partial_t \tau_{t,0}^{\mathrm{s,c}}(A) = \delta_{\mathrm{s,c}}(t)\big(\tau_{t,0}^{\mathrm{s,c}}(A)\big), \qquad \tau_{0,0}^{\mathrm{s,c}}(A) = A. \tag{3}$$

The map $A \mapsto \tau_{t,0}^{\mathrm{s,c}}(A)$ is a $*$–automorphism on $\mathcal{B}(\mathbb{C}^d)$. We have the representation

$$\tau_{t,0}^{\mathrm{s,c}}(A) = V_{\mathrm{c}}(t)\mathrm{e}^{\mathrm{i}t H_{\mathrm{s}}} A \mathrm{e}^{-\mathrm{i}t H_{\mathrm{s}}} V_{\mathrm{c}}(t)^*, \tag{4}$$

where the unitary $V_{\mathrm{c}}(t)$ is given by

$$V_{\mathrm{c}}(t) = \mathrm{id}_{\mathbb{C}^d} + \sum_{n \geq 1} i^n \int_{0 < t_1 < \ldots < t_n < t} H_{\mathrm{c}}(t_n) \ldots H_{\mathrm{c}}(t_1) \mathrm{d}t_1 \ldots \mathrm{d}t_n. \tag{5}$$

Consider now the reservoir $\mathcal{R}$. Its observable algebra is the canonical anticommutation relation (CAR) C^*–algebra $\mathcal{V}_{\mathcal{R}}$, generated by the annihilation and

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creation operators $a(f)$, $a^*(f)$, $f \in L^2(\mathbb{R}^3)$, acting on the antisymmetric Fock space over the one-particle space $L^2(\mathbb{R}^3)$. The creation and annihilation operators fulfill the CAR

$$a(f_1)a^*(f_2) + a^*(f_2)a(f_1) = \langle f_1, f_2 \rangle, \quad a(f_1)a(f_2) + a(f_2)a(f_1) = 0.$$

The reservoir dynamics is the Bogoliubov automorphism group

$$\tau_t^{\mathcal{R}}(a(f)) = a(\mathrm{e}^{ith_1}f), \quad f \in L^2(\mathbb{R}^3), \ t \in \mathbb{R}. \tag{6}$$

Here, h_1 is the diagonalized one–particle Hamiltonian, represented by the multiplication operator $f(\mathbf{p}) \mapsto |\mathbf{p}|f(\mathbf{p})$ on $L^2(\mathbb{R}^3)$. We denote by δ_{r} the generator of the strongly continuous group $\tau_t^{\mathcal{R}}$, i.e., $\partial_t \tau_t^{\mathcal{R}}(A) = \delta_{\mathrm{r}}(\tau_t^{\mathcal{R}}(A))$ for all $A \in \mathcal{V}_{\mathcal{R}}$. The thermal reservoir state $\omega_{\mathcal{R}}$ (the $(\beta, \tau_t^{\mathcal{R}})$–KMS state at inverse temperature $0 < \beta < \infty$) is the quasi-free state satisfying $\omega_{\mathcal{R}}(a(f)a(g)) = \omega_{\mathcal{R}}(a(f)) = 0$ and

$$\omega_{\mathcal{R}}(a^*(f)a(g)) = \langle g, [1 + \mathrm{e}^{\beta h_1}]^{-1}f \rangle. \tag{7}$$

The observable algebra of the total system is the C^*–algebra $\mathcal{V} := \mathcal{B}(\mathbb{C}^d) \otimes \mathcal{V}_{\mathcal{R}}$. The total dynamics is the solution of the differential equation

$$\partial_t \tau_{t,0}(A) = \delta(t)\big(\tau_{t,0}(A)\big)$$

with initial condition $\tau_{0,0}(A) = A$, where

$$\delta(t) := \delta_{\mathrm{s}} + \delta_{\mathrm{c}}(t) + \delta_{\mathrm{r}} + \lambda \delta_{\mathrm{s,r}} \tag{8}$$

is a time-dependent, symmetric derivation of the interacting and control-driven total system. Here, $\lambda \in \mathbb{R}$ is a (small) coupling constant and the interaction with the reservoir is

$$\delta_{\mathrm{s,r}} := i\,[Q \otimes \Phi(f), \cdot], \tag{9}$$

where Q is a self-adjoint operator on $\mathbb{C}^d$ and $\Phi(f) = \frac{1}{\sqrt{2}}[a^*(f) + a(f)]$ is the field operator (with $f \in L^2(\mathbb{R}^3)$ a "form factor"). Note that $\delta_{\mathrm{s,r}}$ is a bounded operator because the fields obey the Fermi statistics.

We make the following assumptions.

(A1) We have $T\|H_{\mathrm{s}}\| < \pi/2$.

(A2) For a form factor $f \in L^2(\mathbb{R}^3)$ define $g_f \in L^2(\mathbb{R} \times S^2)$ by

$$g_f(p,\vartheta) := |p| \left(1 + \mathrm{e}^{-\beta p}\right)^{-1/2} \begin{cases} f(p\vartheta) & , \quad p \geq 0 \,, \\ \overline{f(-p\vartheta)} & , \quad p < 0. \end{cases} \tag{10}$$

We assume that g_f and $\widetilde{g}_f(p,\vartheta) := i\overline{g_f(-p,\vartheta)}$ have analytic $L^2(S^2)$-valued continuations, in the variable p, to the strip $\mathbb{R} + i(-r_{\max}, r_{\max})$ for some $r_{\max} > 8\|H_{\mathrm{s}}\|$. Furthermore,

$$\sup_{|\theta| < r_{\max}} \int_{\mathbb{R} \times S^2} (1 + p^2)\left(|g_f(p+i\theta, \vartheta)|^2 + |\mathrm{e}^{-\frac{\beta}{2}(p+i\theta)}\widetilde{g}_f(p+i\theta, \vartheta)|^2\right)\mathrm{d}p\mathrm{d}\vartheta < \infty. \tag{11}$$

The method of (translation) analytic spectral deformation goes back to [8, 10] and requires the technical condition (A2) on the form factor - see Remark 1.8. Condition (A1) guarantees that resonances in this spectral deformation method do not overlap (see the discussion before (82)). We also assume the following *Dynamical Decoupling Condition*,

$$\int_t^{T+t} V_{\mathrm{c}}^*(s)QV_{\mathrm{c}}(s)\mathrm{d}s = 0, \qquad \text{for all } t \in \mathbb{R}. \tag{12}$$

This condition has already appeared in [3, 4]. It forces the system-reservoir interaction to allow for energy exchanges, for in the opposite case, the integral is simply TQ, which vanishes only in trivial cases. We refer to Remarks 1.5, 1.6 for further explanations about the dynamical decoupling condition.

We consider initial states

$$\omega_0 = \omega_{\mathcal{S}} \otimes \omega_{\mathcal{R}}, \tag{13}$$

where $\omega_{\mathcal{S}}$ is any state on $\mathcal{B}(\mathbb{C}^d)$ and $\omega_{\mathcal{R}}$ is the equilibrium state of the reservoir, determined by (7). Here is our main result.

Theorem 1.1 (Effective system dynamics)**.** *Suppose the T-periodic control term H_{c} satisfies the dynamical decoupling condition (12). Then there are constants $0 < c, C < \infty$, independent of $t, \lambda, T, H_{\mathrm{c}}$, and $\omega_{\mathcal{S}}$, such that*

$$\left|\omega_0\left(\tau_{t,0}(A \otimes \mathbf{1}_{\mathcal{R}})\right) - \omega_{\mathcal{S}}\left(\tau_{t,0}^{\mathrm{s,c}}(A)\right)\right|$$
$$\leq C\|A\|\left(|\lambda| + (D|\lambda| + 1)T + 1 - \mathrm{e}^{-ct|\lambda|T}\right) \tag{14}$$

for all $t \geq 0$, $A \in \mathcal{B}(\mathbb{C}^d)$ and all initial states $\omega_0 = \omega_{\mathcal{S}} \otimes \omega_{\mathcal{R}}$. Here,

$$D := T \max_{t \in [0,T]} \|H_{\mathrm{c}}(t)\|. \tag{15}$$

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We point out that $H_c(t)$ is of size T^{-1}, for T small (see Remarks 1.3, 1.5, and 1.6 below), hence the definition of D, (15). The bound (14) shows that up to times $t < (c|\lambda|T)^{-1}$, the reduced dynamics of the system is approximated by the automorphism group $\tau_{t,0}^{s,c}(A) = V_c(t)e^{itH_s}Ae^{-itH_s}V_c(t)^*$ on $\mathcal{B}(\mathbb{C}^d)$, which is the dynamics of the periodically driven system not subjected to the interaction with the reservoir, see (4).

Since $V_c(t)$ commutes with H_s, the approximated dynamics leaves eigenspaces of H_s invariant. In particular, if E is a simple eigenvalue of H_s with eigenvector φ, then $H_c(t)\varphi = \Theta(t)\varphi$ for some real $\Theta(t)$, and hence (see (5)) $V_c(t)\varphi = e^{i\int_0^t \Theta(s)ds}\varphi$. We thus obtain the following result for the dynamics of the reduced density matrix $\rho_s(t)$, which is defined by $\mathrm{Tr}(\rho_s(t)A) = \omega_0(\tau_{t,0}(A \otimes \mathbf{1}_\mathcal{R}))$.

Corollary 1.2 (Suppression of decoherence). *Suppose the eigenvalues $\{E_k\}_{k=1}^d$ of H_s are simple, with orthonormal eigenvectors $\{\varphi_k\}_{k=1}^d$. Then*

$$
\begin{aligned}
\langle \varphi_m, \rho_s(t)\varphi_n \rangle &= e^{-it(E_m - E_n)+i[\Theta_m(t)-\Theta_n(t)]} \langle \varphi_m, \rho_s(0)\varphi_n \rangle \\
&\quad + \mathcal{O}(|\lambda| + T + 1 - e^{-ct|\lambda|T}),
\end{aligned}
$$

where the $\Theta_k(t)$ are real-valued and depend only on $H_c(t)$.

Remark 1.3. Corollary 1.2 shows that, for small $|\lambda|$ and T, decoherence is suppressed for times $t < (c|\lambda|T)^{-1}$: the off-diagonal density matrix elements do not decay on this time-scale (modulo a small error of size $|\lambda| + T$). On the same time-scale, populations (diagonal elements) are constant, modulo small errors.

The time scale $|\lambda|^{-1}T^{-1}$ is is to be compared to the usual time scale λ^{-2} for decoherence and thermalization deriving in the limit of weak coupling. We hence conclude that the control term induces a noticable reduction of decoherence provided that

$$
T \ll |\lambda|. \tag{16}
$$

Remark 1.4. If the spectral subspaces of H_s are not one-dimensional, then the statement of the corollary is slightly modified. The reduced density matrix evolves "clusterwise": matrix elements belonging to a given cluster, i.e., those associated to a given spectral subspace, evolve jointly. Different clusters evolve independently [5, 6, 7].

Remark 1.5. The dynamical decoupling condition (12) implies $H_c \gtrsim T^{-1}$. Indeed, if $Q \neq 0$, then we obtain from (12) $\max_{0 \leq t \leq T} \|V_c(t) - \mathbf{1}\| \geq -1 + \sqrt{2}$, uniformly in T and Q (write Q as the sum of four terms using $1 = V + (1 - V)$

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and integrate). Thus (5) gives $\exp\{T\max_{0\leq t\leq T}\|H_{\mathrm{c}}(t)\|\} - 1 \geq -1 + \sqrt{2}$. This implies

$$\max_{0\leq t\leq T}\|H_{\mathrm{c}}(t)\| \geq \frac{\ln(2)}{2T}. \tag{17}$$

Remark 1.6. Let $T, T' > 0$. It is easy to see that the T-periodic $H_{\mathrm{c}}(t)$ satisfies (12) with T if and only if the T'-periodic $H_{\mathrm{c}}'(t) = \frac{T}{T'}H_{\mathrm{c}}(tT/T')$ satisfies (12) with T'. We also have $T\max_{1\leq t\leq T}\|H_{\mathrm{c}}(t)\| = T'\max_{1\leq t\leq T'}\|H_{\mathrm{c}}'(t)\|$. Therefore, by comparing with $T' = 1$, we see that $T\max_{1\leq t\leq T}\|H_{\mathrm{c}}(t)\| = \mathrm{const}$, independent of T. So the estimate $\max_{1\leq t\leq T}\|H_{\mathrm{c}}(t)\| = \mathcal{O}(T^{-1})$ is sharp.

Remark 1.7. Let $Q(t) := V_{\mathrm{c}}^*(t)QV_{\mathrm{c}}(t)$, $t \in \mathbb{R}$. In our analysis, we require the two conditions

$$Q(T) = Q \quad \text{and} \quad \widehat{Q}(0) := \frac{1}{T}\int_0^T Q(s)\mathrm{d}s = 0. \tag{18}$$

The condition $Q(T) = Q$ is equivalent to $Q(t + T) = Q(t)$ for all $t \in \mathbb{R}$. Indeed, due to the periodicity of $H_{\mathrm{c}}(t)$ (see (1)), both functions $t \mapsto Q(t + T)$ and $t \mapsto Q(t)$ satisfy the same differential equation $\frac{\mathrm{d}}{\mathrm{d}t}X(t) = -i[H_{\mathrm{c}}(t), X(t)]$ and thus the initial condition determines the solution uniquely. We show now that (12) and (18) are equivalent. Assume that (18) holds. Let $f(t) = \int_t^{t+T} Q(s)\mathrm{d}s$. Then $f'(t) = Q(t + T) - Q(t) = 0$ and therefore, $f(t) = f(0) = 0$, which implies condition (12). Therefore (18) implies (12). Conversely, assume that (12) holds. By taking the derivative w.r.t. t at $t = 0$, we obtain the first equality in (18). By setting $t = 0$ in (12) we obtain the second equality in (18). Therefore (12) implies (18). This shows that the dynamical decoupling condition (12) is *equivalent* to the two conditions (18). We need $Q(T) = Q$ in order to have periodicity of the dynamics in the interaction picture, see (22). The condition of a vanishing zero Fourier mode, $\widehat{Q}(0) = 0$, is used in the proof of Theorem 3.8, see (103).

Remark 1.8. In many applications, the interaction is isotropic, and the form factor $f \in L^2(\mathbb{R}^3)$ is a radial function, $f(p\vartheta) \equiv f(p)$. Then (A2) implies locally at $p = 0$ that $\mathbb{R}^+ \ni p \mapsto pf(p)$ is an even, real, regular function (which can be expanded in a power series about $p = 0$).

Remark 1.9. We are deriving our results for fermion fields because these are mathematically easier to deal with than boson fields, for the interaction is a bounded operator in this case. In contrast, the unboundedness of the interaction operator in

case of boson fields brings along several additional difficulties: The existence of the unitary time-evolution operator is not trivial anymore, especially, not for the explicitly time-dependent perturbation we are considering here. Analyticity of the family of Liouvilleans under consideration w.r.t. complex translation of the field momenta is not easy to see, either.

In anticipation of a very similar analysis for bosons, the Hamiltonian we study has the form of the spin-boson Hamiltonian - the model for the main application we have in mind -, but the fields fulfill the canonical anticommutation relations, rather than canonical commutation relations. With this replacement, however, the model as such is physically not meaningful because it mixes the even and the odd sectors of the fermion algebra. We note, however, that it is easy to extend our analysis from $\delta_{\mathrm{s,r}} = i \left[Q \otimes \Phi(f), \ \cdot \right]$ in (9) to $\delta_{\mathrm{s,r},\mathcal{P}} := i \left[Q \otimes \mathcal{P}(\Phi), \ \cdot \right]$, where $\mathcal{P}(\Phi)$ is an arbitrary real polynomial in the field Φ. We consider a coupling linear in Φ to keep technical aspects as simple as possible. See also [11, 1.3. Remarks.].

Remark 1.10. On a technical level, Eq. (2) is a key assumption for the derivation of our results, as it allows to move the explicit time-dependence of the forcing, which is big in magnitude, into an explicit time-dependence of the interaction, which is small in magnitude. Physically, if the small system is an atom held in a magnetic trap and the forcing is a magnetic force, this means that the directions of these two magnetic fields are parallel. It would be desirable, however, to also treat the case in which the forcing is perpendicular to the magnetic trap field. We do not study that case in the present paper. Indeed, note that a forcing parallel to the trap field corresponds to a control term $H_{\mathrm{c}}(t)$ commuting with the atom Hamiltonian H_{s}. In this situation the populations (diagonal terms of the density matrix) of the initial state are not affected by the forcing. By contrast, if the forcing is perpendicular to the trap field, then $H_{\mathrm{c}}(t)$ does not commute with H_{s} and even a very small forcing can cause drastic changes of populations of the atom, by a so-called "pumping" mechanism, at large enough times (as compared to $\|H_{\mathrm{c}}(t)\|^{-1}$). Hence, in this case, the problem of conservation of quantum information is more subtle.

Organization of the paper. In Section 1.1 we outline the main steps of the proof of Theorem 1.1. Technical details are presented in Section 3. In Section 2 we apply our results to the spin-fermion model. We obtain explicit expressions for the times until which coherence is preserved (the 'controlled decoherence times'). They coincide with those found by formal computations in [3].

1.1 Outline of proof of Theorem 1.1

The purpose of this section is to explain the main four steps in the proof of Theorem 1.1. The details are given in Section 3.

1.1.1 Interaction picture

By passing to the interaction picture, we transfer the dependence on the period T, which is considered a small parameter, into the interaction term. To this end, let $\mathfrak{H}_{\mathrm{s}}$ be the Hilbert space $\mathcal{B}(\mathbb{C}^d)$ endowed with the Hilbert–Schmidt scalar product $\langle A, B \rangle_{\mathrm{s}} := \mathrm{Tr}(A^*B)$, for $A, B \in \mathcal{B}(\mathbb{C}^d)$. Define the unitaries $\widetilde{V}_{\mathrm{c}}(t)$ on $\mathfrak{H}_{\mathrm{s}}$ by

$$\widetilde{V}_{\mathrm{c}}(t)A := V_{\mathrm{c}}(t)^* A V_{\mathrm{c}}(t), \quad A \in \mathfrak{H}_{\mathrm{s}}, \ t \in \mathbb{R}, \tag{19}$$

where $V_{\mathrm{c}}(t)$ is given in (5). We extend $\widetilde{V}_{\mathrm{c}}(t)$ to all of $\mathcal{V} = \mathcal{B}(\mathbb{C}^d) \otimes \mathcal{V}_{\mathcal{R}}$ by trivial action on $\mathcal{V}_{\mathcal{R}}$ and set, for $s \leq t \in \mathbb{R}$,

$$\tau_{t,s}^I := \widetilde{V}_{\mathrm{c}}(t) \circ \tau_{t,s} \circ \widetilde{V}_{\mathrm{c}}(s)^*. \tag{20}$$

We have

$$\partial_t \tau_{t,s}^I(A) = \delta_t^I \left(\tau_{t,s}^I(A) \right),$$

where the time-dependent symmetric derivation is

$$\delta_t^I := \delta_{\mathrm{s}} + \delta_{\mathrm{r}} + \lambda \delta_{\mathrm{s,r}}^I(t), \quad t \in \mathbb{R},$$

where

$$\begin{aligned} \delta_{\mathrm{s,r}}^I(t) &:= i \left[Q(t) \otimes \Phi(f), \cdot \right], \\ Q(t) &:= V_{\mathrm{c}}(t)^* Q V_{\mathrm{c}}(t). \end{aligned} \tag{21}$$

The dynamical decoupling condition (12) implies that $Q(t+T) = Q(t)$ (see (18)) and consequently, τ^I is T-periodic, i.e.,

$$\tau_{t+T,s+T}^I = \tau_{t,s}^I, \quad t \geq s. \tag{22}$$

1.1.2 Howland-Yajima Hilbert space

We represent the total system on its Gelfand-Naimark-Segal (GNS) Hilbert space $\mathfrak{H}$ with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$,

$$\omega_0(\tau_{t,0}^I(A)) = \langle \Omega_0, U_{t,0} \pi(A) \Omega_0 \rangle_{\mathfrak{H}}. \tag{23}$$

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Here, Ω_0 is the vector in $\mathfrak{H}$ representing the initial state ω_0, see (13), π is the representation map (sending observables to bounded operators on $\mathfrak{H}$), and $U_{t,0}$ is the implemented periodic dynamics, satisfying $U_{t+T,s+T} = U_{t,s}$ (see Theorem 3.2). Next, we pass to the Howland-Yajima Hilbert space

$$\mathfrak{H}_{\mathrm{per}} = L^2_{\mathrm{per}}(\mathbb{R}/T\mathbb{Z}, \mathfrak{H})$$

of T-periodic, $\mathfrak{H}$-valued functions, with inner product

$$\langle \psi, \chi \rangle_{\mathrm{per}} = \frac{1}{T} \int_0^T \langle \psi(x), \chi(x) \rangle_{\mathfrak{H}} \, \mathrm{d}x$$

[12, 13, 14]. We denote $\mathbb{T}_T := \mathbb{R}/T\mathbb{Z}$. For all $t \geq s$, we define $U^{\mathrm{per}}_{t,s}$ acting on $\mathfrak{H}_{\mathrm{per}}$ by

$$(U^{\mathrm{per}}_{t,s} \psi)(x) = U_{t,s}(\psi(x)), \qquad \psi \in \mathfrak{H}_{\mathrm{per}}, \ x \in \mathbb{T}_T.$$

Moreover, by

$$Z : \mathfrak{H} \to \mathfrak{H}_{\mathrm{per}}, \qquad (Z\psi)(x) := \psi,$$

we map vectors in $\mathfrak{H}$ to constant $\mathfrak{H}$-valued functions on $\mathbb{T}_T$. Then (23) reads

$$\omega_0(\tau^I_{t,0}(A)) = \left\langle Z\Omega_0, U^{\mathrm{per}}_{t,0} Z(\pi(A)\Omega_0) \right\rangle_{\mathrm{per}}.$$

The Howland, or Floquet generator $\mathcal{G}$ is defined on $\mathfrak{H}_{\mathrm{per}}$ by

$$(\mathrm{e}^{i\alpha\mathcal{G}} \psi)(t) = U^{\mathrm{per}}_{t,t-\alpha} \psi(t - \alpha), \tag{24}$$

the r.h.s. being a strongly continuous semigroup in $\alpha \geq 0$. In the remaining part of this section, we denote by A a system observable, acting on $\mathfrak{H}_{\mathrm{s}}$ (instead of writing $A \otimes \mathbf{1}_{\mathcal{R}}$). We have (Lemma 3.3)

$$\left| \langle \Omega_0, U_{\alpha,0} \pi(A)\Omega_0 \rangle_{\mathfrak{H}} - \left\langle Z\Omega_0, \mathrm{e}^{i\alpha\mathcal{G}} Z(\pi(A)\Omega_0) \right\rangle_{\mathrm{per}} \right| = \mathcal{O}(T). \tag{25}$$

This estimate can be understood as follows. Since

$$\langle \Omega_0, U_{\alpha,0} \pi(A)\Omega_0 \rangle_{\mathfrak{H}} = \langle Z\Omega_0, U^{\mathrm{per}}_{\alpha,0} Z(\pi(A)\Omega_0) \rangle_{\mathrm{per}},$$

the left side of (25) is given by $\frac{1}{T} \int_0^T \langle \Omega_0, (U_{\alpha,0} - U_{t,t-\alpha})\pi(A)\Omega_0 \rangle_{\mathfrak{H}} \, \mathrm{d}t$, and the difference $(U_{\alpha,0} - U_{t,t-\alpha})\pi(A)\Omega_0$ is small, for small T. That is, given α and t, we have $t + nT = \alpha + \epsilon$ for some integer n and some $0 \leq \epsilon < T$, and so by periodicity, $U_{t,t-\alpha}\pi(A)\Omega_0 = U_{\alpha+\epsilon,\epsilon}\pi(A)\Omega_0 = U_{\alpha,0}\pi(A)\Omega_0 + O(\epsilon)$.

1.1.3 Spectral deformation, resonances, extraction of main term

Let $\mathcal{G}_0$ denote the Howland generator in (24), for $\lambda = 0$ (no system-reservoir interaction). We show in Theorem 3.4 that

$$\left| \left\langle Z\Omega_0, (e^{i\alpha\mathcal{G}_0} - e^{i\alpha\mathcal{G}})Z(\pi(A)\Omega_0)\right\rangle_{\text{per}} \right| = \mathcal{O}(|\lambda| + T + e^{\alpha c|\lambda|T} - 1). \qquad (26)$$

We explain how to arrive at this bound. The Fourier transform $\mathfrak{F}$ with respect to t,

$$(\mathfrak{F}\psi)(k) \equiv \widehat{\psi}(k) := \frac{1}{T}\int_0^T e^{-2\pi ikt/T}\psi(t)\mathrm{d}t, \qquad (27)$$

where $k \in \mathbb{Z}$, maps $\mathfrak{H}_{\text{per}}$ into $\mathfrak{F}\mathfrak{H}_{\text{per}}\mathfrak{F}^* = \widehat{\mathfrak{H}}_{\text{per}} = l^2(\mathbb{Z}, \mathfrak{H})$. We denote the inner product of $\widehat{\mathfrak{H}}_{\text{per}}$ by $\langle\cdot,\cdot\rangle_{\widehat{\text{per}}}$ and set $\widehat{Z} = \mathfrak{F} \circ Z : \mathfrak{H} \to \widehat{\mathfrak{H}}_{\text{per}}$. By unitarity of the Fourier transform and deformation analyticity (translations), it suffices to prove the bound (26) in Fourier space and for the spectrally deformed generators $\widehat{\mathcal{G}}_0(\theta)$ and $\widehat{\mathcal{G}}(\theta)$, where

$$\widehat{\mathcal{G}}_0(\theta) := \mathfrak{F}\mathcal{G}_0(\theta)\mathfrak{F}^* = -\frac{2\pi}{T}k + L + \theta\widehat{N}. \qquad (28)$$

Here, k is the operator of multiplication by the argument in $l^2(\mathbb{Z}, \mathfrak{H})$, $L = L_{\text{s}} + L_{\mathcal{R}}$ is the self-adjoint Liouvillian on $\mathfrak{H}$ (acting fiber-wise on $l^2(\mathbb{Z}, \mathfrak{H})$, for each k), associated to the free motion of the system and of the reservoir. We have $L_{\text{s}} = [H_{\text{s}}, \cdot]$ (commutator) and $L_{\mathcal{R}}$ has a simple eigenvalue at the origin (the associated eigenvector being the KMS state), embedded in continuous spectrum covering all of $\mathbb{R}$. The $\widehat{N}$ is a "number operator", acting on $\mathfrak{H}$ (fiber-wise), commuting with L and having spectrum $\{0, 1, 2, \ldots\}$. The spectrum of $\widehat{\mathcal{G}}_0(\theta)$ consists of eigenvalues $\frac{2\pi}{T}\mathbb{Z} + \text{spec}([H_{\text{s}}, \cdot])$ on the real axis and continuous spectrum on the horizontal lines $\mathbb{R} + i\mathbb{N}\,\text{Im}\theta$. For $\text{Im}\,\theta > 0$, the branches of continuous spectrum are pushed into the upper complex half-plane $\mathbb{C}^+$, moved away from the eigenvalues on the real axis. Eigenvalues of $\widehat{\mathcal{G}}_0(\theta)$ associated to different k (see (28)) are separated from each other due to the condition (A1) of non-overlapping resonances (see before Theorem 1.1). For $|\lambda|$ small, this separation persists by perturbation theory. Denote by $\mathcal{P}_k(\lambda)$ the Riesz projection of $\widehat{\mathcal{G}}(\theta)$ associated to the group of real eigenvalues associated to $k \in \mathbb{Z}$, i.e., those lying in the vicinity of $\frac{2\pi}{T}k$. By the Laplace inversion formula, $e^{i\alpha\widehat{\mathcal{G}}(\theta)}$ is expressed as a complex line integral over the resolvent $(\mathcal{G}(\theta) - z)^{-1}$ along the horizontal path $\mathbb{R} - i$. Deforming this contour to the horizontal line $z = \mathbb{R} + ir/2$, where $0 < r \equiv \text{Im}\,\theta < r_{\max}$, one generates

residue contributions at the eigenvalues of $\mathcal{G}(\theta)$, which migrate (with varying λ) into the upper complex half-plane,

$$\left\langle \widehat{Z}\Omega_0, e^{i\alpha\widehat{\mathcal{G}}}\widehat{Z}(\pi(A)\Omega_0)\right\rangle_{\widehat{\mathrm{per}}} = \sum_{k\in\mathbb{Z}}\left\langle \widehat{Z}\Omega_0, e^{i\alpha\widehat{\mathcal{G}}(ir)}\mathcal{P}_k(\lambda)\widehat{Z}(\pi(A)\Omega_0)\right\rangle_{\widehat{\mathrm{per}}}$$
$$+\mathcal{O}(\lambda^2 e^{-\alpha r/2}). \tag{29}$$

The series on the right side of (29) converges since

$$\|\mathcal{P}_k(\lambda)\widehat{Z}(\pi(A)\Omega_0)\| \leq C\left(\frac{T}{k}\right)^2 \max_{0\leq t\leq T}\|H_{\mathrm{c}}(t)\| = k^{-2}\mathcal{O}(T). \tag{30}$$

To understand the bound (30), we note that $\widehat{Z}(\pi(A)\Omega_0)$ belongs to the domain of $\widehat{\mathcal{G}}(\theta)^2$, so

$$\|\widehat{\mathcal{G}}(\theta)^2\widehat{Z}(\pi(A)\Omega_0)\|^2_{\widehat{\mathrm{per}}} \approx \sum_{k\in\mathbb{Z}}\left(\frac{2\pi k}{T}\right)^4 \|P_k(\lambda)\widehat{Z}(\pi(A)\Omega_0)\|^2_{\widehat{\mathrm{per}}}$$

converges, which implies the decay $\|P_k(\lambda)\widehat{Z}(\pi(A)\Omega_0)\|_{\widehat{\mathrm{per}}} \sim (T/k)^2$. The bound (30) implies that, for small T, the main contribution to the sum in (29) comes from $k = 0$,

$$\left\langle \widehat{Z}\Omega_0, e^{i\alpha\widehat{\mathcal{G}}}\widehat{Z}(\pi(A)\Omega_0)\right\rangle_{\widehat{\mathrm{per}}} = \left\langle \widehat{Z}\Omega_0, e^{i\alpha\widehat{\mathcal{G}}(ir)}\mathcal{P}_0(\lambda)\widehat{Z}(\pi(A)\Omega_0)\right\rangle_{\widehat{\mathrm{per}}} + \mathcal{O}(\lambda^2 + T). \tag{31}$$

The final step in showing (26) is the estimate

$$e^{i\alpha\widehat{\mathcal{G}}_0(ir)}\mathcal{P}_0(0) - e^{i\alpha\widehat{\mathcal{G}}(ir)}\mathcal{P}_0(\lambda) = \mathcal{O}(|\lambda| + e^{c\alpha|\lambda|T} - 1). \tag{32}$$

We show (32) in Theorem 3.8 by comparing, with the help of Kato's method of similar projections, the two dynamics generated by $\widehat{\mathcal{G}}(ir)\mathcal{P}_0(\lambda)$ and $\widehat{\mathcal{G}}_0(ir)\mathcal{P}_0(0)$, respectively. The proof of this theorem uses the vanishing of the zero Fourier mode, $\widehat{Q}(0) = 0$, see also (18).

1.1.4 Proof of Theorem 1.1

Putting together (23), (25) and (26) we obtain

$$\omega_0(\tau^I_{t,0}(A)) = \left\langle \widehat{Z}\Omega_0, e^{it\mathcal{G}_0}\widehat{Z}(\pi(A)\Omega_0)\right\rangle_{\mathrm{per}} + \mathcal{O}(|\lambda| + T + e^{ct|\lambda|T} - 1). \tag{33}$$

We have $\tau^I_{t,0}(A) = V_{\mathrm{c}}(t)^*\tau_{t,0}(A)V_{\mathrm{c}}(t)$ and $e^{it\mathcal{G}_0}\pi(A)\Omega_0 = \pi(e^{itH_{\mathrm{s}}}Ae^{-itH_{\mathrm{s}}})\Omega_0$. Finally, since the l.h.s. and the first term on the r.h.s. of (33) are bounded uniformly in t, we may replace $e^{ct|\lambda|T} - 1$ in the remainder term by $\min\{1, e^{ct|\lambda|T} - 1\}$. The latter minimum is bounded above by $2(1 - e^{-ct|\lambda|T})$. The bound (14) follows.

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2 Spin-fermion model

For the concrete model of a two-level system ('qubit') for our system, we are able to obtain more detailed information on the dynamics. We present the analysis in this section, and we recover a bound on the decoherence time that has been derived in the physics literature by heuristic means before in [3].

The Hilbert space of the spin is $\mathbb{C}^2$, on which the spin Hamiltonian H_{s}, the forcing term $H_{\mathrm{c}}(t)$, and the interaction operator Q act as

$$H_{\mathrm{s}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad H_{\mathrm{c}}(t) = \frac{\mu}{T}\varkappa(t/T) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

respectively. Here, μ is a real coupling constant and $\varkappa(\cdot)$ is a smooth, real function of period one.

Example. The choice $\varkappa(t) = \cos(2\pi t)$ gives

$$Q(t) = \begin{bmatrix} 0 & \exp(-\frac{i\mu}{\pi}\sin(\frac{2\pi}{T}t)) \\ \exp(\frac{i\mu}{\pi}\sin(\frac{2\pi}{T}t)) & 0 \end{bmatrix} \otimes \Phi(\mathrm{f}), \tag{34}$$

see (21), and the dynamical decoupling condition (12) becomes

$$\int_{-\pi}^{\pi} \cos\left[\frac{\mu}{\pi}\sin(s)\right] \mathrm{d}s = 0. \tag{35}$$

A simple stationary phase argument shows that, as μ varies throughout $\mathbb{R}$, the set of real numbers $\int_{-\pi}^{\pi} \cos\left[\frac{\mu}{\pi}\sin(s)\right] \mathrm{d}s$ contains negative and positive values. In particular, by continuity, (35) holds for some μ. (Condition (35) is actually related to the existence of zeros of certain Bessel functions.)

We will assume from now on that $\varkappa$ is chosen so that the dynamical decoupling condition is satisfied, but we do not have to take it as in the example. We now show how one can obtain finer information about the dynamics than the one given in Theorem 1.1, by choosing a "corrected" uncoupled dynamics. The decoherence time estimate comes from an estimate on the difference of the total dynamics to a coherence preserving one, $\mathrm{e}^{i\alpha\widehat{\mathcal{G}}_0(ir)}\mathcal{P}_0(0) - \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)}\mathcal{P}_0(\lambda)$, see (32). The idea now is that by modifying $\widehat{\mathcal{G}}_0(ir)$ to $\widehat{\mathcal{G}}_0(ir) + \Delta$, where $\Delta = [S, \cdot]$ with a self-adjoint operator S *which commutes with* H_{s}, one can achieve two things. Firstly, coherences are still preserved, as S commutes with H_{s} – in Corollary 1.2, S simply changes the phases. Secondly, the difference between the generators $\widehat{\mathcal{G}}_0(ir) + \Delta$

and $\widehat{\mathcal{G}}(ir)$ becomes smaller for an appropriate Δ, which gives a sharper estimate on the difference of the dynamics they generate. We now implement this idea.

Recall that $\widehat{\mathcal{G}} = \widehat{\mathcal{G}}(ir)$ is the spectrally deformed Howland operator and $\mathcal{P}_0(\lambda)$ its Riesz projection associated to the group of discrete eigenvalues emerging from the eigenvalue 0 of $\widehat{\mathcal{G}}_0 = \widehat{\mathcal{G}}_0(ir)$, as $\lambda \neq 0$. The map $\lambda \mapsto \widehat{\mathcal{G}}\mathcal{P}_0(\lambda)$ is analytic at $\lambda = 0$. We write

$$\widehat{\mathcal{G}}\mathcal{P}_0(\lambda) - \widehat{\mathcal{G}}_0\mathcal{P}_0(0) = \sum_{m \geq 1} \lambda^m A_m. \tag{36}$$

It is not hard to see that $A_m = 0$ for all m odd, and that $\|A_m\| \leq C^m$ for a constant C independent of $0 < T \leq 1$. The second order term is the *level shift operator*

$$\lambda^2 A_2 = -\lambda^2 \lim_{\varepsilon \searrow 0} \sum_{e \in \sigma(\delta_s)} \mathcal{P}_{0,e}(0)\, \widehat{V}_{\mathrm{per}}(ir)\, (\widehat{\mathcal{G}}_0 - e + i\varepsilon)^{-1}\, \widehat{V}_{\mathrm{per}}(ir)\, \mathcal{P}_{0,e}(0)$$

(see also e.g. [5, 6, 7, 15]). Here, we have defined, for $e \in \sigma(\delta_s) = \{-2, 0, 2\}$, the orthogonal projections $\mathcal{P}_{0,e}(0)$ with range

$$\mathrm{Ran}\, \mathcal{P}_{0,e}(0) = \{\Psi \in \mathrm{Ran}\, \mathcal{P}_0(0) \; : \; \widehat{\mathcal{G}}_0 \Psi = e\Psi\}.$$

We have also defined the interaction operator $\widehat{V}_{\mathrm{per}}(ir) = \widehat{\mathcal{G}}(ir) - \widehat{\mathcal{G}}_0(ir)$ (c.f. sections 3.4 and 3.5). Let $A \in \mathcal{B}(\mathbb{C}^d)$ and define $\underrightarrow{A}$ and $\underleftarrow{A}$ as the left and right multiplication on operators $B \in \mathcal{B}(\mathbb{C}^d)$,

$$B \mapsto \underrightarrow{A}B := AB \quad \text{and} \quad B \mapsto \underleftarrow{A}B := BA. \tag{37}$$

Note that, as $\mathcal{B}(\mathbb{C}^d)$ is isomorphic to $\mathbb{C}^d \otimes \mathbb{C}^d$, we may naturally identify $\underrightarrow{A}$ and $\underleftarrow{A}$ with $A \otimes \mathbf{1}$ and $\mathbf{1} \otimes A \in \mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d)$, respectively.

A straightforward, but lengthy and tedious, calculation leads us to

$$\lambda^2 A_2 = -i\frac{\lambda^2}{2} \sum_{a=\pm 1} \sum_{k \in \mathbb{Z} \setminus \{0\}} \left[\pi G_f(k/T + 2a)(2\underrightarrow{Q_{k,a}^*}\underleftarrow{Q_{k,a}} - \underrightarrow{Q_{k,a}^*}\underrightarrow{Q_{k,a}} - \underleftarrow{Q_{k,a}}\,\underleftarrow{Q_{k,a}^*}) \right.$$

$$\left. + i\mathrm{PV}\left[\frac{1}{p - (k/T + 2a)}\right](G_f)(\underleftarrow{Q_{k,a}}\,\underleftarrow{Q_{k,a}^*} - \underrightarrow{Q_{k,a}^*}\underrightarrow{Q_{k,a}}) \right], \tag{38}$$

where

$$Q_{k,a} := \int_0^1 \mathrm{e}^{-2\pi ikt}\, \mathrm{e}^{-iH_s \int_0^t \varkappa(s)\mathrm{d}s}\, q_a \mathrm{e}^{iH_s \int_0^t \varkappa(s)\mathrm{d}s}\mathrm{d}t,$$

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with

$$q_{-1} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad q_1 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Here, the function $G_f : \mathbb{R} \to \mathbb{R}_0^+$ is defined by

$$G_f(p) := \int_{S^2} \frac{p^2}{1 + e^{-\beta p}} \, |g_f(p, \vartheta)|^2 \, d\vartheta$$

for all $p \in \mathbb{R}$, and

$$\mathrm{PV}\Big[\frac{1}{p - (k/T + 2a)}\Big](G_f) := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R} \backslash (-\varepsilon, \varepsilon)} \frac{G_f(p + k/T + 2a)}{p} \, dp$$

is the principal value of the function $p \mapsto \frac{G_f(p)}{p - (k/T + 2a)}$ at the singularity $p = k/T + 2a$.

Remark 2.1. In (38) we recognize that $\lambda^2 A_2$ assumes the standard form of a Lindblad generator, the first three terms in the sum giving the dissipative part and the last two corresponding to the Hamiltonian part.

Remark 2.2. The term $k = 0$ is absent in the above sum over the k, due to the dynamical decoupling condition (12).

The self-adjoint operators $Q_{k,a}^* Q_{k,a}$ commute with H_s. We define

$$\Delta = \frac{\lambda^2}{2} \sum_{a=\pm 1} \sum_{k \in \mathbb{Z} \backslash \{0\}} (\overleftarrow{Q_{k,a} \, Q_{k,a}^*} - \overrightarrow{Q_{k,a}^* Q_{k,a}}) \mathrm{PV}\Big[\frac{1}{1 - (k/T + 2a)}\Big](G_f), \quad (39)$$

so that, from (36) and (38), we obtain

$$\widehat{\mathcal{G}} \mathcal{P}_0(\lambda) - (\widehat{\mathcal{G}}_0 + \Delta) \mathcal{P}_0(0) = \tag{40}$$
$$-i \frac{\lambda^2}{2} \sum_{a=\pm 1} \sum_{k \in \mathbb{Z} \backslash \{0\}} \pi G_f(k/T + 2a)(2\overrightarrow{Q_{k,a}^*} \overleftarrow{Q_{k,a}} - \overrightarrow{Q_{k,a}^* Q_{k,a}} - \overleftarrow{Q_{k,a}^* Q_{k,a}}) + \mathcal{O}(\lambda^4).$$

Therefore,
$$\|\widehat{\mathcal{G}} \mathcal{P}_0(\lambda) - (\widehat{\mathcal{G}}_0 + \Delta) \mathcal{P}_0(0)\| \le 2\pi \lambda^2 \xi(T) + c\lambda^4, \tag{41}$$

for some constant c independent of $0 < T \le 1$, and where

$$\xi(T) = \sum_{a=\pm 1} \sum_{k \in \mathbb{Z} \backslash \{0\}} \|Q_{k,a}\|^2 \, |G_f(k/T + 2a)|^2. \tag{42}$$

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We pass from estimate (41) on the difference of the generators to an estimate on the difference of the propagators,

$$e^{i\alpha\widehat{\mathcal{G}}}\mathcal{P}_0(\lambda) - e^{i\alpha(\widehat{\mathcal{G}}_0+\Delta)}\mathcal{P}_0(0) \tag{43}$$

$$= e^{i\alpha\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)}\mathcal{P}_0(\lambda) - e^{i\alpha(\widehat{\mathcal{G}}_0+\Delta)\mathcal{P}_0(0)}\mathcal{P}_0(0)$$

$$= (e^{i\alpha\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)} - e^{i\alpha(\widehat{\mathcal{G}}_0+\Delta)\mathcal{P}_0(0)})\mathcal{P}_0(\lambda) + e^{i\alpha(\widehat{\mathcal{G}}_0+\Delta)\mathcal{P}_0(0)}(\mathcal{P}_0(\lambda) - \mathcal{P}_0(0)).$$

The norm of the second summand on the right side is bounded above by $C|\lambda|$, since $\|e^{i\alpha(\widehat{\mathcal{G}}_0+\Delta)\mathcal{P}_0(0)}\| = 1$ and $\mathcal{P}_0(\lambda) - \mathcal{P}_0(0) = \mathcal{O}(\lambda)$. By first iterating the Duhamel formula, and then using (41), we obtain the estimates

$$\|e^{i\alpha\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)} - e^{i\alpha(\widehat{\mathcal{G}}_0+\Delta)\mathcal{P}_0(0)}\| \le e^{\alpha\|\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)-(\widehat{\mathcal{G}}_0+\Delta)\mathcal{P}_0(0)\|} - 1 \le e^{2\pi\alpha\lambda^2[\xi(T)+c\lambda^2]} - 1.$$

This gives the bound

$$\left\|e^{i\alpha(\widehat{\mathcal{G}}_0(ir)+\Delta)}\mathcal{P}_0(0) - e^{i\alpha\widehat{\mathcal{G}}(ir)}\mathcal{P}_0(\lambda)\right\| \le C\left(|\lambda| + e^{2\pi\alpha\lambda^2[\xi(T)+c\lambda^2]} - 1\right), \quad (44)$$

for constants c, C independent of $0 < T \le 1$, λ sufficiently small, and where $\xi(T)$ is given in (42). The bound (44) shows that decoherence is suppressed for times up to

$$t_{\text{dec}} = \frac{1}{2\pi\lambda^2[\xi(T) + c\lambda^2]}. \tag{45}$$

For $\lambda^2\xi(T) + \lambda^4 \ll |\lambda|T$, the bound (45) on the decoherence time is *better* than the general one obtained in Theorem 1.1. Also, (45) shows that to leading order in λ, the decoherence time can be very large even at moderate control frequencies. Indeed, with a suitable choice of $\varkappa$, one can achieve that $\|Q_{k,a}\|G_f(k/T + 2a)$ is very small, and hence so is $\xi(T)$ (see (42)), driving t_{dec} in (45) to very large values. We refer to [3] for more discussions about this point. It has to be noted that the analysis leading to Theorem 1.1 was carried out with small values of T in mind. For $T \sim 0$, t_{dec} in (45) behaves as $1/\lambda^4$, while the bound on the decoherence time in Theorem 1.1 is $(|\lambda|T)^{-1}$, which is better (larger).

Bang-bang control. Take $\varkappa$ to be a 1-periodic sequence of delta functions

$$\varkappa(t) := \sum_{j\in\mathbb{Z}}\sum_{l=1}^{n} c_l\,\delta(t - (j + \alpha_l)),$$

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with fixed $0 < \alpha_1 < \cdots < \alpha_n < 1$ and real constants c_j satisfying $c_1, \ldots, c_n < 1$ with $c_1 + \cdots + c_n = 0$ (this ensures that the dynamical decoupling condition (12) is satisfied). Then we have

$$
\int_0^t \varkappa(s)\,\mathrm{d}s = \begin{cases} \sum_{j \in \mathbb{Z}} \sum_{k=1}^n c_l \mathbf{1}[j + \alpha_l \in [0, t)] & , \quad t > 0, \\ 0 & , \quad t = 0, \\ -\sum_{j \in \mathbb{Z}} \sum_{l=1}^n c_l \mathbf{1}[j + \alpha_l \in [t, 0)] & , \quad t < 0, \end{cases}
$$

and so $Q_a(t) = \mathrm{e}^{-iH_\mathrm{s} \int_0^t \varkappa(s)\mathrm{d}s} q_a \mathrm{e}^{iH_\mathrm{s} \int_0^t \varkappa(s)\mathrm{d}s}$ is also a 1-periodic, piece-wise constant functions of the variable $t \in \mathbb{R}$. The Fourier coefficients become (see (38))

$$
Q_{k,a} := \int_0^1 Q_a(t)\, \mathrm{e}^{-2\pi i k t}\, \mathrm{d}t = -\frac{i}{2\pi k}\left(\sum_{l=1}^n \mathrm{e}^{-2\pi i \alpha_l k}\, \delta Q_l \right)
$$

if $k \in \mathbb{Z}\backslash\{0\}$, and $Q_{0,a} = 0$ (because of the dynamical decoupling condition). Here, $\delta Q_l := \lim_{\varepsilon \searrow 0}(Q_a(t + \varepsilon) - Q_a(t - \varepsilon))$. Thus $\|Q_{k,a}\|$ is proportional to $1/|k|$ and the decoherence rates to lowest order in λ is

$$
\lambda^2 \xi(T) \sim \lambda^2 \sum_{k \in \mathbb{Z}\backslash\{0\}} \frac{1}{k^2} \left\{ |G_f(k/T + 2)|^2 + G_f(k/T - 2)|^2 \right\}.
$$

This behaviour of the decoherence rate was obtained in [3], for a model describing one spin interacting with a generic reservoir at inverse temperature β (see equation (110) in that reference). The analysis of [3] is formal, however, and based on functional integrals and Markov approximations.

3 Proof of Theorem 1.1

3.1 The GNS representation

Let $\omega_\mathcal{S}$ be a faithful state on $\mathcal{B}(\mathbb{C}^d)$. Its GNS representation $(\mathfrak{H}_\mathrm{s}, \pi_\mathrm{s}, \Omega_\mathrm{s})$ is given explicitly as follows. The representation Hilbert space is the linear space $\mathfrak{H}_\mathrm{s} = \mathcal{B}(\mathbb{C}^d)$ with inner product $\langle A, B \rangle_\mathrm{s} = \mathrm{Tr}(A^* B)$. The representation map $\pi_\mathrm{s} : \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathfrak{H}_\mathrm{s})$ is the left multiplication, $\pi_\mathrm{s}(A) = \overrightarrow{A}$, where for $A \in \mathcal{B}(\mathbb{C}^d)$, we define the left and right multiplication operators $\overrightarrow{A}$ and $\overleftarrow{A}$, acting on $\mathfrak{H}_\mathrm{s}$, as in

(37). The system von Neumann algebra of observables (acting on the GNS Hilbert space) is

$$\mathfrak{M}_{\mathrm{s}} = \underrightarrow{\mathcal{B}(\mathbb{C}^d)} = \{\underrightarrow{A} \; : \; A \in \mathcal{B}(\mathbb{C}^d)\}. \tag{46}$$

The cyclic vector of the GNS representation of $\omega_{\mathcal{S}}$ is given by the vector $\Omega_{\mathrm{s}} = \rho_{\mathrm{s}}^{1/2} \in \mathfrak{H}_{\mathrm{s}}$, where $\rho_{\mathrm{s}} \in \mathcal{B}(\mathbb{C}^d)$ is the density matrix associated to $\omega_{\mathcal{S}}$. We have $\omega_{\mathcal{S}}(A) = \langle \Omega_{\mathrm{s}}, \underrightarrow{A}\Omega_{\mathrm{s}}\rangle_{\mathrm{s}}$ for $A \in \mathcal{B}(\mathbb{C}^d)$. The dynamics τ_t^{s}, generated by the derivation $\delta_{\mathrm{s}} = i[H_{\mathrm{s}}, \cdot]$ (see also (3)), is implemented by the self-adjoint Liouville operator

$$L_{\mathrm{s}} := \left(\underrightarrow{H_{\mathrm{s}}} - \underleftarrow{H_{\mathrm{s}}}\right) = [H_{\mathrm{s}}, \cdot], \tag{47}$$

we have $\pi(\tau_t^{\mathrm{s}}(A)) = \mathrm{e}^{itL_{\mathrm{s}}}\pi(A)\mathrm{e}^{-itL_{\mathrm{s}}}$. Note that $L_{\mathrm{s}}\Omega_{\mathrm{s}} = 0$.

We now present the GNS representation of the infinitely extended free Fermi reservoir at inverse temperature β. It is given by the Araki–Wyss representation of the CAR algebra $\mathcal{V}_{\mathcal{R}}$ [11, 16]. The representation Hilbert space is

$$\mathfrak{H}_{\mathcal{R}} := \mathcal{F}_-(L^2(\mathbb{R} \times S^2)),$$

the anti-symmetric Fock space over the one-particle space $L^2(\mathbb{R} \times S^2)$ (equipped with the product of Lebesgue measure on $\mathbb{R}$ times the uniform measure on the two-sphere S^2). This representation is customarily referred to as *Jaksic-Pillet glueing* and appeared for the first time in [8]. The cyclic vector $\Omega_{\mathcal{R}}$ is the vacuum of $\mathcal{F}_-(L^2(\mathbb{R} \times S^2))$. The representation map $\pi_{\mathcal{R}}$ maps an annihilation operator $a(f) \in \mathcal{V}_{\mathcal{R}}$, $f \in L^2(\mathbb{R}^3)$, to an annihilation operator acting on $\mathfrak{H}_{\mathcal{R}}$, $\pi_{\mathcal{R}}(a(f)) = a(g_f)$, where $g_f \in L^2(\mathbb{R} \times S^2)$ is given by (10). We have $\omega_{\mathcal{R}}(A) = \langle \Omega_{\mathcal{R}}, \pi_{\mathcal{R}}(A)\Omega_{\mathcal{R}}\rangle$ for all $A \in \mathcal{V}_{\mathcal{R}}$.

The Bogoliubov automorphism (6), defined on $\mathcal{V}_{\mathcal{R}}$, extends to a $*$–automorphism group on the von Neumann algebra

$$\mathfrak{M}_{\mathcal{R}} = \mathcal{V}_{\mathcal{R}}'' \tag{48}$$

(weak closure of $\mathcal{V}_{\mathcal{R}}$). We denote the extension again by $\tau_t^{\mathcal{R}}$. The thermal state $\omega_{\mathcal{R}}$ is a $(\beta, \tau_t^{\mathcal{R}})$–KMS state on $\mathfrak{M}_{\mathcal{R}}$. The dynamics is implemented by the self-adjoint Liouville operator

$$L_{\mathcal{R}} = \mathrm{d}\Gamma(p), \tag{49}$$

the second quantization of the multiplication operator by the radial variable $p \in \mathbb{R}$ of functions in $L^2(\mathbb{R} \times S^2)$. We have $\pi_{\mathcal{R}}(\tau_t^{\mathcal{R}}(A)) = \mathrm{e}^{itL_{\mathcal{R}}}\pi_{\mathcal{R}}(A)\mathrm{e}^{-itL_{\mathcal{R}}}$ for all $A \in \mathcal{V}_{\mathcal{R}}$, and $L_{\mathcal{R}}\Omega_{\mathcal{R}} = 0$.

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The GNS representation $(\mathfrak{H}, \pi, \Omega)$ of the initial state (13) is given by

$$\mathfrak{H} := \mathfrak{H}_{\mathrm{s}} \otimes \mathfrak{H}_{\mathcal{R}}, \quad \pi := \pi_{\mathrm{s}} \otimes \pi_{\mathcal{R}}, \quad \Omega_0 := \Omega_{\mathrm{s}} \otimes \Omega_{\mathcal{R}}. \tag{50}$$

We denote by

$$\mathfrak{M} = \mathfrak{M}_{\mathrm{s}} \otimes \mathfrak{M}_{\mathcal{R}}$$

the von Neumann algebra of observables of the joint system, see also (46) and (48). The self-adjoint Liouville operator

$$L = L_{\mathrm{s}} \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}} + \mathbf{1}_{\mathfrak{H}_{\mathrm{s}}} \otimes L_{\mathcal{R}} = (\underrightarrow{H_{\mathrm{s}}} - \underleftarrow{H_{\mathrm{s}}}) \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}} + \mathbf{1}_{\mathfrak{H}_{\mathrm{s}}} \otimes \mathrm{d}\Gamma(p), \tag{51}$$

defines the uncoupled Heisenberg dynamics $\mathrm{e}^{itL} A \mathrm{e}^{-itL}$ on $\mathfrak{M}$.

3.2 Time-dependent Liouville operator

As $\Omega_{\mathcal{R}}$ defines a KMS state on $\mathfrak{M}_{\mathcal{R}}$, the vector $\Omega_{\mathcal{R}}$ is cyclic and separating for $\mathfrak{M}_{\mathcal{R}}$. Let $\Delta_{\mathcal{R}}$ and $J_{\mathcal{R}}$ be the modular operator and conjugation of $(\mathfrak{M}_{\mathcal{R}}, \Omega_{\mathcal{R}})$ [17]. Similarly, since $\omega_{\mathcal{S}}$ is faithful, $\Omega_{\mathrm{s}} = \rho_{\mathrm{s}}^{1/2}$ is cyclic and separating for the von Neumann algebra $\mathfrak{M}_{\mathrm{s}}$. Let Δ_{s} and J_{s} be the associated modular operator and conjugation. We set $J = J_{\mathrm{s}} \otimes J_{\mathcal{R}}$ and $\Delta = \Delta_{\mathrm{s}} \otimes \Delta_{\mathcal{R}}$ and define the time-dependent Liouville operator, for $t \geq 0$, by

$$\begin{aligned} \mathcal{L}_t &= L + V_t, \tag{52} \\ V_t &= W_t - J\Delta^{1/2} W_t \Delta^{-1/2} J, \tag{53} \end{aligned}$$

where

$$W_t = \lambda \, \underrightarrow{Q(t)} \otimes \frac{1}{\sqrt{2}} \left(a(g_f) + a^*(g_f) \right) \equiv \lambda Q(t) \otimes \Phi(g_f), \tag{54}$$

and

$$\begin{aligned} J\Delta^{1/2} W_t \Delta^{-1/2} J &= \lambda \underleftarrow{\rho_{\mathrm{s}}^{-1/2} Q(t) \rho_{\mathrm{s}}^{1/2}} \otimes \frac{(-1)^{\mathrm{d}\Gamma(\mathbf{1})}}{\sqrt{2}} \left(a(i\mathrm{e}^{-\beta p} g_f) + a^*(ig_f) \right) \\ &\equiv \lambda \underleftarrow{\rho_{\mathrm{s}}^{-1/2} Q(t) \rho_{\mathrm{s}}^{1/2}} \otimes \Phi'(g_f). \tag{55} \end{aligned}$$

Here, $\mathrm{d}\Gamma(\mathbf{1})$ is the second quantization of $\mathbf{1}$, i.e., the particle number operator acting on $\mathfrak{H}_{\mathcal{R}}$.

3.3 Implementation of the dynamics

A family $\{U_{t,s}\}_{t \geq s}$ of bounded operators on $\mathfrak{H}$ is called an evolution family if it satisfies (i) $U_{t,s} = U_{t,r}U_{r,s}$ for all $t \geq r \geq s$ (cocycle, or Chapman–Kolmogorov property) and (ii) $U_{t,s}$ is a strongly continuous two–parameter family.

Lemma 3.1. *There is an evolution family $\{U_{t,s}\}_{t \geq s} \subset \mathcal{B}(\mathfrak{H})$ solving the following non–autonomous evolution equations on* $\mathrm{Dom}(L)$,

$$\forall t > s: \quad \partial_t U_{t,s} = i\mathcal{L}_t U_{t,s}, \quad \partial_s U_{t,s} = -iU_{t,s}\mathcal{L}_s, \quad U_{s,s} := \mathbf{1}.$$

For any $t \geq s$, $U_{t,s}$ possesses a bounded inverse $U_{t,s}^{-1}$. Moreover, we have $U_{t,s} = U_{t+T,s+T}$ for all $t \geq s$.

A proof of this result is not difficult. For instance, one can follow the ideas of [18, Theorem X.69 and comments thereafter].

Theorem 3.2 (Implementing the dynamics). *We have for all $t \geq s$, $A \in \mathfrak{M}$,*

$$U_{t,s}\Omega_0 = \Omega_0 \qquad and \qquad \pi\left(\tau_{t,s}^I(A)\right) = U_{t,s}\pi(A)U_{t,s}^{-1}, \tag{56}$$

where $\tau_{t,s}^I$ is defined in (20).

Proof of Theorem 3.2. We show that for $t > s$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \phi, U_{t,s}^{-1}\pi(\tau_{t,s}^I(A))U_{t,s}\psi \right\rangle = 0, \tag{57}$$

for all $\phi, \psi \in \mathrm{Dom}(L_{\mathcal{R}})$ and all $A \in \mathrm{Dom}(\delta_{\mathrm{r}})$. It then follows ($t \downarrow s$) that $U_{t,s}^{-1}\pi(\tau_{t,s}^I(A))U_{t,s} = \pi(A)$ for all $A \in \mathrm{Dom}(\delta_{\mathrm{r}})$ and hence for all $A \in \mathfrak{M}$, which is the statement to be proven. Note that $U_{t,s}\Omega_0 = \Omega_0$ follows from $\mathcal{L}_t\Omega_0 = 0$. By using $\partial_t U_{t,s} = i\mathcal{L}_t U_{t,s}$ and the ensuing equation $\partial_t(U_{t,s}^{-1})^* = i\mathcal{L}_t^*(U_{t,s}^{-1})^*$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \phi, U_{t,s}^{-1}\pi(\tau_{t,s}^I(A))U_{t,s}\psi \right\rangle &= \left\langle i\mathcal{L}_t^*(U_{t,s}^{-1})^*\phi, \pi(\tau_{t,s}^I(A))U_{t,s}\psi \right\rangle \\
&+ \left\langle (U_{t,s}^{-1})^*\phi, \pi(\tau_{t,s}^I(A))i\mathcal{L}_t U_{t,s}\psi \right\rangle \\
&+ \left\langle (U_{t,s}^{-1})^*\phi, \pi\left(\delta_t^I(\tau_{t,s}^I(A))\right)U_{t,s}\psi \right\rangle. \tag{58}
\end{aligned}$$

The term $J\Delta^{1/2}W_t\Delta^{-1/2}J$ in $\mathcal{L}_t$ and the corresponding part in the adjoint $\mathcal{L}_t^*$ cancel out in the first two terms in (58), as they commute with $\pi(\tau_{t,s}^I(A))$. Thus we can replace both $\mathcal{L}_t$ and $\mathcal{L}_t^*$ by $L_{\mathrm{s}} + L_{\mathcal{R}} + W_t$ in (58). The contributions coming

from $L_\mathrm{s} + W_t$ in the first two terms cancel the commutator term in $\delta_t^I(\tau_{t,s}^I(A)) = \delta_\mathrm{r}(\tau_{t,s}^I(A)) + i[H_\mathrm{s} + \lambda Q(t) \otimes \Phi(\mathrm{f}), \tau_{t,s}^I(A)]$. It follows that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \langle \phi, U_{t,s}^{-1} \pi(\tau_{t,s}^I(A)) U_{t,s} \psi \rangle = \langle i L_\mathcal{R} (U_{t,s}^{-1})^* \phi, \pi(\tau_{t,s}^I(A)) U_{t,s} \psi \rangle \tag{59}
$$
$$
+ \langle (U_{t,s}^{-1})^* \phi, \pi(\tau_{t,s}^I(A)) i L_\mathcal{R} U_{t,s} \psi \rangle
$$
$$
+ \langle (U_{t,s}^{-1})^* \phi, \pi\big(\delta_\mathrm{r}(\tau_{t,s}^I(A))\big) U_{t,s} \psi \rangle .
$$

The last term in (59) equals

$$
\frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0} \langle (U_{t,s}^{-1})^* \phi, \pi\big(\tau_\alpha^\mathcal{R}(\tau_{t,s}^I(A))\big) U_{t,s} \psi \rangle
$$
$$
= \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=0} \langle (U_{t,s}^{-1})^* \phi, \mathrm{e}^{i\alpha L_\mathcal{R}} \pi\big(\tau_{t,s}^I(A)\big) \mathrm{e}^{-i\alpha L_\mathcal{R}} U_{t,s} \psi \rangle ,
$$

which is exactly the negative of the sum of the first two terms on the right side in that equation. ∎

3.4   Howland Generator for $U_{t,s}$

The generator $\mathcal{G}$, acting on $\mathfrak{H}_\mathrm{per} = L^2(\mathbb{T}_T, \mathfrak{H})$ and defined by (24), is given explicitly by

$$
\mathcal{G} = i\frac{\mathrm{d}}{\mathrm{d}t} + L + V_\mathrm{per} \tag{60}
$$

There is a core of $\mathcal{G}$ whose elements are differentiable functions $t \mapsto f(t)$ with $f(t) \in \mathrm{Dom}(L)$. The operators on the right side are understood as follows:

$$
\begin{aligned}
\left(\tfrac{\mathrm{d}}{\mathrm{d}t} f\right)(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t} f(t) \\
(Lf)(t) &= L(f(t)) \\
(V_\mathrm{per} f)(t) &= [W_t - J\Delta^{1/2} W_t \Delta^{-1/2} J] (f(t)) .
\end{aligned} \tag{61}
$$

Recall the definition of the Fourier transform (27) and the notation introduced after it. The operator $\widehat{\mathcal{G}} = \mathfrak{F}\mathcal{G}\mathfrak{F}^*$ on $\ell^2(\mathbb{Z}, \mathfrak{H})$ is

$$
\widehat{\mathcal{G}} = -\frac{2\pi}{T} k + L + \widehat{V}_\mathrm{per}. \tag{62}
$$

Its domain consists of $\hat{f}$ satisfying $\sum_{k \in \mathbb{Z}} k^2 \|\hat{f}(k)\|^2 < \infty$ and $f(k) \in \mathrm{Dom}(L)$. Here, for $k \in \mathbb{Z}$,

$$
(k\hat{f})(k) := k\hat{f}(k) \quad \text{and} \quad (L\hat{f})(k) := L(\hat{f}(k)).
$$

The operator $\widehat{V}_{\mathrm{per}} := \mathfrak{F} V_{\mathrm{per}} \mathfrak{F}^*$ acts as a convolution,

$$[\widehat{V}_{\mathrm{per}} \hat{f}](k) = \sum_{n \in \mathbb{Z}} \widehat{v}_{\mathrm{per}}(n-k) \hat{f}(n), \tag{63}$$

with convolution kernel

$$\widehat{v}_{\mathrm{per}}(\ell) = \frac{1}{T} \int_0^T \mathrm{e}^{-2\pi i \ell t / T} \{W_t - J\Delta^{1/2} W_t \Delta^{-1/2} J\} \mathrm{d}t. \tag{64}$$

The operators W_t and $J\Delta^{1/2} W_t \Delta^{-1/2} J$ are given in (54) and (55), respectively.

Lemma 3.3. *For any initial state $\omega_{\mathcal{S}}$ and all $A \in \mathcal{B}(\mathbb{C}^d)$,*

$$\left| \langle \Omega_0, U_{\alpha,0} \pi(A \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}}) \Omega_0 \rangle_{\mathfrak{H}} - \left\langle \widehat{Z}\Omega_0, \mathrm{e}^{i\alpha\widehat{\mathcal{G}}} \widehat{Z}((\pi(A \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}})\Omega_0)) \right\rangle_{\widehat{\mathrm{per}}} \right|$$
$$\leq C(1 + |\lambda|) \, \|A\| \, T, \tag{65}$$

where C is some finite constant not depending on $\omega_{\mathcal{S}}$, A, λ, T, and α.

Proof of Lemma 3.3. By unitarity of the Fourier transform, we can replace $\mathrm{e}^{i\alpha\widehat{\mathcal{G}}}$ by $\mathrm{e}^{i\alpha\mathcal{G}}$. We then have

$$\left| \langle \Omega_0, U_{\alpha,0} \pi(A \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}}) \Omega_0 \rangle_{\mathfrak{H}} - \left\langle Z\Omega_0, \mathrm{e}^{i\alpha\mathcal{G}} Z((\pi(A \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}})\Omega_0)) \right\rangle_{\mathrm{per}} \right|$$
$$= \left| \left\langle Z\Omega_0, U_{\alpha,0}^{\mathrm{per}} Z(\pi(A \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}})\Omega_0) \right\rangle_{\mathrm{per}} - \left\langle Z\Omega_0, \mathrm{e}^{i\alpha\mathcal{G}} Z((\pi(A \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}})\Omega_0)) \right\rangle_{\mathrm{per}} \right|$$
$$\leq \frac{1}{T} \int_0^T \left| \langle \Omega_0, (U_{\alpha,0} - U_{t,t-\alpha}) \pi(A \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}}) \Omega_0 \rangle_{\mathfrak{H}} \right| \mathrm{d}t. \tag{66}$$

We show that (66) is small. For this, we want to shift the time-indices of the $U_{t,t-\alpha}$ close to those of $U_{\alpha,0}$ by adding multiples of T and using periodicity. The approximation is better the smaller T is. Given any $t \in [0, T)$ and $\alpha \geq 0$, there is an integer $n = n(t, \alpha)$ and an $\epsilon = \epsilon(t, \alpha)$ s.t. $n \geq 0$, $0 \leq \epsilon < T$ and $t + nT = \alpha + \epsilon$. Since $U_{t,t-\alpha} = U_{t+nT, t+nT-\alpha} = U_{\alpha+\epsilon, \epsilon}$, it suffices to show that

$$\left| \langle \Omega_0, (U_{\alpha,0} - U_{\alpha+\epsilon, \epsilon}) \pi(A \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}}) \Omega_0 \rangle_{\mathfrak{H}} \right| \leq C(1 + |\lambda|) \|A\| \epsilon, \tag{67}$$

uniformly in $\alpha \geq 0$. Uniformity in α holds true because U implements a norm-preserving map (dynamics) on the algebra $\mathfrak{M}$. We have

$$\langle \Omega_0, (U_{\alpha,0} - U_{\alpha+\epsilon, \epsilon}) \pi(A \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}}) \Omega_0 \rangle_{\mathfrak{H}} = - \int_0^\epsilon \langle \Omega_0, \partial_s U_{\alpha+s, s} \pi(A \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}}) \Omega_0 \rangle_{\mathfrak{H}} \, \mathrm{d}s. \tag{68}$$

Now $\partial_s U_{\alpha+s,s} = i\mathcal{L}_{\alpha+s}U_{\alpha+s,s} - U_{\alpha+s,s}(i\mathcal{L}_s)$, with $\mathcal{L}_t$ given in (52). We analyze the term $-U_{\alpha+s,s}(i\mathcal{L}_s)$, the other one is dealt with similarly. We have

$$
\begin{aligned}
&\langle \Omega_0, U_{\alpha+s,s}\mathcal{L}_s\pi(A \otimes \mathbf{1}_{\mathfrak{H}_\mathcal{R}})\Omega_0\rangle_{\mathfrak{H}} \\
&= \langle \Omega_0, U_{\alpha+s,s}\pi([H_s \otimes \mathbf{1}_{\mathfrak{H}_\mathcal{R}} + \lambda Q(s) \otimes \Phi(\mathrm{f}), A \otimes \mathbf{1}_{\mathfrak{H}_\mathcal{R}}])\Omega_0\rangle_{\mathfrak{H}} \\
&= \langle \Omega_0, \pi\big(\tau^I_{\alpha+s,s}([H_s \otimes \mathbf{1}_{\mathfrak{H}_\mathcal{R}} + \lambda Q(s) \otimes \Phi(\mathrm{f}), A \otimes \mathbf{1}_{\mathfrak{H}_\mathcal{R}}])\big)\Omega_0\rangle_{\mathfrak{H}},
\end{aligned}
$$

which has modulus bounded above by

$$
\big(2\|H_s\| + 2|\lambda|\,\|Q\|\,\|f\|\big)\|A\| \leq C(1 + |\lambda|)\|A\|,
$$

uniformly in α, s. Using this bound in (68) we obtain (67). ∎

The next result implies our main result, Theorem 1.1 (see details given after Theorem 3.4). We give the proof of Theorem 3.4 in the next section.

Denote by $\widehat{\mathcal{G}}_0$ the Howland generator for $\lambda = 0$ (uncoupled system).

Theorem 3.4. *Suppose the T–periodic control term H_c satisfies the dynamical decoupling condition (12). Then*

$$
\left| \left\langle \widehat{Z}\Omega_0, (\mathrm{e}^{i\alpha\widehat{\mathcal{G}}_0} - \mathrm{e}^{i\alpha\widehat{\mathcal{G}}})\widehat{Z}((\pi(A \otimes \mathbf{1}_{\mathfrak{H}_\mathcal{R}})\Omega_0)) \right\rangle_{\widehat{\mathrm{per}}} \right|
$$
$$
\leq C\,\|A\| \left(|\lambda| + (D|\lambda| + 1)T + \mathrm{e}^{c\alpha|\lambda|T} - 1 \right)
$$

for all $\alpha \geq 0$ and all $A \in \mathcal{B}(\mathbb{C}^d)$, and where $0 < c, C < \infty$ are constants not depending on α, λ, T, H_c, nor on $\omega_\mathcal{S}$ in the initial state $\omega_0 = \omega_\mathcal{S} \otimes \omega_\mathcal{R}$. Here, D is given in (15).

Proof of Theorem 1.1. Since the left side of the inequality in Theorem 3.4 is bounded above by $C\|A\|$ uniformly in $\alpha \geq 0$, we can replace $\mathrm{e}^{c\alpha|\lambda|T} - 1$ by $\min\{1, \mathrm{e}^{c\alpha|\lambda|T} - 1\}$. Moreover, since we have $\min\{1, \mathrm{e}^a - 1\} \leq 2(1 - \mathrm{e}^{-a})$ for all $a \geq 0$, the upper bound in Theorem 3.4 can be replaced by $C\|A\|\{|\lambda| + (D|\lambda| + 1)T + 1 - \mathrm{e}^{-c\alpha|\lambda|T}\}$. Next, note that

$$
\mathrm{e}^{i\alpha\widehat{\mathcal{G}}_0}\widehat{Z}(\pi(A \otimes \mathbf{1}_{\mathfrak{H}_\mathcal{R}})\Omega_0) = \widehat{Z}\big(\pi(\mathrm{e}^{i\alpha H_s}A\mathrm{e}^{-i\alpha H_s} \otimes \mathbf{1}_{\mathfrak{H}_\mathcal{R}})\Omega_0\big).
$$

Combining Lemma 3.3 and Theorem 3.4 thus gives

$$
\begin{aligned}
&\big|\omega_0(\tau^I_{\alpha,0}(A \otimes \mathbf{1}_{\mathfrak{H}_\mathcal{R}})) - \omega_\mathcal{S}(\mathrm{e}^{i\alpha H_s}A\mathrm{e}^{-i\alpha H_s})\big| \\
&\qquad \leq C\|A\|\big(|\lambda| + (D|\lambda| + 1)T + 1 - \mathrm{e}^{-c\alpha|\lambda|T}\big)
\end{aligned}
\tag{69}
$$

Since $\tau^I_{\alpha,0}(A \otimes \mathbf{1}_{\mathfrak{H}_\mathcal{R}}) = V_c(\alpha)^*\tau_{\alpha,0}(A \otimes \mathbf{1}_{\mathfrak{H}_\mathcal{R}})V_c(\alpha)$ (see (20)), and since the bound (69) in uniform in the initial state $\omega_\mathcal{S}$, we obtain the assertion (14). ∎

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3.5 Resonances of Howland Generator, Proof of Theorem 3.4

Now we perform an analytic deformation of the Howland operator $\widehat{\mathcal{G}} = \mathfrak{F}\mathcal{G}\mathfrak{F}^*$ (62) acting on $\widehat{\mathfrak{H}}_{\mathrm{per}} = \ell^2(\mathbb{Z}, \mathfrak{H})$. For all $\theta \in \mathbb{C}$, define

$$\widehat{\mathcal{G}}_0(\theta) := -\frac{2\pi}{T}k + L + \theta\widehat{N}, \tag{70}$$

where $\widehat{N}(\hat{f})(k) := (\mathbf{1}_{\mathfrak{H}_{\mathrm{s}}} \otimes \mathrm{d}\Gamma(\mathbf{1}))\hat{f}(k)$ for all integers k. For θ with $\mathrm{Im}\theta > 0$, $\widehat{\mathcal{G}}_0(\theta)$ is a normal operator with spectrum contained in the closed upper complex half–plane $\mathbb{C}^+$ and domain $\mathrm{Dom}(\widehat{\mathcal{G}}_0(\theta)) = \mathrm{Dom}(\widehat{\mathcal{G}}_0) \cap \mathrm{Dom}(\widehat{N})$. In particular, $\widehat{\mathcal{G}}_0(\theta)$ is the generator of a strongly continuous contraction semigroup for such θ.

Recall that $\widehat{V}_{\mathrm{per}}$ is defined as $\mathfrak{F}V_{\mathrm{per}}\mathfrak{F}^*$, see (62). This operator acts as a convolution, (63), having the convolution kernel $\widehat{v}_{\mathrm{per}}(\ell)$ given in (64). We define the spectrally deformed kernel by

$$\widehat{v}_{\mathrm{per}}(\ell; \theta) = \frac{1}{T}\int_0^T e^{-2\pi i\ell t/T}\{W_t(\theta) - (J\Delta^{1/2}W_t\Delta^{-1/2}J)(\theta)\}\mathrm{d}t, \tag{71}$$

where (see (54), (55))

$$\begin{aligned}
W_t(\theta) &= \lambda \underrightarrow{Q(t)} \otimes \frac{1}{\sqrt{2}}\left(a(g_{f,\bar{\theta}}) + a^*(g_{f,\theta})\right) \\
&\equiv \lambda \underrightarrow{Q(t)} \otimes \Phi_\theta(g_f) \tag{72} \\
(J\Delta^{1/2}W_t\Delta^{-1/2}J)(\theta) &= \lambda\rho_{\mathrm{s}}^{-1/2}\underleftarrow{Q(t)}\rho_{\mathrm{s}}^{1/2} \\
&\quad \otimes \frac{(-1)^{\mathrm{d}\Gamma(\mathbf{1})}}{\sqrt{2}}\left(a(ie^{-\beta(p+\bar{\theta})}g_{f,\bar{\theta}}) + a^*(ig_{f,\theta})\right) \\
&\equiv \lambda\rho_{\mathrm{s}}^{-1/2}\underleftarrow{Q(t)}\rho_{\mathrm{s}}^{1/2} \otimes \Phi_\theta'(g_f). \tag{73}
\end{aligned}$$

Here,

$$g_{f,\theta}(p, \vartheta) := g_f(p + \theta, \vartheta). \tag{74}$$

We have put complex conjugates on θ in (72) and (73) in an appropriate way, so that $\theta \mapsto \widehat{v}_{\mathrm{per}}(\ell; \theta)$ is analytic in θ, for $0 < \mathrm{Im}\theta < r_{\mathrm{max}}$. Combining (71), (72) and (73) we have the representation

$$\widehat{v}_{\mathrm{per}}(\ell; \theta) = \lambda\underrightarrow{\widehat{Q}(\ell)} \otimes \Phi_\theta(g_f) + \lambda\rho_{\mathrm{s}}^{-1/2}\underleftarrow{\widehat{Q}(\ell)}\rho_{\mathrm{s}}^{1/2} \otimes \Phi_\theta'(g_f), \tag{75}$$

where $\widehat{Q}(\ell) = \frac{1}{T}\int_0^t e^{-2\pi i\ell t/T}Q(t)\mathrm{d}t$, see (27).

Let $\widehat{V}_{\mathrm{per}}(\theta)$ be the operator on $\widehat{\mathfrak{H}}_{\mathrm{per}}$ given by the convolution with kernel $\widehat{v}_{\mathrm{per}}(\ell;\theta)$. Then $\theta \mapsto \widehat{V}_{\mathrm{per}}(\theta)$ is (bounded operator valued) analytic in θ, for $0 < \mathrm{Im}\theta < r_{\mathrm{max}}$, and we have the bound

$$\max_{\theta:0\leq \mathrm{Im}\theta\leq r_{\mathrm{max}}} \left\|\widehat{V}_{\mathrm{per}}(\theta)\right\| \leq C\,|\lambda|, \tag{76}$$

for some $C < \infty$ (see Assumption (A1) before (11)). Since $\widehat{V}_{\mathrm{per}}(\theta)$ is bounded for any $\theta \in \mathbb{C}$ with $0 < \mathrm{Im}\theta < r_{\mathrm{max}}$, the deformed Howland generator

$$\widehat{\mathcal{G}}(\theta) := \widehat{\mathcal{G}}_0(\theta) + \widehat{V}_{\mathrm{per}}(\theta) \tag{77}$$

is the generator of a strongly continuous semigroup, for each such θ. It is easy to see that $\|\mathrm{e}^{i\alpha\widehat{\mathcal{G}}_0(\theta)}\| = 1$ and therefore [19, Chapter III, 1.3],

$$\|\mathrm{e}^{i\alpha\widehat{\mathcal{G}}(\theta)}\| \leq \mathrm{e}^{\alpha\|\widehat{V}_{\mathrm{per}}(\theta)\|}. \tag{78}$$

The subspace

$$\widehat{\mathfrak{D}}_0 := \left\{ \hat{f}:\ \hat{f}(\mathbb{Z}) \subset \mathrm{Dom}(L) \text{ and } \hat{f}(k) \neq 0,\ \text{only for finitely many } k \in \mathbb{Z} \right\}$$

is a core of $\widehat{\mathcal{G}}_0(\theta)$ for all $\theta \in \mathbb{C}$ with $0 \leq \mathrm{Im}\theta$. $\widehat{\mathfrak{D}}_0$ is also a core of $\widehat{\mathcal{G}}(\theta)$ for such θ, since $\widehat{V}_{\mathrm{per}}(\theta)$ is a bounded operator.

Theorem 3.5 (Deformation invariance of evolution). *For all $\alpha \geq 0$, $\widehat{\psi}_1, \widehat{\psi}_2 \in \widehat{Z}(\mathfrak{H}_{\mathrm{s}} \otimes \Omega_{\mathcal{R}}) \subset \widehat{\mathfrak{H}}_{\mathrm{per}}$, and all $\theta \in \mathbb{C}$ with $\mathrm{Im}\theta \in [0, r_{\mathrm{max}})$, we have*

$$\langle \widehat{\psi}_1, \mathrm{e}^{i\alpha\widehat{\mathcal{G}}}\widehat{\psi}_2 \rangle_{\widehat{\mathrm{per}}} = \langle \widehat{\psi}_1, \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(\theta)}\widehat{\psi}_2 \rangle_{\widehat{\mathrm{per}}}. \tag{79}$$

Proof of Theorem 3.5. We show that

$$\langle \widehat{\psi}_1, \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(\theta)}\widehat{\psi}_2 \rangle_{\widehat{\mathrm{per}}} = \langle \widehat{\psi}_1, \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(\theta')}\widehat{\psi}_2 \rangle_{\widehat{\mathrm{per}}} \tag{80}$$

for all $\theta, \theta' \in \mathbb{C}$ with $\mathrm{Im}\theta, \mathrm{Im}\theta' \in [0, r_{\mathrm{max}})$. The result follows since $\widehat{\mathcal{G}} = \widehat{\mathcal{G}}(0)$.

Note that $\widehat{\mathcal{G}}(\theta) \to \widehat{\mathcal{G}}$ strongly on $\widehat{\mathfrak{D}}_0$, as $\theta \to 0$ in the upper complex half plane. It follows from the first Trotter-Kato approximation theorem [19, Chapter III, 4.8] that for all $\hat{f} \in \widehat{\mathfrak{H}}_{\mathrm{per}}$,

(a) $\mathrm{e}^{i\alpha\widehat{\mathcal{G}}}\hat{f} = \lim_{\theta\to 0} \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(\theta)}\hat{f}$

(b) $(\zeta - \widehat{\mathcal{G}})^{-1}\widehat{f} = \lim_{\theta \to 0}(\zeta - \widehat{\mathcal{G}}(\theta))^{-1}\widehat{f}$, for $\zeta \in \mathbb{C}$ with $\mathrm{Im}\zeta < -\|\widehat{V}_{\mathrm{per}}\|$ (the contraction constant for $\theta = 0$, see also (78)).

The integral representation $(\zeta - \widehat{\mathcal{G}}(\theta))^{-1}\widehat{f} = -i \int_0^{\infty} \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(\theta)} \mathrm{e}^{i\zeta\alpha}\widehat{f} \, \mathrm{d}\alpha$ is valid for all $\zeta \in \mathbb{C}$ with $\mathrm{Im}\zeta < -\|\widehat{V}_{\mathrm{per}}(\theta)\|$ [19, Chapter II, 1.10]. This representation, together with the injectivity of the Laplace transform implies that in order to prove (80), we only need to show

$$\langle \widehat{\psi}_1, (\zeta - \widehat{\mathcal{G}}(\theta))^{-1}\widehat{\psi}_2 \rangle_{\widehat{\mathrm{per}}} = \langle \widehat{\psi}_1, (\zeta - \widehat{\mathcal{G}}(\theta'))^{-1}\widehat{\psi}_2 \rangle_{\widehat{\mathrm{per}}} \tag{81}$$

for any $\widehat{\psi}_1, \widehat{\psi}_2 \in \mathfrak{F}(\mathfrak{H}_{\mathrm{s}} \otimes \Omega_{\mathcal{R}}) \subset \widehat{\mathfrak{H}}_{\mathrm{per}}$, all $\theta, \theta' \in \mathbb{C}$ with $\mathrm{Im}\theta, \mathrm{Im}\theta' \in [0, \mathrm{r}_{\max})$ and every $\zeta \in \mathbb{C}$ with imaginary part sufficiently large and negative.

Let $\theta \in \mathbb{R}$. Define the translation operator $u(\theta)$ on $L^2(\mathbb{R}^3)$ by $(u(\theta)f)(p, \vartheta) = f(p + \theta, \vartheta)$ and lift its action to $\widehat{\mathfrak{H}}_{\mathrm{per}}$ by setting

$$\forall \widehat{f} \in \widehat{\mathfrak{H}}_{\mathrm{per}}, \; k \in \mathbb{Z}: \quad (U(\theta)\widehat{f})(k) := (\mathbf{1}_{\mathfrak{H}_{\mathrm{s}}} \otimes \Gamma(u(\theta)))(\widehat{f}(k)),$$

where $\Gamma(u(\theta))$ is the second quantization of $u(\theta)$. We have $U(\theta) = U(-\theta)^*$ for all $\theta \in \mathbb{R}$. Observe that $(\zeta - \widehat{\mathcal{G}}(\theta))^{-1} = U(\theta)(\zeta - \widehat{\mathcal{G}}(0))^{-1}U(\theta)^*$ and $U(\theta)\widehat{\psi} = \widehat{\psi}$ for $\theta \in \mathbb{R}$ and $\widehat{\psi} \in \widehat{Z}(\mathfrak{H}_{\mathrm{s}} \otimes \Omega_{\mathcal{R}})$. It follows that the function

$$g(\theta) := \langle \widehat{\psi}_1, (\zeta - \widehat{\mathcal{G}}(\theta))^{-1}\widehat{\psi}_2 \rangle_{\widehat{\mathrm{per}}}$$

is constant on $\mathbb{R}$, i.e., $g(\theta) = g(0)$ for all $\theta \in \mathbb{R}$. The family $\{\widehat{\mathcal{G}}(\theta)\}_{\theta \in \mathbb{R} + i(0, \mathrm{r}_{\max})}$ of closed operators is of type A. Take ζ with $\mathrm{Im}\zeta < -\sup_{\theta \in \mathbb{R} + i(0, \mathrm{r}_{\max})}\|\widehat{V}_{\mathrm{per}}(\theta)\|$. Then $\theta \mapsto g(\theta)$ is analytic for all $\theta \in \mathbb{R} + i(0, r_{\max})$. By (b) above, the function is continuous as θ approaches the real axis from above. Using the Schwarz reflection principle, we deduce that g must be constant on all of $\mathbb{R} + i[0, r_{\max})$. $\blacksquare$

Theorem 3.5 shows that the evolution of the system is expressed by the C_0-semigroup $\{\mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)}\}_{\alpha \geq 0}$ at any fixed $r \in (8\|H_{\mathrm{s}}\|, r_{\max})$. See again condition (A2). For $\lambda = 0$, the spectrum of the normal operator $\widehat{\mathcal{G}}_0(ir)$ is

$$\sigma(\widehat{\mathcal{G}}_0(ir)) = \sigma_{\mathrm{d}}(\widehat{\mathcal{G}}_0(ir)) \cup \{\mathbb{R} + ir\mathbb{N}\},$$

where

$$\sigma_{\mathrm{d}}(\widehat{\mathcal{G}}_0(ir)) = \frac{2\pi}{T}\mathbb{Z} + \sigma([H_{\mathrm{s}}, \cdot])$$

is the set of discrete eigenvalues of $\widehat{\mathcal{G}}_0(ir)$. The spectrum of $[H_{\mathrm{s}}, \cdot]$ consists of all differences of eigenvalues of H_{s} (the so-called Bohr energies). The spectral

deformation separates the discrete and continuous spectrum by a distance r. This property allows us to apply standard analytic perturbation theory to follow the eigenvalues under perturbation ($\lambda \neq 0$).

Let $\gamma_{k,\varepsilon}$, $k \in \mathbb{Z}$, be the positively oriented circle with center $\frac{2\pi}{T}k$ and radius ε satisfying $2\|H_{\mathrm{s}}\| < \varepsilon < \min\{\frac{\pi}{T}, \frac{r}{2}\}$. We want the circles $\gamma_{k,\varepsilon}$ to contain exactly the eigenvalues of $\widehat{\mathcal{G}}(ir)$ bifurcating (for $\lambda \neq 0$) out of the eigenvalues $2\pi\ell/T + \sigma([H_{\mathrm{s}}, \cdot])$ for $\ell = k$ (no overlapping resonances). Therefore we impose the condition $\pi/T > 2\|H_{\mathrm{s}}\|$, or, condition (A1). The projection valued maps

$$\lambda \mapsto \mathcal{P}_k(\lambda) := \frac{1}{2\pi i} \oint_{\gamma_{k,\varepsilon}} (\zeta - \widehat{\mathcal{G}}(ir))^{-1} \mathrm{d}\zeta \tag{82}$$

are analytic in some ball of radius $c > 0$ and center 0 in the complex plane, with c independent of $k \in \mathbb{Z}$. We have

$$\mathcal{P}_0(0) = \mathbf{1}_{\mathfrak{H}_{\mathrm{s}}} \otimes |\widehat{Z}(\Omega_{\mathcal{R}})\rangle\langle\widehat{Z}(\Omega_{\mathcal{R}})|. \tag{83}$$

To ease the readability of the equations to follow, we define

$$\widehat{\Psi}(A) = \widehat{Z}(\pi(A \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}})\Omega_0), \tag{84}$$

so that

$$\left\langle \widehat{Z}\Omega_0, \mathrm{e}^{i\alpha\widehat{\mathcal{G}}}\widehat{Z}((\pi(A \otimes \mathbf{1}_{\mathfrak{H}_{\mathcal{R}}})\Omega_0)) \right\rangle_{\widehat{\mathrm{per}}} = \left\langle \widehat{\Psi}(\mathbf{1}), \mathrm{e}^{i\alpha\widehat{\mathcal{G}}}\widehat{\Psi}(A) \right\rangle_{\widehat{\mathrm{per}}}. \tag{85}$$

By the Laplace inversion formula [19, Chapter III, Corollary 5.15],

$$\left\langle \widehat{\Psi}(\mathbf{1}), \mathrm{e}^{i\alpha\widehat{\mathcal{G}}}\widehat{\Psi}(A) \right\rangle_{\widehat{\mathrm{per}}}$$
$$= \lim_{\ell \to \infty} \frac{1}{2\pi i} \int_{\tilde{\gamma}_\ell} \mathrm{e}^{i\alpha z} \left\langle \widehat{\Psi}(\mathbf{1}), (z - \widehat{\mathcal{G}}(ir))^{-1}\widehat{\Psi}(A) \right\rangle_{\widehat{\mathrm{per}}} \mathrm{d}z,$$

where $\tilde{\gamma}_\ell$ is the straight contour from $-\ell - i$ to $\ell - i$. Deforming contours we obtain

$$\left\langle \widehat{\Psi}(\mathbf{1}), \mathrm{e}^{i\alpha\widehat{\mathcal{G}}}\widehat{\Psi}(A) \right\rangle_{\widehat{\mathrm{per}}}$$
$$= \lim_{\ell \to \infty} \Bigg[\frac{1}{2\pi i} \int_{\gamma_\ell} \mathrm{e}^{i\alpha z} \left\langle \widehat{\Psi}(\mathbf{1}), (z - \widehat{\mathcal{G}}(ir))^{-1}\widehat{\Psi}(A) \right\rangle_{\widehat{\mathrm{per}}} \mathrm{d}z$$
$$+ \sum_{\substack{k \in \mathbb{Z}, \\ |k\frac{2\pi}{T}| < \ell}} \left\langle \widehat{\Psi}(\mathbf{1}), \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)}\mathcal{P}_k(\lambda)\widehat{\Psi}(A) \right\rangle_{\widehat{\mathrm{per}}} \Bigg], \tag{86}$$

where γ_ℓ is the straight contour from $-\ell + ir/2$ to $\ell + ir/2$. Indeed, observe that, on the domain of $\widehat{\mathcal{G}}(ir)$,

$$(z - \widehat{\mathcal{G}}(ir))^{-1} = \frac{1}{z}\left[(z - \widehat{\mathcal{G}}(ir))^{-1}\widehat{\mathcal{G}}(ir) + \mathbf{1}\right] \tag{87}$$

and thus $(z - \widehat{\mathcal{G}}(ir))^{-1}\widehat{\Psi}(A) = \mathcal{O}(\ell^{-1})$ on the straight vertical paths joining $z = \pm\ell - i$ and $z = \pm\ell + ir/2$.

Lemma 3.6 (Bounds for $\mathcal{P}_k(\lambda)$). *For some $C < \infty$, all $A \in \mathcal{B}(\mathbb{C}^d)$, and all $k \in \mathbb{Z}\backslash\{0\}$, we have*

$$\left\|\mathcal{P}_k(\lambda)\widehat{\Psi}(A)\right\|_{\widehat{\mathrm{per}}} \leq C\,\|A\|\,T^2 k^{-2}\left(1 + |\lambda|\max_{0 \leq t \leq T}\|H_\mathrm{c}(t)\|\right).$$

Proof of Lemma 3.6. Applying (87) twice and observing that $\frac{1}{\zeta}$, $\frac{1}{\zeta^2}$ are analytic away from $\zeta = 0$ we get

$$\begin{aligned}
\mathcal{P}_k(\lambda)\widehat{\Psi}(A) &= \frac{1}{2\pi i}\oint_{\gamma_{k,\varepsilon}} \frac{1}{\zeta}(\zeta - \widehat{\mathcal{G}}(ir))^{-1}\widehat{\mathcal{G}}(ir)\widehat{\Psi}(A)\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\oint_{\gamma_{k,\varepsilon}} \frac{1}{\zeta^2}(\zeta - \widehat{\mathcal{G}}(ir))^{-1}\widehat{\mathcal{G}}(ir)^2\widehat{\Psi}(A)\mathrm{d}\zeta.
\end{aligned} \tag{88}$$

We obtain now the bound $\widehat{\mathcal{G}}(ir)^2\widehat{\Psi}(A) = \mathcal{O}(T^0\lambda^0 + \lambda\max_{0 \leq t \leq T}\|H_\mathrm{c}(t)\|)$, which, together with $\zeta^{-2} = \mathcal{O}(T^2/k^2)$, implies the bound of Lemma 3.6. Observe that, since $L_\mathcal{R}\widehat{\Psi}(A) = 0$, $\widehat{N}\widehat{\Psi}(A) = 0$ and $k\widehat{\Psi}(A) = 0$, we have

$$\begin{aligned}
\widehat{\mathcal{G}}(ir)^2\widehat{\Psi}(A) &= \left(\widehat{\mathcal{G}}_0(ir) + \widehat{V}_{\mathrm{per}}(ir)\right)\left(L_\mathrm{s} + \widehat{V}_{\mathrm{per}}(ir)\right)\widehat{\Psi}(A) \\
&= \left(L_\mathrm{s}^2 + \widehat{V}_{\mathrm{per}}(ir)^2 + \widehat{V}_{\mathrm{per}}(ir)L_\mathrm{s} + \widehat{\mathcal{G}}_0(ir)\widehat{V}_{\mathrm{per}}(ir)\right)\widehat{\Psi}(A),
\end{aligned}$$

from which it follows that

$$\|\widehat{\mathcal{G}}(ir)^2\widehat{\Psi}(A)\|_{\widehat{\mathrm{per}}} \leq C\|A\| + \|\widehat{\mathcal{G}}_0(ir)\widehat{V}_{\mathrm{per}}(ir)\widehat{\Psi}(A)\|_{\widehat{\mathrm{per}}}.$$

To prove the lemma it suffices thus to show that

$$\|\widehat{\mathcal{G}}_0(ir)\widehat{V}_{\mathrm{per}}(ir)\widehat{\Psi}(A)\|_{\widehat{\mathrm{per}}} \leq C\|A\|\,|\lambda|\left(1 + |\lambda|\max_{0 \leq t \leq T}\|H_\mathrm{c}(t)\|\right). \tag{89}$$

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We have

$$\widehat{\mathcal{G}}_0(ir)\widehat{V}_{\mathrm{per}}(ir)\,\widehat{\Psi}(A) = (L_{\mathrm{s}} + L_{\mathcal{R}} + ir - 2\pi k/T)\widehat{V}_{\mathrm{per}}(ir)\,\widehat{\Psi}(A). \qquad (90)$$

We use here that the vector $\widehat{V}_{\mathrm{per}}(ir)\widehat{\Psi}(A)$ is in the sector $\widehat{N} = 1$, since $\widehat{V}_{\mathrm{per}}(ir)$ is linear in creation and annihilation operators and $\widehat{\Psi}(A)$ is proportional to the vacuum, see (84) and (50). The summand with $L_{\mathrm{s}} + ir$ in (90) is bounded above by $C|\lambda|\,\|A\|$ (uniformly in T). Next, $\widehat{V}_{\mathrm{per}}(ir) = \widehat{V}_{\mathrm{per},1}(ir) + \widehat{V}_{\mathrm{per},2}(ir)$ is the sum of two terms, each one given as the convolution operator with kernel given by one of the summands in (75). We treat

$$\widehat{V}_{\mathrm{per},1}(ir) = \lambda\mathfrak{F}Q(t)\mathfrak{F}^* \otimes \Phi_{ir}(g_f),$$

the other one is estimated in the same fashion. We have

$$\|L_{\mathcal{R}}\widehat{V}_{\mathrm{per},1}(ir)\widehat{\Psi}(A)\| \leq |\lambda|\,\|Q\|\,\|A\|\,\|L_{\mathcal{R}}\Phi_{ir}(g_f)\Omega_{\mathcal{R}}\| \leq C|\lambda|\,\|A\|,$$

due to condition (A2) and since $\|\mathfrak{F}Q(t)\mathfrak{F}^*\| = \|Q(t)\| = \|Q\|$. Consider now the term containing the factor $-2\pi k/T$ in (90),

$$\|2\pi kT^{-1}\widehat{V}_{\mathrm{per}}(ir)\| \leq C|\lambda|\,\|2\pi kT^{-1}\mathfrak{F}Q\mathfrak{F}^*\| = C|\lambda|\,\|\mathfrak{F}\partial_t Q\mathfrak{F}^*\| = C|\lambda|\,\|\partial_t Q\|.$$

Finally,

$$\|\partial_t Q(t)\| \leq C\|\partial_t V_{\mathrm{c}}(t)\| \leq C\|H_{\mathrm{c}}(t)\|,$$

see (21) and (5). This shows (89). $\blacksquare$

Lemma 3.6 implies that the sum in (86) converges as $\ell \to \infty$, and thus

$$
\begin{aligned}
\Big\langle \widehat{\Psi}(\mathbf{1}), \mathrm{e}^{i\alpha\widehat{\mathcal{G}}}\widehat{\Psi}(A)\Big\rangle_{\widehat{\mathrm{per}}} \\
= \ &\frac{1}{2\pi i}\lim_{\ell\to\infty}\int_{\gamma_\ell}\mathrm{e}^{i\alpha z}\Big\langle \widehat{\Psi}(\mathbf{1}), (z - \widehat{\mathcal{G}}(ir))^{-1}\widehat{\Psi}(A)\Big\rangle_{\widehat{\mathrm{per}}}\,\mathrm{d}z \\
&+ \sum_{k\in\mathbb{Z}}\Big\langle \widehat{\Psi}(\mathbf{1}), \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)}\mathcal{P}_k(\lambda)\widehat{\Psi}(A)\Big\rangle_{\widehat{\mathrm{per}}} \\
= \ &\sum_{k\in\mathbb{Z}}\Big\langle \widehat{\Psi}(\mathbf{1}), \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)}\mathcal{P}_k(\lambda)\widehat{\Psi}(A)\Big\rangle_{\widehat{\mathrm{per}}} + \mathcal{O}(\mathrm{e}^{-\frac{\alpha r}{2}}\lambda^2). \qquad (91)
\end{aligned}
$$

To see that the integral above is $\mathcal{O}(\mathrm{e}^{-\frac{\alpha r}{2}}\lambda^2)$ we observe that by expanding the resolvent in the integrand in powers of λ, the terms of order λ^0 and λ^1 vanish: the former vanishes since L_{s} has no spectrum below γ_ℓ, the latter vanishes since the interaction $\widehat{V}_{\mathrm{per}}$ is linear in creation and annihilation operators.

The next result shows that the term $k = 0$ in (91) is dominant.

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Lemma 3.7. *We have for all $\alpha \geq 0$*

$$\left| \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} \left\langle \widehat{\Psi}(\mathbf{1}), \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)} \mathcal{P}_k(\lambda) \widehat{\Psi}(A) \right\rangle_{\widehat{\mathrm{per}}} \right|$$

$$\leq\ C \, \|A\| \, T^2 \left(1 + \left\| \mathrm{e}^{i\alpha\widehat{\mathcal{G}}_0(ir)} \mathcal{P}_0(0) - \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)} \mathcal{P}_0(\lambda) \right\| \right) \left(1 + |\lambda| \max_{0 \leq t \leq T} \|H_{\mathrm{c}}(t)\| \right)$$

for some $C < \infty$ not depending on $A \in \mathcal{B}(\mathbb{C}^d)$.

Proof of Lemma 3.7. Define the family of unitaries $\{U_\delta\}_{\delta \in \mathbb{Z}}$ acting on $\widehat{\mathfrak{H}}_{\mathrm{per}}$ by $(U_\delta \widehat{\psi})(k) := \widehat{\psi}(k + \delta)$. We have $U_\delta \widehat{\mathcal{G}}(ir) U_\delta^* = -\frac{2\pi\delta}{T} + \widehat{\mathcal{G}}(ir)$ and hence

$$U_\delta \mathcal{P}_k(\lambda) = \mathcal{P}_{k+\delta}(\lambda) U_\delta. \tag{92}$$

Using this relation we obtain

$$
\begin{aligned}
\mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)} &\mathcal{P}_k(\lambda) \widehat{\Psi}(A) \\
&= \mathrm{e}^{-2\pi i k\alpha/T} U_k \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)} \mathcal{P}_0(\lambda) U_k^* \mathcal{P}_k(\lambda) \widehat{\Psi}(A) \\
&= -\mathrm{e}^{-2\pi i k\alpha/T} U_k [\mathrm{e}^{i\alpha\widehat{\mathcal{G}}_0(ir)} \mathcal{P}_0(0) - \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)} \mathcal{P}_0(\lambda)] U_k^* \mathcal{P}_k(\lambda) \widehat{\Psi}(A) \\
&\quad + \mathrm{e}^{-2\pi i k\alpha/T} U_k \mathrm{e}^{i\alpha\widehat{\mathcal{G}}_0(ir)} \mathcal{P}_0(0) U_k^* \mathcal{P}_k(\lambda) \widehat{\Psi}(A).
\end{aligned}
$$

This equality together with Lemma 3.6 gives the result of Lemma 3.7. ∎

Combining (91) with Lemma 3.7 gives

$$
\begin{aligned}
\left\langle \widehat{\Psi}(\mathbf{1}), \mathrm{e}^{i\alpha\widehat{\mathcal{G}}} \widehat{\Psi}(A) \right\rangle_{\widehat{\mathrm{per}}} &= \left\langle \widehat{\Psi}(\mathbf{1}), \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)} \mathcal{P}_0(\lambda) \widehat{\Psi}(A) \right\rangle_{\widehat{\mathrm{per}}} \\
&\quad + \|A\| \left(1 + \left\| \mathrm{e}^{i\alpha\widehat{\mathcal{G}}_0(ir)} \mathcal{P}_0(0) - \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)} \mathcal{P}_0(\lambda) \right\| \right) \mathcal{O}\left(D|\lambda|T + T^2 + \lambda^2 \right),
\end{aligned}
$$

where D is defined in Theorem 1.1. Noticing that $\mathrm{e}^{i\alpha\widehat{\mathcal{G}}_0(ir)} \mathcal{P}_0(0) = \mathrm{e}^{i\alpha L_{\mathrm{s}}} \mathcal{P}_0(0)$, due to (83), we see that Theorem 3.4 follows from the following result:

Theorem 3.8. *Suppose the T–periodic control term H_{c} satisfies the dynamical decoupling condition (12). Then*

$$\left\| \mathrm{e}^{i\alpha L_{\mathrm{s}}} \mathcal{P}_0(0) - \mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)} \mathcal{P}_0(\lambda) \right\| \leq C(|\lambda| + \mathrm{e}^{c\alpha|\lambda|T} - 1)$$

for all $\alpha \geq 0$, where $c, C < \infty$ are constants independent of α, λ, T and H_{c}.

Proof of Theorem 3.8. In order to compare $\mathrm{e}^{i\alpha\widehat{\mathcal{G}}(ir)}\mathcal{P}_0(\lambda)$ with $\mathrm{e}^{i\alpha\widehat{\mathcal{G}}_0(ir)}\mathcal{P}_0(0) = \mathrm{e}^{i\alpha L_\mathrm{s}}\mathcal{P}_0(0)$, we use Kato's representation [20, Chapter I, Section 4.6]

$$\mathcal{P}_0(\lambda) = U\mathcal{P}_0(0)V, \tag{93}$$

valid for λ small enough so that $\|\mathcal{P}_0(\lambda) - \mathcal{P}_0(0)\| < 1$. The operators U, V are defined as follows: let $R := (\mathcal{P}_0(\lambda) - \mathcal{P}_0(0))^2$, then

$$
\begin{aligned}
U &:= (1-R)^{-\frac{1}{2}}U' = U'(1-R)^{-\frac{1}{2}}, \\
V &:= (1-R)^{-\frac{1}{2}}V' = V'(1-R)^{-\frac{1}{2}}, \\
U' &:= \mathcal{P}_0(\lambda)\mathcal{P}_0(0) + (1-\mathcal{P}_0(\lambda))(1-\mathcal{P}_0(0)), \\
V' &:= \mathcal{P}_0(0)\mathcal{P}_0(\lambda) + (1-\mathcal{P}_0(0))(1-\mathcal{P}_0(\lambda)).
\end{aligned}
$$

The operator R has norm $\|R\| < 1$ and $(1-R)^{-\frac{1}{2}}$ is defined as a bounded operator by the Taylor series of $z \mapsto (1-z)^{-\frac{1}{2}}$, centered at the origin and having radius of convergence one.

Note that $V = U^{-1}$ and that R commutes with both $\mathcal{P}_0(0)$ and $\mathcal{P}_0(\lambda)$. Using relation (93) we obtain

$$
\begin{aligned}
\exp[i\alpha\widehat{\mathcal{G}}(ir)]\mathcal{P}_0(\lambda) &= \mathcal{P}_0(\lambda)\exp[i\alpha\widehat{\mathcal{G}}(ir)]\mathcal{P}_0(\lambda) \\
&= U\mathcal{P}_0(0)V\mathcal{P}_0(\lambda)\exp[i\alpha\widehat{\mathcal{G}}(ir)]\mathcal{P}_0(\lambda)U\mathcal{P}_0(0)V \\
&= U\mathcal{P}_0(0)\exp\left[i\alpha\mathcal{P}_0(0)V\widehat{\mathcal{G}}(ir)\mathcal{P}_0(\lambda)U\mathcal{P}_0(0)\right]\mathcal{P}_0(0)V. \tag{94}
\end{aligned}
$$

To understand the last equality, it is important to notice that U maps the range of $\mathcal{P}_0(0)$ into the range of $\mathcal{P}_0(\lambda)$ and V maps the range of $\mathcal{P}_0(\lambda)$ back into the range of $\mathcal{P}_0(0)$. Recall that $\widehat{\mathcal{G}}_0(ir)\mathcal{P}_0(0) = L_\mathrm{s}\mathcal{P}_0(0)$. We prove below that

$$\left\|\mathcal{P}_0(0)V\widehat{\mathcal{G}}(ir)\mathcal{P}_0(\lambda)U\mathcal{P}_0(0) - L_\mathrm{s}\mathcal{P}_0(0)\right\| \leq c|\lambda|T, \tag{95}$$

for some $c < \infty$ independent of T, H_c. Iterating the Duhamel formula and using the fact that $\exp[i\alpha L_\mathrm{s}\mathcal{P}_0(0)]$ is a contraction group, we obtain from (95) that

$$\left\|\exp\left[i\alpha\mathcal{P}_0(0)V\widehat{\mathcal{G}}(ir)\mathcal{P}_0(\lambda)U\mathcal{P}_0(0)\right] - \exp[i\alpha L_\mathrm{s}\mathcal{P}_0(0)]\right\| \leq \mathrm{e}^{c\alpha|\lambda|T} - 1. \tag{96}$$

Notice that $U, V = \mathbf{1} + \mathcal{O}(|\lambda|)$, so Theorem 3.8 follows from (94) and (96).

It remains to show (95). We will write $\widehat{\mathcal{G}}$ for $\widehat{\mathcal{G}}(ir)$ in the remaining part of the proof. Using the definitions of U and V above and the fact that R commutes with

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$\mathcal{P}_0(\lambda)$ and $\mathcal{P}_0(0)$, we obtain

$$
\begin{aligned}
\mathcal{P}_0(0)&V\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)U\mathcal{P}_0(0) \\
&= \mathcal{P}_0(0)V'(1-R)^{-\frac{1}{2}}\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)(1-R)^{-\frac{1}{2}}U'\mathcal{P}_0(0) \\
&= \mathcal{P}_0(0)\mathcal{P}_0(\lambda)(1-R)^{-\frac{1}{2}}\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)(1-R)^{-\frac{1}{2}}\mathcal{P}_0(\lambda)\mathcal{P}_0(0) \\
&= \mathcal{P}_0(0)(1-R)^{-\frac{1}{2}}\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)(1-R)^{-\frac{1}{2}}\mathcal{P}_0(0) \\
&\quad +\mathcal{P}_0(0)(\mathcal{P}_0(\lambda)-\mathcal{P}_0(0))(1-R)^{-\frac{1}{2}}\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)(1-R)^{-\frac{1}{2}}\mathcal{P}_0(0) \\
&\quad +\mathcal{P}_0(0)\mathcal{P}_0(\lambda)(1-R)^{-\frac{1}{2}}\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)(1-R)^{-\frac{1}{2}}(\mathcal{P}_0(\lambda)\mathcal{P}_0(0)-\mathcal{P}_0(0)). \quad (97)
\end{aligned}
$$

The first term on the r.h.s. can be written as

$$
\begin{aligned}
\mathcal{P}_0(0)(1-R)^{-\frac{1}{2}}\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)(1-R)^{-\frac{1}{2}}\mathcal{P}_0(0) &= \mathcal{P}_0(0)\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)\mathcal{P}_0(0) \\
&\quad +\mathcal{P}_0(0)[(1-R)^{-\frac{1}{2}}-1]\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)\mathcal{P}_0(0) \\
&\quad +\mathcal{P}_0(0)(1-R)^{-\frac{1}{2}}\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)[(1-R)^{-\frac{1}{2}}-1]\mathcal{P}_0(0). \quad (98)
\end{aligned}
$$

By Taylor expanding about $R=0$ and remembering that $R=(\mathcal{P}_0(\lambda)-\mathcal{P}_0(0))^2$, we see that $(1-R)^{-\frac{1}{2}}-1=(\mathcal{P}_0(\lambda)-\mathcal{P}_0(0))^2\Upsilon(\lambda)=\Upsilon(\lambda)(\mathcal{P}_0(\lambda)-\mathcal{P}_0(0))^2$ for an operator $\Upsilon(\lambda)$ uniformly bounded in norm w.r.t. to λ. It follows from this and (97), (98), together with the fact that $\|\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)\|\le C$, that

$$
\left\|\mathcal{P}_0(0)V\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)U\mathcal{P}_0(0)-\mathcal{P}_0(0)\widehat{\mathcal{G}}\mathcal{P}_0(\lambda)\mathcal{P}_0(0)\right\| \tag{99}
$$

$$
\le C\left\{\|\mathcal{P}_0(0)[\mathcal{P}_0(\lambda)-\mathcal{P}_0(0)]\|+\|[\mathcal{P}_0(\lambda)-\mathcal{P}_0(0)]\mathcal{P}_0(0)\|\right\}
$$

for a constant $C<\infty$ independent of λ, T, H_c. Finally, (95) follows from (99) and this result:

Lemma 3.9. *For all $T>0$ and any control term H_c satisfying the dynamical decoupling condition (12) we have*

$$
\left.\begin{aligned}
&\|\mathcal{P}_0(\lambda)\mathcal{P}_0(0)-\mathcal{P}_0(0)\| \\
&\|\mathcal{P}_0(0)\mathcal{P}_0(\lambda)-\mathcal{P}_0(0)\| \\
&\left\|\widehat{\mathcal{G}}(ir)\mathcal{P}_0(\lambda)\,\mathcal{P}_0(0)-\widehat{\mathcal{G}}_0(ir)\mathcal{P}_0(0)\right\|
\end{aligned}\right\} \le CT|\lambda|
$$

where $C<\infty$ is a constant not depending on λ,T,H_c.

Proof of Lemma 3.9. We start with the first inequality. We have

$$
\begin{aligned}
&\mathcal{P}_0(\lambda)\mathcal{P}_0(0) - \mathcal{P}_0(0) \\
&= \frac{1}{2\pi i} \oint_{\gamma_{0,\varepsilon}} (\zeta - \widehat{\mathcal{G}}(ir))^{-1}\widehat{V}_{\mathrm{per}}(ir)(\zeta - \widehat{\mathcal{G}}_0(ir))^{-1}\mathcal{P}_0(0)\mathrm{d}\zeta, \\
&= \frac{1}{2\pi i} \oint_{\gamma_{0,\varepsilon}} \Xi(\zeta)(\zeta - \widehat{\mathcal{G}}_0(ir))^{-1}\widehat{V}_{\mathrm{per}}(ir)\mathcal{P}_0(0)(\zeta - \widehat{\mathcal{G}}_0(ir))^{-1}\mathrm{d}\zeta,
\end{aligned}
$$

where

$$
\Xi(\zeta) = \sum_{n \geq 0}\left[(\zeta - \widehat{\mathcal{G}}_0(ir))^{-1}\widehat{V}_{\mathrm{per}}(ir)\right]^n \tag{100}
$$

is uniformly bounded for $\zeta \in \gamma_{0,\varepsilon}$ (see before (82) for the definition of $\gamma_{0,\varepsilon}$). To establish the first bound in the Lemma it suffices to show that

$$
\left\|(\zeta - \widehat{\mathcal{G}}_0(ir))^{-1}\widehat{V}_{\mathrm{per}}(ir)\mathcal{P}_0(0)\right\| \leq C|\lambda|T. \tag{101}
$$

Let $\varphi_{ij} := |\varphi_i\rangle\langle\varphi_j|$, $i, j = 1, \ldots, d$, be an orthonormal basis of $\mathfrak{H}_{\mathrm{s}}$ (see before (19)). For each $\hat{f} \in \widehat{\mathfrak{H}}_{\mathrm{per}}$ and $k \in \mathbb{Z}$ we have

$$
[\mathcal{P}_0(0)\hat{f}](k) = \delta_{k,0}\sum_{i,j}\langle\varphi_{ij}\otimes\Omega_{\mathcal{R}}, \hat{f}(0)\rangle_{\mathfrak{H}}\,\varphi_{ij}\otimes\Omega_{\mathcal{R}},
$$

where $\delta_{k,0}$ is the Kronecker symbol, and

$$
[\widehat{V}_{\mathrm{per}}(ir)\varphi_{ij}\otimes\Omega_{\mathcal{R}}](k) = \widehat{v}_{\mathrm{per}}(-k; ir)\varphi_{ij}\otimes\Omega_{\mathcal{R}}, \tag{102}
$$

see (75). We thus have

$$
\left\|(\zeta - \widehat{\mathcal{G}}_0(ir))^{-1}\widehat{V}_{\mathrm{per}}(ir)\mathcal{P}_0(0)\hat{f}\right\|_{\widehat{\mathfrak{H}}_{\mathrm{per}}}^2
$$

$$
\leq \|\hat{f}(0)\|_{\mathfrak{H}}^2 \sum_{k\in\mathbb{Z}, k\neq 0}\left\|\sum_{i,j}(\zeta + \tfrac{2\pi}{T}k - L - ir\widehat{N})^{-1}\widehat{v}_{\mathrm{per}}(-k; ir)\,\varphi_{ij}\otimes\Omega_{\mathcal{R}}\right\|^2
$$

$$
\leq \|\hat{f}\|_{\widehat{\mathfrak{H}}_{\mathrm{per}}}^2 d^2 \max_{i,j}\sum_{k\in\mathbb{Z}, k\neq 0}\left\|(\zeta + \tfrac{2\pi}{T}k - L - ir)^{-1}\widehat{v}_{\mathrm{per}}(-k; ir)\,\varphi_{ij}\otimes\Omega_{\mathcal{R}}\right\|^2. \tag{103}
$$

The sum is only over $k \neq 0$ since we have, by the dynamical decoupling condition (12) (see also (18)), that $\widehat{Q}(0) = 0$, which in turn implies that $\widehat{v}_{\mathrm{per}}(0; ir) = 0$, see

(75). Again, the number operator is replaced by $\widehat{N} = 1$, since $\widehat{v}_{\mathrm{per}}(-k, ir)\varphi_{ij} \otimes \Omega_{\mathcal{R}}$ belongs to the one-particle sector. Next we examine the general term of the sum in (103). The operator $\widehat{v}_{\mathrm{per}}(-k; ir)$ has two terms, see (75). We deal with the first one, $\lambda\widehat{Q}(-k) \otimes \Phi_{ir}(g_f)$, the second one is handled in the same way. Writing the square of the norm in (103) as an inner product, we obtain the following expression (see (72)-(74))

$$
\left\| (\zeta + \tfrac{2\pi}{T}k - L - ir)^{-1}\lambda\overrightarrow{\widehat{Q}(-k)} \otimes \Phi_{ir}(g_f)\,\varphi_{ij} \otimes \Omega_{\mathcal{R}} \right\|^2
$$
$$
= \frac{\lambda^2}{2}\left\langle \varphi_{ij} \otimes \Omega_{\mathcal{R}},\,\overrightarrow{\widehat{Q}(k)} \otimes a(g_{f,ir})\left| \zeta + \tfrac{2\pi}{T}k - L - ir \right|^{-2} \right.
$$
$$
\left. \times\, \overrightarrow{\widehat{Q}(-k)} \otimes a^*(g_{f,ir})\,\varphi_{ij} \otimes \Omega_{\mathcal{R}} \right\rangle
$$
$$
= \max_{e\in\sigma(L_{\mathrm{s}})}\frac{\lambda^2}{2}\int_{\mathbb{R}\times S^2} |g_f(p+ir,\vartheta)|^2 \left\langle \varphi_{ij},\,\overrightarrow{\widehat{Q}(k)}\left| \zeta + \tfrac{2\pi}{T}k - e - p - ir \right|^{-2} \right.
$$
$$
\left. \times\, \overrightarrow{\widehat{Q}(-k)}\,\varphi_{ij} \right\rangle \mathrm{d}p\,\mathrm{d}\vartheta.
$$

Recall the notation (74). We have used the standard "pull-through" method to arrive at the last expression (e.g., $a(p,\vartheta)L_{\mathcal{R}} = (L_{\mathcal{R}} + p)a(p,\vartheta)$). With $\|\overrightarrow{\widehat{Q}(k)}\| \leq \|Q\|$ we obtain

$$
\left\| (\zeta - \widehat{\mathcal{G}}_0(ir))^{-1}\widehat{V}_{\mathrm{per}}(ir)\mathcal{P}_0(0)\hat{f} \right\|_{\widehat{\mathfrak{H}}_{\mathrm{per}}}^2 \tag{104}
$$
$$
\leq \lambda^2 d^2\|\hat{f}\|_{\widehat{\mathfrak{H}}_{\mathrm{per}}}^2\|Q\|^2 \sum_{k\in\mathbb{Z},k\neq 0}\max_{e\in\sigma(L_{\mathrm{s}})}\int_{\mathbb{R}\times S^2}\left| \frac{g_f(p+ir,\vartheta)}{\zeta + \tfrac{2\pi}{T}k - e - p - ir} \right|^2 \mathrm{d}p\,\mathrm{d}\vartheta.
$$

Set

$$
h(u) = \int_{S^2} |g_f(u - e + \mathrm{Re}\,\zeta + ir,\vartheta)|^2\mathrm{d}\vartheta, \qquad M = \frac{2\pi}{T}k, \qquad a = (r - \mathrm{Im}\,\zeta)^2.
$$

Then

$$
\int_{\mathbb{R}\times S^2}\left| \frac{g_f(p+ir,\vartheta)}{\zeta + \tfrac{2\pi}{T}k - e - p - ir} \right|^2 \mathrm{d}p\,\mathrm{d}\vartheta = \int_{\mathbb{R}}\frac{h(u)}{(u - M)^2 + a^2}\mathrm{d}u. \tag{105}
$$

We have

$$
\sup_{M\in\mathbb{R}}\frac{M^2}{(u - M)^2 + a^2} = \frac{(u^2 + a^2)^2}{a^4 + u^2 a^2}
$$

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and therefore

$$\int_{\mathbb{R}} \frac{h(u)}{(u-M)^2 + a^2} \mathrm{d}u \leq M^{-2} \int_{\mathbb{R}} \frac{(u^2 + a^2)^2}{a^4 + u^2 a^2} h(u) \mathrm{d}u \leq CM^{-2} = C\left(\frac{T}{2\pi k}\right)^2,$$

due to condition (A2), equation (11). Combining this with (103), (104) and (105), we obtain (101).

The second bound of Lemma 3.9, $\|\mathcal{P}_0(0)\mathcal{P}_0(\lambda) - \mathcal{P}_0(0)\| \leq CT|\lambda|$ is established analogously.

To show the last bound in Lemma 3.9, write (see also (100))

$$
\begin{aligned}
\widehat{\mathcal{G}}(ir)\mathcal{P}_0(\lambda)\,\mathcal{P}_0(0) - \widehat{\mathcal{G}}_0(ir)\mathcal{P}_0(0) &= \mathcal{P}_0(\lambda)\widehat{\mathcal{G}}(ir)\,\mathcal{P}_0(0) - \widehat{\mathcal{G}}_0(ir)\mathcal{P}_0(0) \\
&= \mathcal{P}_0(\lambda)(\widehat{V}_{\mathrm{per}}(ir) + L_{\mathrm{s}})\mathcal{P}_0(0) - \mathcal{P}_0(0)L_{\mathrm{s}}\mathcal{P}_0(0) \\
&= \frac{1}{2\pi i}\oint_{\gamma_{0,\varepsilon}} \Xi(\zeta)(\zeta - \widehat{\mathcal{G}}_0(ir))^{-1}\widehat{V}_{\mathrm{per}}\,(ir)\,\mathcal{P}_0(0)\mathrm{d}\zeta \\
&\quad + \frac{1}{2\pi i}\oint_{\gamma_{0,\varepsilon}} \Xi(\zeta)(\zeta - \widehat{\mathcal{G}}_0(ir))^{-1}\widehat{V}_{\mathrm{per}}\,(ir)\,\mathcal{P}_0(0)(\zeta - \widehat{\mathcal{G}}_0(ir))^{-1}L_{\mathrm{s}}\mathcal{P}_0(0)\mathrm{d}\zeta.
\end{aligned}
$$

By the bound (101) on $(\zeta - \widehat{\mathcal{G}}_0(ir))^{-1}\widehat{V}_{\mathrm{per}}\,(ir)\,\mathcal{P}_0(0)$, the third inequality in Lemma 3.9 holds. This completes the proof of Lemma 3.9 and with that the proof of Theorem 3.8. ∎

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