Repeated and continuous interactions in open quantum systems

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October 5, 2009

Abstract

We consider a finite quantum system $\mathcal{S}$ coupled to two environments of different nature. One is a heat reservoir $\mathcal{R}$ (continuous interaction) and the other one is a chain $\mathcal{C}$ of independent quantum systems $\mathcal{E}$ (repeated interaction). The interactions of $\mathcal{S}$ with $\mathcal{R}$ and $\mathcal{C}$ lead to two simultaneous dynamical processes. We show that for generic such systems, any initial state approaches an asymptotic state in the limit of large times. We express the latter in terms of the resonance data of a reduced propagator of $\mathcal{S}+\mathcal{R}$ and show that it satisfies a second law of thermodynamics. We analyze a model where both $\mathcal{S}$ and $\mathcal{E}$ are two-level systems and obtain the asymptotic state explicitly (lowest order in the interaction strength). Even though $\mathcal{R}$ and $\mathcal{C}$ are not direcly coupled, we show that they exchange energy, and we find the dependence of this exchange in terms of the thermodynamic parameters.

We formulate the problem in the framework of W^*-dynamical systems and base the analysis on a combination of spectral deformation methods and repeated interaction model techniques. We analyze the full system via rigorous perturbation theory in the coupling strength, and do not resort to any scaling limit, like e.g. weak coupling limits, or any other approximations in order to derive some master equation.

1 Introduction

Over the last years, the rigorous study of equilibrium and non-equilibrium quantum systems has received much and renewed attention. While this topic of fundamental interest has a long tradition in physics and mathematics, conventionally explored via

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‡Supported partially from Insitute for Mathematical Sciences, National University of Singapore, through the program "Mathematical Horizons for Quantum Physics", during which parts of this work have been performed. Partially supported by the Ministère Français des Affaires Étrangères through a séjour scientifique haut niveau.

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¶Supported partially from Insitute for Mathematical Sciences, National University of Singapore, through the program "Mathematical Horizons for Quantum Physics", during which parts of this work have been performed. Partially supported by the Ministère Français des Affaires Étrangères through a séjour scientifique haut niveau.

master equations [9, 6], dynamical semi-groups [3, 6] and algebraic scattering theory [33, 16], many recent works focus on a quantum resonance theory approach. The latter has been applied successfully to systems close to equilibrium [18, 27, 28, 29] and far from equilibrium [19, 26]. In both situations, one of the main questions is the (time-) asymptotic behaviour of a quantum system consisting of a subsystem $\mathcal{S}$ interacting with one or several other subsystems, given by thermal reservoirs $\mathcal{R}_1, \ldots, \mathcal{R}_n$. It has been shown that if $\mathcal{S} + \mathcal{R}$ starts in a state in which the reservoir is in a thermal state at temperature $T > 0$ far away from the system $\mathcal{S}$, then $\mathcal{S} + \mathcal{R}$ converges to the joint equilibrium state at temperature T, as time $t \to \infty$. This phenomenon is called *return to equilibrium*. (See also [23] for the situation where several equilibrium states at a fixed temperature coexist.) In case $\mathcal{S}$ is in contact with several reservoirs having different temperatures (or different other macroscopic properties), the whole system converges to a *non-equilibrium stationary state* (NESS). The success of the resonance approach is measured not only by the fact that the above-mentioned phenomena can be described rigorously and quantitatively (convergence rates), but also by that the asymptotic states can be constructed (via perturbation theory in the interaction) and their physical and mathematical structure can be examined explicitly (entropy production, heat- and matter fluxes). One of the main advantages of this method over the usual master equation approach (and the related van Hove limit) is that it gives a perturbation theory of the dynamics which is *uniform* in time $t \geq 0$. While the initial motivation for the development of the dynamical resonance theory was the investigation of the time-asymptotics, the method is becoming increasingly refined. It has been extended to give a precise picture of the dynamics of open quantum systems for all times $t \geq 0$, with applications to the phenomena of decoherence, disentanglement, and their relation to thermalization [28, 27, 29, 24]. An extension to systems with rather arbitrary time-dependent Hamiltonians has been presented in [30] (see also [2] for time-periodic systems). A further direction of development is a quantum theory of linear response and of fluctuations [21].

In certain physical setups, the reservoir has a structure of *a chain of independent elements*, $\mathcal{C} = \mathcal{E}_1 + \mathcal{E}_2 + \cdots$. An example of such a system is the so-called "one-atom maser" [25], where $\mathcal{S}$ describes the modes of the electromagnetic field in a cavity, interacting with a beam $\mathcal{C}$ of atoms $\mathcal{E}_j$, shot one by one into the cavity and interacting for a duration $\tau_j > 0$ with it. A mathematical treatment of the one-atom maser is provided in [14]. Another instance of the use of such systems is the construction of reservoirs made of "quantum noises" by means of adequate scaling limits of the characteristics of the chain $\mathcal{C}$ and its coupling with $\mathcal{S}$, which lead to certain types of master equations as well as Quantum Langevin equations [1, 7, 6, 4, 5]. The central feature of such systems is that $\mathcal{S}$ interacts successively with independent elements $\mathcal{E}_j$ constituting a reservoir. This independence implies a markovian property which simplifies the mathematical treatment considerably. In essence it enables one to express the dynamics of $\mathcal{S}$ at time $t = \tau_1 + \cdots + \tau_N$ by a propagator of product form $M_1(\tau_1) \cdots M_N(\tau_N)$, where each $M_j(\tau_j)$ encodes the dynamics of $\mathcal{S}$ with a fixed element $\mathcal{E}_j$. In case each element $\mathcal{E}_j$ is physically the same and each interaction is governed by a fixed duration τ (and a fixed interaction operator), the dynamics is given by $M(\tau)^N$ and the asymptotics is encoded in the spectrum of the *reduced dynamics operator* $M(\tau)$ [11]. An analysis for non-

constant interactions is more involved. It has been carried out in [12, 13] for systems with random characteristics (e.g. random interaction times). See also [31] for related issues. In both the deterministic and the random settings, the system approaches a limit state as $t \to \infty$, called a *repeated interaction asymptotic state* (RIAS), whose physical and mathematical properties have been investigated explicitly.

In the present work we make the synthesis of the above two situations. We consider a system $\mathcal{S}$ interacting with two environments of distinct nature (we are thus in a non-equilibrium situation). On the one hand, $\mathcal{S}$ is coupled in the repeated interaction way to a chain $\mathcal{C} = \mathcal{E} + \mathcal{E} + \cdots$, and on the other hand, $\mathcal{S}$ is in continuous contact with a heat reservoir $\mathcal{R}$. Such a system describes for example a "one-atom maser" in which one also takes into account some losses in the cavity, the latter being not completely isolated from the exterior world, e.g. from the laboratory [15]. It is assumed that $\mathcal{C}$ and $\mathcal{R}$ do not interact directly. This assumption is physically reasonable. Indeed, again for the "one-atom maser" experiment, the idea is that the atoms are ejected from an oven one by one just before they interact with the cavity and moreover the atom-field interaction time τ is typically much smaller than the damping time due to the presence of the heat reservoir. Therefore, the atoms do not have enough time to feel the effects of the reservoir before and during their interaction with the field.

Our goal is to construct the asymptotic state of the system and to analyze its physical properties. The paper is organized as follows. We present in Section 1.2 our results on the convergence to, and form of the asymptotic state, in Section 1.3 the thermodynamic properties of it, and in Section 1.4 we present the analysis of an explicit model. The proofs are given in the next sections. Namely, in Section 2 we prove the results of Section 1.2, i.e. Theorem 1.3. In Section 3 we show how to reduce the analysis of the Fermi Golden Rule (one of the main assumptions of Theorem 1.3) to standard perturbation theory of discrete eigenvalues. In Section 4 we prove the results on thermodynamic properties of the asymptotic state. Finally, in Section 5 we give some details about the explicit model presented in Section 1.4.

1.1 Description of the system

The following is a unified description of $\mathcal{S}$, $\mathcal{R}$, $\mathcal{C}$ in the language of algebraic quantum statistical mechanics (we refer the reader to e.g. [32] for a more detailed exposition). For the reader's convenience, we start from the C^*-dynamical systems formalism. A C^* dynamical system is a pair $(\mathfrak{A}, \alpha)$ where $\mathfrak{A}$ is a C^*-algebra (describing the observables of the pysical system under consideration) and $t \to \alpha^t$ is a strongly continuous group of $*$-automorphisms of $\mathfrak{A}$ (describing the evolution of the observables). A state of the system is described by a positive linear functional ω on $\mathfrak{A}$ satisfying $\omega(\mathbb{1}) = 1$. Following [19], a triple $(\mathfrak{A}, \alpha, \omega)$, where ω is an invariant state (i.e. $\omega \circ \alpha^t \equiv \omega$), is called a quantum dynamical system. Concrete examples of such quantum dynamical systems are given in Section 1.4.

Each subsystem $\# = \mathcal{S}, \mathcal{R}, \mathcal{E}$ is described by a quantum dynamical system $(\mathfrak{A}_\#, \alpha_\#, \omega_\#)$. The "reference" state $\omega_\#$ determine the macroscopic properties of the systems[1], e.g.

[1] In other words, it determines the folium of normal states. If the Hilbert space is finite-dimensional

they are KMS states at some inverse temperature $\beta_\#$. We also assume that they are faithful states, i.e. for any $A \in \mathfrak{A}_\#$, $\omega_\#(A^*A) = 0 \Rightarrow A = 0$.

In our paper, we will study the (time) asymptotic behaviour of the system using a specral approach. For that purpose, it is convenient to have a "Hilbert space description" of the system. Such a description is easy to obtain via the GNS-representation $(\mathcal{H}_\#, \pi_\#, \Psi_\#)$ of the algebras $\mathfrak{A}_\#$ associated to the states $\omega_\#$. Since the $\omega_\#$ are faithful, the $\pi_\#$ are injections and we can identify $\mathfrak{A}_\#$ and $\pi_\#(\mathfrak{A}_\#)$ (in the rest of the paper we will therefore simply write A for $\pi(A)$). We set $\mathfrak{M}_\# = \pi_\#(\mathfrak{A}_\#)'' \subset \mathcal{B}(\mathcal{H}_\#)$. The $\mathfrak{M}_\#$ form the von Neumann algebras of observables. Finally, by construction the representative vectors $\Psi_\#$ are cyclic for $\mathfrak{M}_\#$ [10], and we assume that they are also seperating vectors for $\mathfrak{M}_\#$, i.e. $A\Psi_\# = 0 \ \Rightarrow A = 0$ for any $A \in \mathfrak{M}_\#$ (note that since $\omega_\#$ is faithful, this is automatic when $A \in \pi_\#(\mathfrak{A}_\#)$). Typically, the $\Psi_\#$ describe the equilibrium states at any fixed temperature $T_\# > 0$.

We assume that $\dim \mathcal{H}_\mathcal{S} < \infty$ (i.e. $\mathfrak{A}_\mathcal{S}$ was a matrix algebra $M_n(\mathbb{C})$) and $\dim \mathcal{H}_\mathcal{E}$ may be finite or infinite. $\mathcal{R}$ being a reservoir, its Hilbert space is assumed to be infinite-dimensional, $\dim \mathcal{H}_\mathcal{R} = \infty$. The free dynamics $\alpha_\#$ of each constituent is implemented in the GNS-representation by the so-called Liouville operators $L_\#$, i.e., the Heisenberg evolution of an observable $A \in \mathfrak{M}_\#$ at time t is given by $\mathrm{e}^{\mathrm{i}tL_\#} A \mathrm{e}^{-\mathrm{i}tL_\#}$. In other words we have $\pi_\#(\alpha^t_\#(A)) = \mathrm{e}^{\mathrm{i}tL_\#}\pi_\#(A)\mathrm{e}^{-\mathrm{i}tL_\#}$. Since the $\omega_\#$ were invariant states, we can also chose the Liouville operators $L_\#$ so that $L_\#\Psi_\# = 0$ (actually such an $L_\#$ is unique, see e.g. [10]).

The Hilbert space $\mathcal{H}_\mathcal{C}$ of the chain is the infinite tensor product of factors $\mathcal{H}_\mathcal{E}$, taken with respect to the stabilizing sequence $\Psi_\mathcal{C} = \otimes_{j\geq 1}\Psi_\mathcal{E}$, i.e. $\mathcal{H}_\mathcal{C}$ is obtained by taking the completion of the vector space of finite linear combinations of the form $\otimes_{j\geq 1}\psi_j$, where $\psi_j \in \mathcal{H}_\mathcal{E}$, $\psi_j = \Psi_\mathcal{E}$ except for finitely many indices, in the norm induced by the inner product

$$\langle \otimes_j \psi_j, \otimes_j \phi_j \rangle = \prod_j \langle \psi_j, \phi_j \rangle_{\mathcal{H}_\mathcal{E}} .$$

The algebra of observables $\mathfrak{M}_\mathcal{C}$ of the chain is the von Neumann algebra

$$\mathfrak{M}_\mathcal{C} = \otimes_{m\geq 1}\mathfrak{M}_\mathcal{E}$$

acting on $\mathcal{H}_\mathcal{C}$, which is obtained by taking the weak closure of finite linear combinations of operators $\otimes_{j\geq 1}A_j$, where $A_j \in \mathfrak{M}_\mathcal{E}$ and $A_j = \mathbb{1}_{\mathcal{H}_\mathcal{E}}$ except for finitely many indices.

In summary, the non-interacting system is given by a von Neumann algebra

$$\mathfrak{M} = \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_\mathcal{R} \otimes \mathfrak{M}_\mathcal{C},$$

acting on the Hilbert space

$$\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R} \otimes \mathcal{H}_\mathcal{C},$$

and its dynamics is generated by the Liouvillian

$$L_0 = L_\mathcal{S} + L_\mathcal{R} + \sum_{k\geq 1} L_{\mathcal{E}_k}. \tag{1.1}$$

then the set of normal states is unique, but for infinite systems different classes of normal states are determined by different macroscopic parameters, such as the temperature.

Here we understand that $L_{\mathcal{E}_k}$ acts as the fixed operator $L_{\mathcal{E}}$ on the k-th factor of $\mathcal{H}_{\mathcal{C}}$, and we do not display obvious factors $\mathbb{1}$.

The operators governing the couplings between $\mathcal{S}$ and $\mathcal{E}$ and $\mathcal{S}$ and $\mathcal{R}$ are given by

$$V_{\mathcal{SE}} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{E}} \quad \text{and} \quad V_{\mathcal{SR}} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}}$$

respectively, and the total interaction is

$$V(\lambda) = \lambda_1 V_{\mathcal{SR}} + \lambda_2 V_{\mathcal{SE}} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}} \otimes \mathfrak{M}_{\mathcal{E}}, \tag{1.2}$$

where λ_1, λ_2 are coupling constants ($\lambda = (\lambda_1, \lambda_2)$). The full (Schrödinger) dynamics is

$$\psi \mapsto U(m)\psi, \tag{1.3}$$

where $U(m)$ is the unitary map

$$U(m) = \mathrm{e}^{-\mathrm{i}\tau(L_0+V_m)}\mathrm{e}^{-\mathrm{i}\tau(L_0+V_{m-1})} \cdots \mathrm{e}^{-\mathrm{i}\tau(L_0+V_1)}, \tag{1.4}$$

$\tau > 0$ being the time-scale of the repeated interaction and V_k being the operator $V(\lambda)$, (1.2), acting nontrivially on $\mathcal{H}_{\mathcal{S}}$, $\mathcal{H}_{\mathcal{R}}$ and the k-th factor $\mathcal{H}_{\mathcal{E}}$ of $\mathcal{H}_{\mathcal{C}}$ (we will also write $L_m = L_0 + V_m$). We discuss here the dynamics (1.3) at discrete time steps $m\tau$ only, a discussion for arbitrary continuous times follows in a straightforward manner by decomposing $t = m\tau + s$, $s \in [0, \tau)$, see [11].

Explicit form of finite systems and thermal reservoirs.

(A) *Finite systems.* We take $\mathcal{S}$ (and possibly $\mathcal{E}$) to be finite, i.e. $\mathfrak{h}_{\mathcal{S}} = \mathbb{C}^n$ for some n. The Hamiltonian of $\mathcal{S}$ is given by $h_{\mathcal{S}}$, acting on $\mathfrak{h}_{\mathcal{S}}$. In other words, $\mathfrak{A}_{\mathcal{S}} = M_n(\mathbb{C})$ and $\alpha^t_{\mathcal{S}}(A) = \mathrm{e}^{ith_{\mathcal{S}}} A \mathrm{e}^{-ith_{\mathcal{S}}}$. The (Gelfand-Naimark-Segal) Hilbert space, the observable algebra and the Liouville operator are given by

$$\mathcal{H}_{\mathcal{S}} = \mathfrak{h}_{\mathcal{S}} \otimes \mathfrak{h}_{\mathcal{S}}, \qquad \mathfrak{M}_{\mathcal{S}} = \mathcal{B}(\mathcal{H}_{\mathcal{S}}) \otimes \mathbb{1}, \qquad L_{\mathcal{S}} = h_{\mathcal{S}} \otimes \mathbb{1} - \mathbb{1} \otimes h_{\mathcal{S}}.$$

The reference state is chosen to be the trace state, represented by

$$\Psi_{\mathcal{S}} = \frac{1}{\sqrt{\dim \mathcal{H}_{\mathcal{S}}}} \sum_{j=1}^{\dim \mathcal{H}_{\mathcal{S}}} \varphi_j \otimes \varphi_j,$$

where $\{\varphi_j\}$ is an orthonormal basis of $\mathfrak{h}_{\mathcal{S}}$ diagonalizing $h_{\mathcal{S}}$.

(B) *Thermal reservoirs.* We take $\mathcal{R}$ (and possibly $\mathcal{E}$) to be a thermal reservoir of free Fermi particles at a temperature $T > 0$, in the thermodynamic limit. Its description was originally given in the work by Araki and Wyss [8]; see also [18] and [30], Appendix A, for an exposition close to ours. We give directly the descritpion in the GNS-representation, we provide a precise derivation of this formalism starting from the usual description of a reservoir of non-interacting and non-relativistic fermions via C^*-dynamical systems in Section 1.4.

The Hilbert space is the anti-symmetric Fock space

$$\mathcal{H}_{\mathcal{R}} = \Gamma_-(\mathfrak{h}) := \bigoplus_{n \geq 0} \mathcal{P}_- [L^2(\mathfrak{h})]^{\otimes_{j=1}^n}$$

over the one-particle space

$$\mathfrak{h} = L^2(\mathbb{R}, \mathfrak{G}), \tag{1.5}$$

where $\mathcal{P}_-$ is the orthonormal projection onto the subspace of anti-symmetric functions, and $\mathfrak{G}$ is an 'auxiliary space' (typically an angular part like $L^2(S^2)$). In this representation, the one-particle Hamiltonian h is the operator of multiplication by the radial variable (extended to negative values) $s \in \mathbb{R}$ of (1.5), i.e. for $\varphi \in L^2(\mathbb{R}, \mathfrak{G})$

$$(h\varphi)(s) = s\varphi(s).$$

The Liouville operator is the second quantization of h,

$$L_{\mathcal{R}} = \mathrm{d}\Gamma(h) := \bigoplus_{n\geq 0} \sum_{j=1}^{n} h_j, \tag{1.6}$$

where h_j is understood to act as h on the j-th factor of $\mathcal{P}_-[L^2(\mathfrak{h})]^{\otimes_{j=1}^n}$ and trivially on the other ones.

The von Neumann algebra $\mathfrak{M}_{\mathcal{R}}$ is the subalgebra of $\mathcal{B}(\mathcal{H}_{\mathcal{R}})$ generated by the thermal fermionic field operators (at inverse temperature β), represented on $\mathcal{H}_{\mathcal{R}}$ by

$$\varphi(g_\beta) = \frac{1}{\sqrt{2}}\big[a^*(g_\beta) + a(g_\beta)\big].$$

Here, we define for $g \in L^2(\mathbb{R}_+, \mathfrak{G})$

$$g_\beta(s) = \sqrt{\frac{1}{\mathrm{e}^{-\beta s}+1}} \begin{cases} g(s) & \text{if } s \geq 0 \\ \overline{g}(-s) & \text{if } s < 0. \end{cases}$$

We choose the reference state to be thermal equilibrium state, represented by the vacuum vector of $\mathcal{H}_{\mathcal{R}}$,

$$\Psi_{\mathcal{R}} = \Omega.$$

1.2 Convergence to asymptotic state

One of our main interests is the behaviour of averages $\rho(U(m)^* O_m U(m))$ as $m \to \infty$, where ρ is any (normal) initial state of the total system, and where O_m is a so-called *instantaneous observable* [11, 12, 13].

Definition 1.1 *An observable O_m is called an instantaneous observable if there exist $A_{\mathcal{SR}} \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}}$ and $B_j \in \mathfrak{M}_{\mathcal{E}}$, $j = -l, \ldots, r$, where $l, r \geq 0$ are integers such that*

$$O_m = A_{\mathcal{SR}} \otimes_{j=m-l}^{m+r} \vartheta_j(B_{j-m}) \in \mathfrak{M}, \tag{1.7}$$

where $\vartheta_j(B)$ is the observable of $\mathfrak{M}$ which acts as B on the j-th factor of $\mathcal{H}_{\mathcal{C}}$, and trivially everywhere else (ϑ_j is the translation to the j-th factor).

Note that an instantaneous observable is a time-dependent one. It may be viewed as a train of fixed observables moving with time along the chain $\mathcal{C}$ so that at time m it is "centered" at the m-th factor $\mathcal{H}_\mathcal{E}$ of $\mathcal{H}_\mathcal{C}$, on which it acts as B_0. If O acts trivially on the elements of the chain, then the corresponding instantaneous observable is constant and $O_m = O$. However, in order to be able to reveal interesting physical properties of the system, instantaneous observables are needed. For instance observables measuring fluxes of physical quantities (like energy, entropy) between $\mathcal{S}$ and the chain involve instantaneous observables acting non-trivially on $\mathcal{E}_m$ and on $\mathcal{E}_{m+1}$, which corresponds to nontrivial $A_{\mathcal{SR}}$ and B_0, B_1. We denote the Heisenberg dynamics of observables by (see (1.4))

$$\alpha^m(O_m) = U(m)^* O_m U(m). \tag{1.8}$$

The total reference vector

$$\Psi_0 = \Psi_\mathcal{S} \otimes \Psi_\mathcal{R} \otimes \Psi_\mathcal{C} \tag{1.9}$$

is cyclic and separating for the von Neumann algebra $\mathfrak{M}$. We also introduce, for later purposes, the projector $P_{\mathcal{SR}} = \mathbb{1}_{\mathcal{SR}} \otimes |\Psi_\mathcal{C}\rangle\langle\Psi_\mathcal{C}|$, which range we often identify with $\mathcal{H}_{\mathcal{SR}} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{R}$. Let J and Δ be the modular conjugation and the modular operator associated to the pair $(\mathfrak{M}, \Psi_0)$ [10]. In order to represent the dynamics in a convenient way (using a so-called C-Liouville operator), we make the following assumption.

H1 The interaction operator $V(\lambda)$, (1.2), satisfies $\Delta^{1/2} V(\lambda) \Delta^{-1/2} \in \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_\mathcal{R} \otimes \mathfrak{M}_\mathcal{E}$.

Since we will be using analytic spectral deformation methods on the factor $\mathcal{H}_\mathcal{R}$ of $\mathcal{H}$, we need to make a regularity assumption on the interaction. Let $\mathbb{R} \ni \theta \mapsto T(\theta) \in \mathcal{B}(\mathcal{H}_\mathcal{R})$ be the unitary group defined by

$$T(\theta) = \Gamma(e^{-\theta\partial_s}) \quad \text{on} \quad \Gamma_-(L^2(\mathbb{R}, \mathfrak{G})), \tag{1.10}$$

where for any $f \in L^2(\mathbb{R}, \mathfrak{G})$,

$$(e^{-\theta\partial_s} f)(s) = f(s - \theta).$$

In the following, we will use the notation

$$T(\theta) = \mathbb{1}_\mathcal{S} \otimes T(\theta) \otimes \mathbb{1}_\mathcal{E}$$

for simplicity. Note that $T(\theta)$ commutes with all observables acting trivially on $\mathcal{H}_\mathcal{R}$, in particular with $P_{\mathcal{SR}}$. Also, we have $T(\theta)\Psi_\mathcal{R} = \Psi_\mathcal{R}$ for all θ. The spectral deformation technique relies on making the parameter θ complex.

H2 The coupling operator $W_{\mathcal{SR}} := V_{\mathcal{SR}} - J\Delta^{1/2} V_{\mathcal{SR}} \Delta^{-1/2} J$ is translation analytic in a strip $\kappa_{\theta_0} = \{z \ : \ 0 < \Im z < \theta_0\}$ and strongly continuous on the real axis. More precisely, there is a $\theta_0 > 0$ such that the map

$$\mathbb{R} \ni \theta \mapsto T^{-1}(\theta) W_{\mathcal{SR}} T(\theta) = W_{\mathcal{SR}}(\theta) \in \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_\mathcal{R},$$

admits an analytic continuation into $\theta \in \kappa_{\theta_0}$ which is strongly continuous as $\Im\theta \downarrow 0$, and which satisfies

$$\sup_{0 \le \Im\theta < \theta_0} \|W_{\mathcal{SR}}(\theta)\| < \infty.$$

Definition 1.2 *An observable O is called analytic if the map*

$$\theta \to T(\theta)^{-1} O_m \Psi_0 \tag{1.11}$$

has an analytic extension to $\theta \in \kappa_{\theta_0}$ which is continuous on the real axis.

Note that for an instantaneous obervable O_m, since T acts on $\mathcal{H}_\mathcal{R}$ only, this is equivalent with $T(\theta)^{-1} A_{\mathcal{SR}} \Psi_0$ having such an extension.

Finally, we present a 'Fermi golden rule condition' which guarantees that the subsystems are well coupled so that the physical phenomena studied are visible at lowest nontrivial order in the pertrubation λ_1, λ_2. This is a very common hypothesis which is most often verified in concrete applications. To state it, we mention that the evolution is generated by so-called *reduced dynamics operators* [11] $M(\lambda)$ acting on the reduced space $\mathcal{H}_{\mathcal{SR}}$ (where the degrees of freedom of $\mathcal{C}$ have been 'traced out'). In this paper, we analyze the spectrally deformed operators

$$M_\theta(\lambda) = T(\theta)^{-1} M(\lambda) T(\theta).$$

We show in Corollary 2.3 that $M_\theta(\lambda)$ has an analytic extension into the strip κ_{θ_0}, in the sense of H2 above. We show that 1 is an eigenvalue of $M_\theta(\lambda)$ for all θ, λ and that, for small couplings $\lambda = (\lambda_1, \lambda_2)$, the spectrum of $M_\theta(\lambda)$ must lie in the closed unit disk. The latter fact is true because $M_\theta(\lambda)$ is the analytically translated (one-step) propagator of a reduced unitary dynamics, although we can only prove it in a perturbative regime. The former fact can be seen as a normalization (the trace of the reduced density matrix of $\mathcal{S} + \mathcal{R}$ equals unity at all times). Since the propagator at time step m is represented by a power of $M_\theta(\lambda)$, it is not surprising that convergence to a final state is related to the peripheral eigenvalues of $M_\theta(\lambda)$. The following Fermi golden rule condition is an ergodicity condition ensuring the existence of a unique limit state.

FGR There is a $\theta_1 \in \kappa_{\theta_0}$ and a $\lambda_0 > 0$ (depending on θ_1 in general) such that for all λ with $0 < |\lambda| < \lambda_0$, $\sigma(M_{\theta_1}(\lambda))$ (spectrum) lies inside the complex unit disk, and $\sigma(M_{\theta_1}(\lambda)) \cap \mathbb{S} = \{1\}$, the eigenvalue 1 being simple and isolated. ($\mathbb{S}$ is the complex unit circle.)

This condition is verified in practice by perturbation theory (small λ). It is also possible to prove that if the spectral radius of $M_{\theta_1}(\lambda)$ is determined by discrete eigenvalues only, then the spectrum of $M_{\theta_1}(\lambda)$ is *automatically* inside the unit disk (see Proposition A.3). Since the spectrum is a closed set the FGR condition implies that apart from the eigenvalue 1 the spectrum is contained in a disk of radius $\mathrm{e}^{-\gamma} < 1$.

Theorem 1.3 (Convergence to asymptotic state) *Assume that assumptions H1, H2 and FGR are satisfied. Then there is a $\lambda_0 > 0$ s.t. if $0 < |\lambda| < \lambda_0$, the following holds. Let ρ be any normal initial state on $\mathfrak{M}$, and let O_m be an analytic instantaneous observable of the form (1.7). Then*

$$\lim_{m\to\infty} \rho\big(\alpha^m(O_m)\big) = \rho_{+,\lambda}\Big(P_{\mathcal{SR}}\ \alpha^{l+1}\big(A_{\mathcal{SR}} \otimes_{j=-l}^{0} B_j\big) P_{\mathcal{SR}}\Big) \prod_{j=1}^{r} \langle \Psi_\mathcal{E} | B_j \Psi_\mathcal{E} \rangle, \tag{1.12}$$

where $\rho_{+,\lambda}$ is a state on $\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}}$, $P_{\mathcal{SR}}$ is the orthogonal projection onto the subspace $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$, and where α^l is the dynamics (1.8). Moreover, for analytic $A \in \mathfrak{M}$, we have the representation

$$\rho_{+,\lambda}(P_{\mathcal{SR}} A P_{\mathcal{SR}}) = \langle \psi^*_{\theta_1}(\lambda) | T(\theta_1)^{-1} P_{\mathcal{SR}} A P_{\mathcal{SR}} \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \rangle, \tag{1.13}$$

where $\psi^*_{\theta_1}(\lambda)$ is the unique invariant vector of the adjoint operator $[M_{\theta_1}(\lambda)]^*$, normalized as $\langle \psi^*_{\theta_1}(\lambda) | \Psi_0 \rangle = 1$.

Remark. The operators B_j with $j \geq 1$ measure quantities on elements $\mathcal{E}_{m+j}$ which, at time m, have not yet interacted with the system $\mathcal{S}$. Therefore, they evolve independently simply under the evolution of $\mathcal{E}_{m+j}$. For large times $m \to \infty$, the elements of the chain approach the reference state $\Psi_{\mathcal{E}}$ (because the initial state is normal), and the latter is stationary w.r.t. the uncoupled evolution. This explains the factorization in (1.12).

As a special case of Theorem 1.3 we obtain the reduced evolution of $\mathcal{S} + \mathcal{R}$.

Corollary 1.4 *Assume the setting of Theorem 1.3. Then*

$$\lim_{m \to \infty} \rho\big(\alpha^m(A_{\mathcal{SR}})\big) = \rho_{+,\lambda}\big(A_{\mathcal{SR}}\big).$$

1.3 Thermodynamic properties of asymptotic state

The total energy of the system is not defined, since $\mathcal{R}$ and $\mathcal{C}$ are reservoirs (and typically have infinite total energy). However, the *energy variation* is well defined. In order to quantify these energy jumps, let us assume for a moment that the various components, i.e. $\mathcal{S}$, $\mathcal{R}$ and the elements $\mathcal{E}$, are described via the usual Hamiltonian framework. We denote by $\mathfrak{h}_{\mathcal{S}}$ and $h_{\mathcal{S}}$ the Hilbert space and Hamiltonian describing the system $\mathcal{S}$, by $\mathfrak{h}_{\mathcal{R}}$ and $h_{\mathcal{R}}$ those for the reservoir and by $\mathfrak{h}_{\mathcal{E}}$ and $h_{\mathcal{E}}$ those for an element $\mathcal{E}$, and let $v_{\mathcal{SR}}$, resp. $v_{\mathcal{SE}}$, be a selfadjoint operator on $\mathfrak{h}_{\mathcal{S}} \otimes \mathfrak{h}_{\mathcal{R}}$, resp. $\mathfrak{h}_{\mathcal{S}} \otimes \mathfrak{h}_{\mathcal{E}}$, describing the interaction between $\mathcal{S}$ and $\mathcal{R}$, resp. $\mathcal{S}$ and $\mathcal{E}$. During each time interval $[(m-1)\tau, m\tau)$, the hamiltonian of the total system writes

$$h_m = h_{\mathcal{S}} + h_{\mathcal{R}} + \sum_k h_{\mathcal{E},k} + \lambda_1 v_{\mathcal{SR}} + \lambda_2 \theta_m(v_{\mathcal{SE}}),$$

where $h_{\mathcal{E},k}$ acts non trivially on the k-th element of the chain on which it equals $h_{\mathcal{E}}$, and $\theta_m(v) = v$ acting on $\mathcal{S}$ and the m-th element of the chain. It is then clear that energy change from time t_1 to time t_2 writes

$$\Delta E(t_2, t_1) = u(t_2)^* h_{m(t_2)+1} u(t_2) - u(t_1)^* h_{m(t_1)+1} u(t_1), \tag{1.14}$$

where

$$u(t) = \mathrm{e}^{-\mathrm{i}s(t) h_{m(t)+1}} \mathrm{e}^{-\mathrm{i}\tau h_{m(t)}} \cdots \mathrm{e}^{-\mathrm{i}\tau h_1},$$

and where we decomposed t_1 and t_2 as $t = m(t)\tau + s(t)$, $m(t) \in \mathbb{N}$ and $s(t) \in [0, \tau)$. Now, for $(m-1)\tau \leq t_1 < m\tau \leq t_2 < (m+1)\tau$, it is easy to see that (1.14) simplifies to

$$\Delta E(t_2, t_1) = \lambda_2 u(m\tau)^* (\theta_{m+1}(v_{\mathcal{SE}}) - \theta_m(v_{\mathcal{SE}})) u(m\tau) =: \delta E(m), \tag{1.15}$$

which we interpret as the energy jump observable as time passes the moment $m\tau$ (note that during each interaction, i.e. in each time interval of the form $[m\tau,(m+1)\tau)$, the full system is autonomous so that there is no energy variation in it).

Starting from a Hamiltonian description of the system and given reference states $\omega_{\mathcal{S}}$ (resp. $\omega_{\mathcal{R}}$ and $\omega_{\mathcal{E}}$) of $\mathcal{S}$ (resp. $\mathcal{R}$ and $\mathcal{E}$), one then performs the GNS representation $(\mathcal{H}_{\mathcal{S}},\pi_{\mathcal{S}},\psi_{\mathcal{S}})$ of $(\mathcal{B}(\mathfrak{h}_{\mathcal{S}}),\omega_{\mathcal{S}})$, and similarly for $\mathcal{R}$ and $\mathcal{E}$. The interaction operator $V_{\mathcal{SR}}$ and $V_{\mathcal{SE}}$ are then given by $V_{\mathcal{SR}}=\pi_{\mathcal{S}}\otimes\pi_{\mathcal{R}}\left(v_{\mathcal{SR}}\right)$ and $V_{\mathcal{SE}}=\pi_{\mathcal{S}}\otimes\pi_{\mathcal{E}}\left(v_{\mathcal{SE}}\right)$.

Now, for any observable O one has $\pi\left(u(m\tau)^{*}Ou(m\tau)\right)=\alpha^{m}(\pi(O))$ so that

$$\Delta E^{\mathrm{tot}}(m):=\pi(\delta E(m))=\lambda_{2}\alpha^{m}(\vartheta_{m+1}(V_{\mathcal{SE}})-\vartheta_{m}(V_{\mathcal{SE}})).$$

In view of the above (formal) discussion, we therefore define the energy jump observable as time passes moment $m\tau$ by

$$\Delta E^{\mathrm{tot}}(m)=\lambda_{2}\alpha^{m}(\vartheta_{m+1}(V_{\mathcal{SE}})-\vartheta_{m}(V_{\mathcal{SE}})). \tag{1.16}$$

The variation $\Delta E^{\mathrm{tot}}(m)$ is thus an instantaneous observable. In applications this observable is analytic and hence we obtain, under the conditions of Theorem 1.3 (see also [11]), that

$$\mathrm{d}E_{+}^{\mathrm{tot}}:=\lim_{m\to\infty}\frac{1}{m}\rho(\Delta E^{\mathrm{tot}}(m))=\rho_{+,\lambda}(j_{+}^{\mathrm{tot}}),$$

where $j_{+}^{\mathrm{tot}}=V-\alpha^{\tau}(V)$ is the total energy flux observable. The quantity $\mathrm{d}E_{+}^{\mathrm{tot}}$ represents the asymptotic energy change per unit time τ of the entire system.

In the same way we define the variation of energy within the system $\mathcal{S}$, the reservoir $\mathcal{R}$ and the chain $\mathcal{C}$ between times m and $m+1$ by

$$\Delta E^{\mathcal{S}}(m) = \alpha^{m+1}(L_{\mathcal{S}})-\alpha^{m}(L_{\mathcal{S}}), \tag{1.17}$$
$$\Delta E^{\mathcal{R}}(m) = \alpha^{m+1}(L_{\mathcal{R}})-\alpha^{m}(L_{\mathcal{R}}), \tag{1.18}$$
$$\Delta E^{\mathcal{C}}(m) = \alpha^{m+1}(L_{\mathcal{E}_{m+1}})-\alpha^{m}(L_{\mathcal{E}_{m+1}}). \tag{1.19}$$

Remark. One would a priori define the various energy variations using the hamiltonians $h_{\#}$ instead of the Liouvilleans $L_{\#}$, i.e. in the same way as for the total energy variation and with the same notation, $\Delta E^{\#}(m)=\pi(\delta E^{\#}(m))$ where

$$\delta E^{\#}(m):=u(m+1)^{*}h_{\#}u(m+1)-u(m)^{*}h_{\#}u(m).$$

It is easy to see that (at least formally) this leads to the same expression as those of (1.17)-(1.19).

The energy variations (1.17)-(1.19) can be expressed in terms of commutators $[V_{\mathcal{SE}},L_{\#}]$ and $[V_{\mathcal{SR}},L_{\#}]$, where $\#=\mathcal{S},\mathcal{E},\mathcal{R}$ (see Section 4). Since $[V_{\mathcal{SE}},L_{\#}]$ acts on $\mathcal{S}+\mathcal{E}$ only, it is an analytic observable (see sentence after (1.2)). We make the following Assumption.

H3 The commutators $[V_{\mathcal{SR}},L_{\#}]$, where $\#=\mathcal{S},\mathcal{E},\mathcal{R}$, are analytic observables in $\mathfrak{M}$.

We can thus apply Theorem 1.3 to obtain (see Section 4)

$$\mathrm{d}E_+^{\#} := \lim_{m\to\infty} \frac{1}{m}\rho(\Delta E^{\#}(m)) = \rho_{+,\lambda}(j_+^{\#}), \qquad \# = \mathcal{S}, \mathcal{R}, \mathcal{C}, \tag{1.20}$$

where $j^{\#}$ are explicit 'flux observables' (c.f. (4.3)-(4.5)). We show in Proposition 4.1 that $j_+^{\mathrm{tot}} = j_+^{\mathcal{S}} + j_+^{\mathcal{R}} + j_+^{\mathcal{C}}$, and that $\rho_{+,\lambda}(j_+^{\mathcal{S}}) = 0$. It follows immediately that

$$\mathrm{d}E_+^{\mathrm{tot}} = \mathrm{d}E_+^{\mathcal{R}} + \mathrm{d}E_+^{\mathcal{C}}. \tag{1.21}$$

The total energy variation is thus the sum of the variations in the energy of $\mathcal{C}$ and $\mathcal{R}$. The details of how the energy variations are shared between the subsystems depends on the particulars of the model considered; see below for an explicit example.

Next, we consider the *entropy production*. Given two normal states ρ and ρ_0 on $\mathfrak{M}$, the relative entropy of ρ with respect to ρ_0 is denoted by $\mathrm{Ent}(\rho|\rho_0)$. (This definition coincides with the one in [11] and differs from certain other works by a sign; here $\mathrm{Ent}(\rho|\rho_0) \geq 0$).

We examine the change of relative entropy of the state of the system as time evolves, relative to the reference state ρ_0 represented by the reference vector Ψ_0, see (1.9). For a thermodynamic interpretation of the entropy, we take the vectors $\Psi_{\#}$, $\# = \mathcal{S}$, $\mathcal{E}$, $\mathcal{R}$ to represent equilibrium states of respective temperatures $\beta_{\mathcal{S}}$, $\beta_{\mathcal{E}}$, $\beta_{\mathcal{R}}$. We analyze the change of relative entropy

$$\Delta S(m) = \mathrm{Ent}(\rho \circ \alpha^m|\rho_0) - \mathrm{Ent}(\rho|\rho_0)$$

proceeding as in [11]. We show in Section 4 (see (4.6)) that

$$\mathrm{d}S_+ := \lim_{m\to\infty} \frac{\Delta S(m)}{m} = (\beta_{\mathcal{R}} - \beta_{\mathcal{E}})\mathrm{d}E_+^{\mathcal{R}} + \beta_{\mathcal{E}}\mathrm{d}E_+^{\mathrm{tot}}.$$

Combining this result with (1.21), we arrive at

Corollary 1.5 *The system satisfies the following* asymptotic 2nd law of thermodynamics*,*

$$\mathrm{d}S_+ = \beta_{\mathcal{E}}\mathrm{d}E_+^{\mathcal{C}} + \beta_{\mathcal{R}}\mathrm{d}E_+^{\mathcal{R}}.$$

1.4 An explicit example

We consider $\mathcal{S}$ and $\mathcal{E}$ to be two-level systems. The observable algebra for $\mathcal{S}$ and for $\mathcal{E}$ is $\mathfrak{A}_{\mathcal{S}} = \mathfrak{A}_{\mathcal{E}} = M_2(\mathbb{C})$. Let $E_{\mathcal{S}}$, $E_{\mathcal{E}} > 0$ be the "excited" energy level of $\mathcal{S}$ and of $\mathcal{E}$, respectively. Accordingly, the Hamiltonians are given by

$$h_{\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & E_{\mathcal{S}} \end{pmatrix} \quad \text{and} \quad h_{\mathcal{E}} = \begin{pmatrix} 0 & 0 \\ 0 & E_{\mathcal{E}} \end{pmatrix}.$$

The dynamics are $\alpha_{\mathcal{S}}^t(A) = \mathrm{e}^{\mathrm{i}th_{\mathcal{S}}}A\mathrm{e}^{-\mathrm{i}th_{\mathcal{S}}}$ and $\alpha_{\mathcal{E}}^t(A) = \mathrm{e}^{\mathrm{i}th_{\mathcal{E}}}A\mathrm{e}^{-\mathrm{i}th_{\mathcal{E}}}$. We choose the reference state of $\mathcal{E}$ to be the Gibbs state at inverse temperature $\beta_{\mathcal{E}}$, i.e.

$$\rho_{\beta_{\mathcal{E}},\mathcal{E}}(A) = \frac{\mathrm{Tr}(\mathrm{e}^{-\beta_{\mathcal{E}}h_{\mathcal{E}}}A)}{Z_{\beta_{\mathcal{E}},\mathcal{E}}}, \quad \text{where} \quad Z_{\beta_{\mathcal{E}},\mathcal{E}} = \mathrm{Tr}(\mathrm{e}^{-\beta_{\mathcal{E}}h_{\mathcal{E}}}),$$

and we choose (for computational convenience) the reference state for $\mathcal{S}$ to be the tracial state, $\rho_{\mathcal{S}}(A) = \frac{1}{2}\mathrm{Tr}(A)$. The interaction operator between $\mathcal{S}$ and an element $\mathcal{E}$ of the chain is defined by $\lambda_2 v_{\mathcal{S}\mathcal{E}}$, where λ_2 is a coupling constant, and

$$v_{\mathcal{S}\mathcal{E}} := a_{\mathcal{S}} \otimes a^*_{\mathcal{E}} + a^*_{\mathcal{S}} \otimes a_{\mathcal{E}}.$$

The above creation and annihilation operators are represented by the matrices

$$a_{\#} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad a^*_{\#} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

To get a Hilbert space description of the system, one performs the Gelfand-Naimark-Segal (GNS) construction of $(\mathfrak{A}_{\mathcal{S}}, \rho_{\mathcal{S}})$ and $(\mathfrak{A}_{\mathcal{E}}, \rho_{\beta_{\mathcal{E}},\mathcal{E}})$ as in Section 1.1, see e.g. [10, 11]. In this representation, the Hilbert spaces are given by

$$\mathcal{H}_{\mathcal{S}} = \mathcal{H}_{\mathcal{E}} = \mathbb{C}^2 \otimes \mathbb{C}^2,$$

the Von Neumann algebras by

$$\mathfrak{M}_{\mathcal{S}} = \mathfrak{M}_{\mathcal{E}} = M_2(\mathbb{C}) \otimes \mathbb{1}_{\mathbb{C}^2} \subset \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2),$$

and the vectors representing $\rho_{\mathcal{S}}$ and $\rho_{\beta_{\mathcal{E}},\mathcal{E}}$ are

$$\Psi_{\mathcal{S}} = \frac{1}{\sqrt{2}}\left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right), \qquad \Psi_{\mathcal{E}} = \frac{1}{\sqrt{\mathrm{Tr}\, \mathrm{e}^{-\beta_{\mathcal{E}} h_{\mathcal{E}}}}}\left(|0\rangle \otimes |0\rangle + \mathrm{e}^{-\beta_{\mathcal{E}} E_{\mathcal{E}}/2}|1\rangle \otimes |1\rangle\right).$$

In other words, $\rho_{\mathcal{S}}(A) = \langle \psi_{\mathcal{S}}, (A \otimes \mathbb{1})\psi_{\mathcal{S}}\rangle$ and $\rho_{\beta_{\mathcal{E}},\mathcal{E}}(A) = \langle \psi_{\mathcal{E}}, (A \otimes \mathbb{1})\psi_{\mathcal{E}}\rangle$. Above, $|0\rangle$ (resp. $|1\rangle$) denotes the ground (resp. excited) state of $h_{\mathcal{S}}$ and $h_{\mathcal{E}}$. For shortness, in the following we will denote $|ij\rangle$ for $|i\rangle \otimes |j\rangle$, $i, j = 0, 1$. The free Liouvilleans $L_{\mathcal{S}}$ and $L_{\mathcal{E}}$ are given by

$$L_{\mathcal{S}} = h_{\mathcal{S}} \otimes \mathbb{1}_{\mathbb{C}^2} - \mathbb{1}_{\mathbb{C}^2} \otimes h_{\mathcal{S}}, \quad L_{\mathcal{E}} = h_{\mathcal{E}} \otimes \mathbb{1}_{\mathbb{C}^2} - \mathbb{1}_{\mathbb{C}^2} \otimes h_{\mathcal{E}}$$

and the interaction operator $V_{\mathcal{S}\mathcal{E}}$ is

$$V_{\mathcal{S}\mathcal{E}} = (a_{\mathcal{S}} \otimes \mathbb{1}_{\mathbb{C}^2}) \otimes (a^*_{\mathcal{E}} \otimes \mathbb{1}_{\mathbb{C}^2}) + (a^*_{\mathcal{S}} \otimes \mathbb{1}_{\mathbb{C}^2}) \otimes (a_{\mathcal{E}} \otimes \mathbb{1}_{\mathbb{C}^2}).$$

For the reservoir, we consider a bath of non-interacting and non-relativistic fermions. The one particle space is $\mathfrak{h}_{\mathcal{R}} = L^2(\mathbb{R}^3, \mathrm{d}^3k)$ and the one-particle energy operator $h_{\mathcal{R}}$ is the multiplication operator by $|k|^2$. The Hilbert space for the reservoir is thus $\Gamma_-(\mathfrak{h}_{\mathcal{R}})$ and the Hamiltonian is the second quantization $\mathrm{d}\Gamma(h_{\mathcal{R}})$ of $h_{\mathcal{R}}$ (see (1.6)). The algebra of observables is the C^*-algebra of operators $\mathfrak{A}$ generated by $\{a^{\#}(f) \,|\, f \in \mathfrak{h}_{\mathcal{R}}\}$ where a/a^* denote the usual annihilation/creation operators on $\Gamma_-(\mathfrak{h}_{\mathcal{R}})$. The dynamics is given by $\tau^t_{\mathrm{f}}(a^{\#}(f)) = a^{\#}(\mathrm{e}^{\mathrm{i}th} f)$, where h is the Hamiltonian of a single particle, acting on $\mathfrak{h}$. It is well known (see e.g. [10]) that for any $\beta_{\mathcal{R}} > 0$ there is a unique $(\tau_{\mathrm{f}}, \beta)-$KMS state $\rho_{\beta_{\mathcal{R}}}$ on $\mathfrak{A}$, determined by the two point function $\rho_{\beta_{\mathcal{R}}}(a^*(f)a(f)) = \langle f, (1+\mathrm{e}^{\beta_{\mathcal{R}} h_{\mathcal{R}}})^{-1} f\rangle$, and which we choose to be the reference state of the reservoir. Finally, the interaction between the small system $\mathcal{S}$ and the reservoir is chosen of electric dipole type,

i.e. of the form $v_{\mathcal{SR}} = (a_{\mathcal{S}} + a_{\mathcal{S}}^*) \otimes \varphi_{\mathcal{R}}(f)$ where $f \in \mathfrak{h}_{\mathcal{R}}$ is a form factor and $\varphi(f) = \frac{1}{\sqrt{2}}(a(f) + a^*(f))$.

We know explain how to get a description of the reservoir similar to the one given in Section 1.1. As for $\mathcal{S}$ and $\mathcal{E}$, the first point is to perform the GNS representation of $(\mathfrak{A}, \rho_{\beta_{\mathcal{R}}})$, so called Araki-Wyss representation [8]. Namely, if Ω denotes the Fock vacuum and N the number operator of $\Gamma_-(\mathfrak{h}_{\mathcal{R}})$, the Hilbert space is given by

$$\tilde{\mathcal{H}}_{\mathcal{R}} = \Gamma_-(L^2(\mathbb{R}^3, \mathrm{d}^3k)) \otimes \Gamma_-(L^2(\mathbb{R}^3, \mathrm{d}^3k)),$$

the Von-Neumann algebra of observables is

$$\tilde{\mathfrak{M}}_{\mathcal{R}} = \pi_\beta(\mathfrak{A})''$$

where

$$\pi_\beta(a(f)) = a\left(\tfrac{\mathrm{e}^{\beta h/2}}{\sqrt{1+\mathrm{e}^{\beta h}}} f\right) \otimes \mathbb{1} + (-1)^N \otimes a^*\left(\tfrac{1}{\sqrt{1+\mathrm{e}^{\beta h}}} \bar{f}\right) =: a_\beta(f),$$
$$\pi_\beta(a^*(f)) = a^*\left(\tfrac{\mathrm{e}^{\beta h/2}}{\sqrt{1+\mathrm{e}^{\beta h}}} f\right) \otimes \mathbb{1} + (-1)^N \otimes a\left(\tfrac{1}{\sqrt{1+\mathrm{e}^{\beta h}}} \bar{f}\right) =: a_\beta^*(f),$$

the reference vector is

$$\tilde{\Psi}_{\mathcal{R}} = \Omega \otimes \Omega,$$

and the Liouvillean is

$$\tilde{L}_{\mathcal{R}} = \mathrm{d}\Gamma(h_{\mathcal{R}}) \otimes \mathbb{1} - \mathbb{1} \otimes \mathrm{d}\Gamma(h_{\mathcal{R}}).$$

We then consider the isomorphism between $L^2(\mathbb{R}^3, \mathrm{d}^3k)$ and $L^2(\mathbb{R}^+ \times S^2, \frac{\sqrt{r}}{2}\mathrm{d}r\mathrm{d}\sigma) \simeq L^2(\mathbb{R}^+, \frac{\sqrt{r}}{2}\mathrm{d}r; \mathfrak{G})$, where $\mathfrak{G} = L^2(S^2, \mathrm{d}\sigma)$, so that the operator $h_{\mathcal{R}}$ (the multiplication by $|k|^2$) becomes multiplication by $r \in \mathbb{R}^+$ (i.e. we have $r = |k|^2$). The Hilbert space $\tilde{\mathcal{H}}_{\mathcal{R}}$ is thus isomorphic to

$$\Gamma_-\left(L^2(\mathbb{R}^+, \frac{\sqrt{r}}{2}\mathrm{d}r; \mathfrak{G})\right) \otimes \Gamma_-\left(L^2(\mathbb{R}^+, \frac{\sqrt{r}}{2}\mathrm{d}r; \mathfrak{G})\right). \tag{1.22}$$

Next, we make use of the maps

$$a^\#(f) \otimes \mathbb{1} \mapsto a^\#(f \oplus 0), \quad (-1)^N \otimes a^\#(f) \mapsto a^\#(0 \oplus f)$$

to define an isometric isomorphism between (1.22) and

$$\Gamma_-\left(L^2(\mathbb{R}^+, \frac{\sqrt{r}}{2}\mathrm{d}r; \mathfrak{G}) \oplus L^2(\mathbb{R}^+, \frac{\sqrt{r}}{2}\mathrm{d}r; \mathfrak{G})\right).$$

A last isometric isomorphism between the above Hilbert space and

$$\mathcal{H}_{\mathcal{R}} := \Gamma_-\left(L^2(\mathbb{R}, \mathrm{d}s; \mathfrak{G})\right)$$

is induced by the following isomorphism between the one-particle spaces $L^2(\mathbb{R}^+, \frac{\sqrt{r}}{2}\mathrm{d}r; \mathfrak{G}) \oplus L^2(\mathbb{R}^+, \frac{\sqrt{r}}{2}\mathrm{d}r; \mathfrak{G})$ and $L^2(\mathbb{R}, \mathrm{d}s; \mathfrak{G}) =: \mathfrak{h}$

$$f \oplus g \mapsto h, \text{ where } h(s) = \frac{|s|^{1/4}}{\sqrt{2}} \begin{cases} f(s) & \text{if } s \geq 0, \\ g(-s) & \text{if} s < 0. \end{cases}$$

Using the above isomorphisms, one gets a description of the form given in Section 1.1 for the reservoir $\mathcal{R}$. In this representation, the interaction operator $v_{\mathcal{SR}}$ becomes

$$V_{\mathcal{SR}} = (\sigma_x \otimes \mathbb{1}_{\mathbb{C}^2}) \otimes \varphi(f_{\beta_\mathcal{R}}) \in \mathfrak{M}_\mathcal{S} \otimes \mathfrak{M}_\mathcal{R},$$

where $\sigma_x = a_\mathcal{S} + a_\mathcal{S}^*$ is the Pauli matrix and $f_{\beta_\mathcal{R}} \in \mathfrak{h} = L^2(\mathbb{R}, \mathrm{d}s; L^2(S^2, \mathrm{d}\sigma))$ is related to the initial form factor $f \in L^2(\mathbb{R}^3, \mathrm{d}^3k)$ as follows

$$(f_{\beta_\mathcal{R}}(s))(\sigma) = \frac{1}{\sqrt{2}} \frac{|s|^{1/4}}{\sqrt{1+\mathrm{e}^{-\beta_\mathcal{R} s}}} \begin{cases} f(\sqrt{s}\,\sigma) & \text{if } s \geq 0, \\ \bar{f}(\sqrt{-s}\,\sigma) & \text{if } s < 0. \end{cases} \tag{1.23}$$

As mentioned at the beginning of the introduction, the situation where $\mathcal{S}$ is interacting with $\mathcal{R}$ or $\mathcal{C}$ alone has been treated in previous works [18, 30] and [11]. If $\mathcal{S}$ is coupled to $\mathcal{R}$ alone, then a normal initial state approaches the joint equilibrium state, i.e. the equilibrium state of the coupled system $\mathcal{S}+\mathcal{R}$ at temperature $\beta_\mathcal{R}^{-1}$, with speed $\mathrm{e}^{-m\tau\gamma_{\mathrm{th}}}$ (we consider discrete moments in time, $t = m\tau$ to compare with the repeated interaction situation). If $\mathcal{S}$ is coupled to $\mathcal{C}$ alone, initial normal states approach a repeated interaction asymptotic state, which turns out to be the equilibrium state of $\mathcal{S}$ at inverse temperature $\beta'_\mathcal{E}$ where

$$\beta'_\mathcal{E} = \beta_\mathcal{E} \frac{E_\mathcal{E}}{E_\mathcal{S}}, \tag{1.24}$$

and with speed $\mathrm{e}^{-m\tau\gamma_{\mathrm{ri}}}$. The convergence rates are given by

$$\gamma_{\mathrm{th}} = \lambda_1^2 \gamma_{\mathrm{th}}^{(2)} + O(\lambda_1^4), \qquad \text{with} \quad \gamma_{\mathrm{th}}^{(2)} = \frac{\pi}{2}\sqrt{E_\mathcal{S}} \|f(\sqrt{E_\mathcal{S}})\|_\mathfrak{G}^2 \tag{1.25}$$

$$\gamma_{\mathrm{ri}} = \lambda_2^2 \gamma_{\mathrm{ri}}^{(2)} + O(\lambda_2^4), \qquad \text{with} \quad \gamma_{\mathrm{ri}}^{(2)} = \tau \mathrm{sinc}^2\left(\frac{\tau(E_\mathcal{E} - E_\mathcal{S})}{2}\right), \tag{1.26}$$

where $\mathrm{sinc}(x) = \sin(x)/x$ and $\|f(\sqrt{E_\mathcal{S}})\|_\mathfrak{G}^2 := \displaystyle\int_{S^2} |f(\sqrt{E_\mathcal{S}}\,\sigma)|^2 \mathrm{d}\sigma$.

In order to satisfy the translation analyticity requirement **H2**, we need to make some assumption on the form factor f. Let $I(\delta) \equiv \{z \in \mathbb{C},\, |\Im(z)| < \delta\}$. We denote by $H^2(\delta)$ the Hardy class of analytic functions $h : I(\delta) \to \mathfrak{G}$ which satisfy

$$\|h\|_{H^2(\delta)} := \sup_{|\theta|<\delta} \int_\mathbb{R} \|h(s+i\theta)\|_\mathfrak{G}^2 \mathrm{d}s < \infty.$$

H4 Let f_0 be defined by (1.23), with $\beta_\mathcal{R} = 0$. There is a $\delta > 0$ s.t. $\mathrm{e}^{-\beta_\mathcal{R} s/2} f_0(s) \in H^2(\delta)$.

Proposition 1.6 (Asymptotic state of $\mathcal{S}$) *Assume f satisfies* **H4**, $\|f(\sqrt{E_\mathcal{S}})\|_\mathfrak{G} \neq 0$ *and* $\tau(E_\mathcal{E} - E_\mathcal{S}) \notin 2\pi\mathbb{Z}^*$. *Then the asymptotic state $\rho_{+,\lambda}$ is given by*

$$\rho_{+,\lambda} = \Big(\gamma \rho_{\beta_\mathcal{R},\mathcal{S}} + (1-\gamma)\rho_{\beta'_\mathcal{E},\mathcal{S}}\Big) \otimes \rho_{\beta_\mathcal{R},\mathcal{R}} + O(\lambda),$$

where $\rho_{\beta,\#}$ is the Gibbs state of $\#$, $\# = \mathcal{S}, \mathcal{R}$, at inverse temperature β and where γ is given by

$$\gamma = \frac{\lambda_1^2 \gamma_{\mathrm{th}}^{(2)}}{\lambda_1^2 \gamma_{\mathrm{th}}^{(2)} + \lambda_2^2 \gamma_{\mathrm{ri}}^{(2)}}.$$

Remark. The fact that the asymptotic state $\rho_{+,\lambda}$ is a convex combination of the two asymptotic states $\rho_{\beta_\mathcal{R},\mathcal{S}}$ and $\rho_{\beta'_\mathcal{E},\mathcal{S}}$ holds only because the system $\mathcal{S}$ is a two-level system and is not true in general.

Using (4.2)-(4.5), Corollary 1.5 and Proposition 1.6, an explicit calculation of the energy fluxes and the entropy production for this concrete model reveals the following result.

Proposition 1.7 *Assume that* $\|f(\sqrt{E_\mathcal{S}})\|_\mathfrak{G} \neq 0$ *and* $\tau(E_\mathcal{E} - E_\mathcal{S}) \notin 2\pi\mathbb{Z}^*$. *Then*

$$\begin{aligned}
\mathrm{d}E_+^\mathcal{C} &= \kappa E_\mathcal{E}\left(\mathrm{e}^{-\beta_\mathcal{R}E_\mathcal{S}} - \mathrm{e}^{-\beta'_\mathcal{E}E_\mathcal{S}}\right) + O(\lambda^3),\\
\mathrm{d}E_+^\mathcal{R} &= \kappa E_\mathcal{S}\left(\mathrm{e}^{-\beta'_\mathcal{E}E_\mathcal{S}} - \mathrm{e}^{-\beta_\mathcal{R}E_\mathcal{S}}\right) + O(\lambda^3),\\
\mathrm{d}E_+^{\mathrm{tot}} &= \kappa(E_\mathcal{E} - E_\mathcal{S})\left(\mathrm{e}^{-\beta_\mathcal{R}E_\mathcal{S}} - \mathrm{e}^{-\beta'_\mathcal{E}E_\mathcal{S}}\right) + O(\lambda^3),\\
\mathrm{d}S_+ &= \kappa(\beta'_\mathcal{E}E_\mathcal{S} - \beta_\mathcal{R}E_\mathcal{S})\left(\mathrm{e}^{-\beta_\mathcal{R}E_\mathcal{S}} - \mathrm{e}^{-\beta'_\mathcal{E}E_\mathcal{S}}\right) + O(\lambda^3),
\end{aligned}$$

where

$$\kappa = Z_{\beta_\mathcal{R},\mathcal{S}}^{-1} Z_{\beta'_\mathcal{E},\mathcal{S}}^{-1} \frac{\lambda_1^2\gamma_{\mathrm{th}}^{(2)}\ \lambda_2^2\gamma_{\mathrm{ri}}^{(2)}}{\lambda_1^2\gamma_{\mathrm{th}}^{(2)} + \lambda_2^2\gamma_{\mathrm{ri}}^{(2)}}.$$

Remarks. 1. The constant κ is positive and of order λ^2. Moreover it is zero if at least one of the two coupling constants vanishes (we are then in an equilibrium situation and there is no energy flux neither entropy production).

2. The energy flux $\mathrm{d}E_+^\mathcal{C}$ is positive (energy flows *into* chain) if and only if the reservoir temperature $T_\mathcal{R} = \beta_\mathcal{R}^{-1}$ is greater than the renormalized temperature $T'_\mathcal{E} = \beta'^{-1}_\mathcal{E}$ of the chain, i.e. if and only if the reservoir is "hotter". A similar statement holds for the energy flux $\mathrm{d}E_+^\mathcal{R}$ of the reservoir. Note that it is not the temperature of the chain which plays a role but its renormalized value (1.24).

3. When both the reservoir and the chain are coupled to the system $\mathcal{S}$ ($\lambda_1\lambda_2 \neq 0$) the entropy production vanishes (at the main order) if and only if the two temperatures $T_\mathcal{R}$ and $T'_\mathcal{E}$ are equal, i.e. if and only if we are in an equilibrium situation. Once again, it is not the initial temperature of the chain which plays a role but the renormalized one.

4. The total energy variation can be either positive or negative depending on the parameters of the model. This is different from the situation considered in [11], where that variation was always non-negative.

2 Proof of Theorem 1.3

2.1 Generator of dynamics K_m

We recall the definition of the so-called 'C-Liouvillean' introduced for the study of open systems out of equilibrium in [19], and further developped in [11, 12, 13, 27, 28, 30] (see also references in the latter papers). Let $(J_\#, \Delta_\#)$ denote the modular data associated with $(\mathfrak{M}_\#, \Psi_\#)$, with # given by $\mathcal{S}, \mathcal{R}$ or $\mathcal{E}$. Then

$$(J, \Delta) = (J_\mathcal{S} \otimes J_\mathcal{R} \otimes J_\mathcal{E}, \Delta_\mathcal{S} \otimes \Delta_\mathcal{R} \otimes \Delta_\mathcal{E})$$

are the modular data associated with $(\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}} \otimes \mathfrak{M}_{\mathcal{E}}, \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}} \otimes \Psi_{\mathcal{E}})$. We will write J_m and Δ_m to mean that these operators are considered on the m-th copy of the infinite tensor product $\mathcal{H}_{\mathcal{C}}$.

We define the C-Liouville operator

$$K_m = L_m - J_m \Delta_m^{1/2} V_m(\lambda) \Delta_m^{-1/2} J_m \equiv L_0 + W_m(\lambda),$$

$m \geq 1$, where $W_m(\lambda) \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}} \otimes \mathfrak{M}_{\mathcal{E}}$ is given by

$$\begin{aligned} W_m(\lambda) &= \lambda_1 (V_{\mathcal{SR}} - (J_{\mathcal{S}} \Delta_{\mathcal{S}}^{1/2} \otimes J_{\mathcal{R}} \Delta_{\mathcal{R}}^{1/2}) V_{\mathcal{SR}} (J_{\mathcal{S}} \Delta_{\mathcal{S}}^{1/2} \otimes J_{\mathcal{R}} \Delta_{\mathcal{R}}^{1/2})) \\ &\quad + \lambda_2 (V_{\mathcal{SE},m} - (J_{\mathcal{S}} \Delta_{\mathcal{S}}^{1/2} \otimes J_m \Delta_m^{1/2}) V_{\mathcal{SE},m} (J_{\mathcal{S}} \Delta_{\mathcal{S}}^{1/2} \otimes J_m \Delta_m^{1/2})) \\ &\equiv \lambda_1 W_{\mathcal{SR}} + \lambda_2 W_{\mathcal{SE},m}. \end{aligned} \tag{2.1}$$

Of course, $W_{\mathcal{SE},m}$ is the operator acting as $W_{\mathcal{SE}}$ on the subspace $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}_m}$ of $\mathcal{H}$, and trivially on its orthogonal complement.

The operators K_m have two crucial properties [19, 11, 27]. The first one is that they implement the same dynamics as the L_m:

$$\mathrm{e}^{\mathrm{i}tL_m} A \mathrm{e}^{-\mathrm{i}tL_m} = \mathrm{e}^{\mathrm{i}tK_m} A \mathrm{e}^{-\mathrm{i}tK_m}, \qquad \forall t \geq 0, \forall A \in \mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}} \otimes \mathfrak{M}_{\mathcal{C}}.$$

The second crucial property is that the reference state Ψ_0, (1.9), is left invariant under the evolution $\mathrm{e}^{\mathrm{i}tK_m}$,

$$K_m \Psi_0 = 0, \qquad \forall m. \tag{2.2}$$

2.2 Reduced Dynamics Operator

We follow the strategy of [11] to reduce the problem to the study of the high powers of an effective dynamics operator. The main difference w.r.t. [11] is that in the present setup, the effective dynamics operator acts now on the *infinite dimensional* Hilbert space $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}$.

We first split off the free dynamics of elements not interacting with $\mathcal{S}$ by writing the product of exponentials in $U(m)$, (1.4), as

$$U(m) = U_m^- e^{-\mathrm{i}\tau L_m} e^{-\mathrm{i}\tau L_{m-1}} \cdots e^{-\mathrm{i}\tau L_1} U_m^+,$$

where

$$L_j = L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}} + V(\lambda)$$

acts nontrivially on the subspace $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}} \otimes \mathcal{H}_{\mathcal{E}_j}$ and

$$\begin{aligned} U_m^- &= \exp\left(-\mathrm{i} \sum_{k=1}^{m} (m-k) \tau L_{\mathcal{E}_k} \right), \\ U_m^+ &= \exp\left(-\mathrm{i} \sum_{k=2}^{m+1} (k-1) \tau L_{\mathcal{E}_k} - im\tau \sum_{k>m+1} L_{\mathcal{E}_k} \right). \end{aligned}$$

Let O_m be an instantaneous observable (see (1.7)). A straightforward computation shows that (see also [13], equation (2.19))

$$\alpha^m(O_m) = (U_m^+)^* \mathrm{e}^{\mathrm{i}\tau L_1} \cdots \mathrm{e}^{\mathrm{i}\tau L_m} \mathcal{N}(O_m) \mathrm{e}^{-\mathrm{i}\tau L_m} \cdots \mathrm{e}^{-\mathrm{i}\tau L_1} U_m^+, \tag{2.3}$$

with

$$\mathcal{N}(O_m) = A_{\mathcal{SR}} \otimes_{j=-l}^{-1} \vartheta_{m+j}(\mathrm{e}^{\mathrm{i}\tau|j|L_\mathcal{E}} B_j \mathrm{e}^{-\mathrm{i}\tau|j|L_\mathcal{E}}) \otimes_{j=0}^{r} \vartheta_{m+j}(B_j). \tag{2.4}$$

As normal states are convex combinations of vector states, it sufficient to consider the latter. Let Ψ_ρ be the GNS vector representing the initial state ρ, i.e., $\rho(\cdot) = \langle \Psi_\rho | \ \cdot \ \Psi_\rho \rangle$. Since every $\Phi \in \mathcal{H}$ is approximated in the norm of $\mathcal{H}$ by finite linear combinations of vectors of the form $\Phi_\mathcal{S} \otimes \Phi_\mathcal{R} \otimes_{m\geq 1} \Phi_m$, where $\Phi_\mathcal{S} \in \mathcal{H}_\mathcal{S}$, $\Phi_\mathcal{R} \in \mathcal{H}_\mathcal{R}$, and $\Phi_m = \Psi_\mathcal{E}$ if $m > N$, for some $N < \infty$, it suffices to prove (1.12) for vector states determined by vectors Ψ_ρ of the form

$$\Psi_\rho = \Phi_\mathcal{S} \otimes \Phi_\mathcal{R} \otimes_{m=1}^{N} \Phi_m \otimes_{m>N} \Psi_\mathcal{E}, \tag{2.5}$$

for some arbitrary $N < \infty$. Finally, since the vectors $\Psi_\mathcal{S}$, $\Psi_\mathcal{R}$, $\Psi_\mathcal{E}$ are cyclic for the commutants $\mathfrak{M}'_\mathcal{S}$, $\mathfrak{M}'_\mathcal{R}$, $\mathfrak{M}'_\mathcal{E}$, any vector of the form (2.5) is approximated by a

$$\Psi = B' \ \Psi_0, \tag{2.6}$$

for some

$$B' = B'_\mathcal{S} \otimes B'_\mathcal{R} \otimes_{m=1}^{N} B'_m \otimes_{m>N} \mathbb{1}_\mathcal{E} \ \in \mathfrak{M}', \tag{2.7}$$

with $B'_\mathcal{S} \in \mathfrak{M}'_\mathcal{S}$, $B'_\mathcal{R} \in \mathfrak{M}'_\mathcal{R}$, $B'_m \in \mathfrak{M}'_\mathcal{E}$. It is therefore sufficient to show (1.12), for vector states with vectors Ψ_ρ of the form (2.6), (2.7).

Let O_m be an instantaneous observable and let us consider the expression

$$\langle \Psi_\rho | \alpha^m(O_m) \Psi_\rho \rangle = \langle B'\Psi_0 | \alpha^m(O_m) B'\Psi_0 \rangle = \langle \Psi_0 | (B')^* B' \alpha^m(O_m) \Psi_0 \rangle.$$

We use expression (2.3) and the properties of the generators K_n to obtain

$$\langle \Psi_0 | (B')^* B' \alpha^m(O_m) \Psi_0 \rangle = \langle \Psi_0 | (B')^* B' (U_m^+)^* \mathrm{e}^{\mathrm{i}\tau K_1} \cdots \mathrm{e}^{\mathrm{i}\tau K_m} \mathcal{N}(O_m) \Psi_0 \rangle.$$

Note that $U_m^+ \Psi_0 = \Psi_0$. Let

$$P_N = \mathbb{1}_\mathcal{S} \otimes \mathbb{1}_\mathcal{R} \otimes \mathbb{1}_{\mathcal{E}_1} \otimes \cdots \mathbb{1}_{\mathcal{E}_N} \otimes P_{\Psi_{\mathcal{E}_{N+1}}} \otimes P_{\Psi_{\mathcal{E}_{N+2}}} \otimes \cdots,$$

where $P_{\Psi_{\mathcal{E}_k}} = |\Psi_{\mathcal{E}_k}\rangle\langle\Psi_{\mathcal{E}_k}|$. Since $(B')^* B'$ acts non-trivially only on the factors of the chain Hilbert space having index $\leq N$, we have

$$\begin{aligned} &\langle \Psi_0 | (B')^* B' \alpha^m(O_m) \Psi_0 \rangle = \\ &\quad \langle \Psi_0 | (B')^* B' (\widetilde{U}_N^+)^* \mathrm{e}^{\mathrm{i}\tau K_1} \cdots \mathrm{e}^{\mathrm{i}\tau K_N} P_N \mathrm{e}^{\mathrm{i}\tau K_{N+1}} \cdots \mathrm{e}^{\mathrm{i}\tau K_m} \mathcal{N}(O_m) \Psi_0 \rangle, \end{aligned} \tag{2.8}$$

where (for $m > N$; we have the limit $m \to \infty$ in mind)

$$\widetilde{U}_N^+ = P_N U_m^+ = P_N \exp\left(-\mathrm{i} \sum_{k=2}^{N} (k-1)\tau L_{\mathcal{E}_k} \right)$$

Recall

$$P_{\mathcal{SR}} = \mathbb{1}_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{R}} \otimes |\Psi_{\mathcal{C}}\rangle\langle\Psi_{\mathcal{C}}|.$$

We have for $m > N + l$

$$\begin{aligned} P_N \mathrm{e}^{\mathrm{i}\tau K_{N+1}} \cdots \mathrm{e}^{\mathrm{i}\tau K_m} \mathcal{N}(O_m)\Psi_0 &= P_{\mathcal{SR}} \mathrm{e}^{\mathrm{i}\tau K_{N+1}} \cdots \mathrm{e}^{\mathrm{i}\tau K_m} \mathcal{N}(O_m)\Psi_0 \\ &= P_{\mathcal{SR}} M^{m-l-N-1} \mathrm{e}^{\mathrm{i}\tau K_{m-l}} \cdots \mathrm{e}^{\mathrm{i}\tau K_m} \mathcal{N}(O_m)\Psi_0, \quad (2.9) \end{aligned}$$

where we have introduced the following *reduced dynamics operator* (RDO), see (2.21) in [13]

$$P_{\mathcal{SR}} \mathrm{e}^{\mathrm{i}\tau K} P_{\mathcal{SR}} = M \otimes |\Psi_{\mathcal{C}}\rangle\langle\Psi_{\mathcal{C}}| \simeq M \quad \text{acting on} \quad \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}}. \tag{2.10}$$

In the last step of (2.9), we use the property

$$P_{\mathcal{SR}} \mathrm{e}^{\mathrm{i}\tau K_s} \mathrm{e}^{\mathrm{i}\tau K_{s+1}} \cdots \mathrm{e}^{\mathrm{i}\tau K_t} P_{\mathcal{SR}} = P_{\mathcal{SR}} \mathrm{e}^{\mathrm{i}\tau K_s} P_{\mathcal{SR}} \mathrm{e}^{\mathrm{i}\tau K_{s+1}} P_{\mathcal{SR}} \cdots P_{\mathcal{SR}} \mathrm{e}^{\mathrm{i}\tau K_t} P_{\mathcal{SR}},$$

which holds for any $1 \leq s < t$. This property follows from the independence of the $\mathcal{E}_j$ for different j, see [11], Proposition 4.1.

Combining (2.8) with (2.9) we obtain

$$\begin{aligned} &\langle\Psi_0|(B')^* B' \alpha^m(O_m)\Psi_0\rangle = \quad (2.11) \\ &\quad \langle\Psi_0|(B')^* B' (\widetilde{U}_N^+)^* \mathrm{e}^{\mathrm{i}\tau K_1} \cdots \mathrm{e}^{\mathrm{i}\tau K_N} P_{\mathcal{SR}} M^{m-l-N-1} \mathrm{e}^{\mathrm{i}\tau K_{m-l}} \cdots \mathrm{e}^{\mathrm{i}\tau K_m} \mathcal{N}(O_m)\Psi_0\rangle. \end{aligned}$$

In order to emphasize the dependence on the coupling constants $\lambda = (\lambda_1, \lambda_2)$, we write $K(\lambda)$ and $M(\lambda)$. The following are general properties of the RDO.

Proposition 2.1 *Let $\lambda \in \mathbb{R}^2$ be arbitrary. We have*

i) $M(\lambda) \in \mathcal{B}(\mathcal{H}_{\mathcal{SR}})$

ii) $M(\lambda)\Psi_{\mathcal{SR}} = \Psi_{\mathcal{SR}}$, *where* $\Psi_{\mathcal{SR}} := \Psi_{\mathcal{S}} \otimes \Psi_{\mathcal{R}}$

iii) For any φ in the dense set $\mathcal{D} = \{A_{\mathcal{SR}}\Psi_{\mathcal{SR}},\ A_{\mathcal{SR}} \in \mathfrak{M}_{\mathcal{SR}}\}$, there exists a constant $C(\varphi) < \infty$ s.t.

$$\sup_{n\in\mathbb{N}} \|M(\lambda)^n \varphi\| \leq C(\varphi). \tag{2.12}$$

Proof: i) follows from the fact that K is a bounded perturbation of a self-adjoint operator and ii) is a consequence of (2.2). To prove iii), first note that $\mathcal{D}$ is dense since $\Psi_{\mathcal{SR}}$ is cyclic for $\mathfrak{M}_{\mathcal{SR}}$. Then note that the following identity holds for all $B_{\mathcal{SR}} \in \mathfrak{M}_{\mathcal{SR}}$

$$\langle B_{\mathcal{SR}}\Psi_0 | \alpha^n(A_{\mathcal{SR}})\Psi_0\rangle = \langle B_{\mathcal{SR}}\Psi_{\mathcal{SR}} | M(\lambda)^n A_{\mathcal{SR}}\Psi_{\mathcal{SR}}\rangle.$$

Statement iii) of the proposition follows from the density of $\mathcal{D}$ and unitarity of the Heisenberg evolution, with $C(A_{\mathcal{SR}}\Psi_{\mathcal{SR}}) = \|A_{\mathcal{SR}}\|$. ∎

Remark. Contrarily to the cases dealt with in [11], [12] and [13], where the underlying Hilbert space is finite dimensional, we cannot conclude from (2.12) that $M(\lambda)$ is power bounded. Hence we do not know *a priori* that $\sigma(M(\lambda)) \subset \{z : |z| \leq 1\}$.

2.3 Translation analyticity

To separate the eigenvalues from the continuous spectrum, we use analytic spectral deformation theory acting on the (radial) variable s of the reservoir $\mathcal{R}$.

Recall the definition (1.10) of the translation. It is not difficult to see that

$$K_\theta := T(\theta)^{-1} K T(\theta) = L_0 + \theta N + \lambda_1 W_{\mathcal{SR}}(\theta) + \lambda_2 W_{\mathcal{SE}}, \tag{2.13}$$

where N is the number operator, and that the right side of (2.13) admits an analytic continuation into $\theta \in \kappa_{\theta_0}$, strongly on the dense domain $D(L_0) \cap D(N)$, defining a family of closed operators (see [18]).

Theorem 2.2 (Analyticity of propagator) *Assume that H1 and H2 hold. Then*

1. $T(\theta)^{-1}\mathrm{e}^{\mathrm{i}\tau K}T(\theta)$ has an analytic continuation from $\theta \in \mathbb{R}$ into the upper strip κ_{θ_0}, and this continuation is strongly continuous as $\Im\theta \downarrow 0$.

2. For each $\theta \in \kappa_{\theta_0} \cup \mathbb{R}$, the analytic continuation of $T(\theta)^{-1}\mathrm{e}^{\mathrm{i}\tau K}T(\theta)$ is given by $\mathrm{e}^{\mathrm{i}\tau K_\theta}$, which is understood as an operator-norm convergent Dyson series (with 'free part' $\mathrm{e}^{\mathrm{i}\tau(L_0+\theta N)}$).

3. For each $\theta \in \kappa_{\theta_0}$, $\lambda_j \mapsto T(\theta)^{-1}\mathrm{e}^{\mathrm{i}\tau K}T(\theta)$, $j = 1, 2$, are analytic entire functions.

Remarks. 1. The proof of this result yields the following bound for all $\theta \in \kappa_{\theta_0} \cup \mathbb{R}$,

$$\|T(\theta)^{-1}\mathrm{e}^{\mathrm{i}\tau K}T(\theta)\| \leq \mathrm{e}^{\tau \sup_{0\leq\Im\theta<\theta_0}\|W_\theta\|}.$$

2. If $\theta_1, \theta_2 \in \kappa_{\theta_0}$ with $\theta_1 + \theta_2 \in \kappa_{\theta_0}$, then

$$T(\theta_1+\theta_2)^{-1}\mathrm{e}^{\mathrm{i}\tau K}T(\theta_1+\theta_2) = T(\theta_2)^{-1}T(\theta_1)^{-1}\mathrm{e}^{\mathrm{i}\tau K}T(\theta_1)T(\theta_2).$$

In particular, $T(\theta)^{-1}\mathrm{e}^{\mathrm{i}\tau K}T(\theta)$ is unitarily equivalent to $T(\mathrm{i}\Im\theta)^{-1}\mathrm{e}^{\mathrm{i}\tau K}T(\mathrm{i}\Im\theta)$, via the unitary $T(\Re\theta)^{-1}$.

Proof of Theorem 2.2. For $\theta \in \mathbb{R}$, the Dyson series expansion of $T(\theta)^{-1}\mathrm{e}^{\mathrm{i}\tau K}T(\theta)$ is given by

$$\begin{aligned} &T(\theta)^{-1}\mathrm{e}^{\mathrm{i}\tau K}T(\theta) \\ &\quad= \sum_{n=0}^{\infty} \mathrm{i}^n \int_0^\tau \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \; \mathrm{e}^{\mathrm{i}t_n\theta N} W_\theta(t_n) \mathrm{e}^{\mathrm{i}(t_{n-1}-t_n)\theta N} W_\theta(t_{n-1}) \mathrm{e}^{\mathrm{i}(t_{n-2}-t_{n-1})\theta N} \cdots \\ &\qquad \cdots \mathrm{e}^{\mathrm{i}(t_1-t_2)\theta N} W_\theta(t_1) \mathrm{e}^{\mathrm{i}(\tau-t_1)\theta N} \mathrm{e}^{\mathrm{i}\tau L_0}, \end{aligned} \tag{2.14}$$

where we define

$$W_\theta(t) = \mathrm{e}^{\mathrm{i}tL_0} W_\theta \mathrm{e}^{-\mathrm{i}tL_0}$$

In the derivation of (2.14), we use that for all $t \in \mathbb{R}$,

$$T(\theta)^{-1}\mathrm{e}^{\mathrm{i}tL_0}T(\theta) = \mathrm{e}^{\mathrm{i}t(L_0+\theta N)} = \mathrm{e}^{\mathrm{i}tL_0}\mathrm{e}^{\mathrm{i}t\theta N} = \mathrm{e}^{\mathrm{i}t\theta N}\mathrm{e}^{\mathrm{i}tL_0}.$$

All the operators $\mathrm{e}^{\mathrm{i}(t_{k-1}-t_k)\theta N}$, as well as $\mathrm{e}^{\mathrm{i}t_n\theta N}$ and $\mathrm{e}^{\mathrm{i}(\tau-t_1)\theta N}$ appearing in the integrand of (2.14) have analytic extensions from real θ to $\theta \in \kappa_{\theta_0}$ which are continuous at $\mathbb{R}$, and

each of those extensions is bounded, having, in fact, norm one (uniformly in $\theta \in \kappa_{\theta_0} \cup \mathbb{R}$ and in the t_j). Consequently, due to assumption H2, for fixed values of $t_1, \ldots, t_n$, the integrand in (2.14) has an analytic extension into κ_{θ_0} which is again continuous on $\mathbb{R}$. Let's call this extension $h_{t_1,\ldots,t_n}(\theta)$. For $\theta \in \kappa_{\theta_0}$, we have

$$\|\partial_\theta h_{t_1,\ldots,t_n}(\theta)\| \leq C(\|\partial_\theta W_\theta\| + [\Im\theta]^{-1}), \tag{2.15}$$

uniformly in $t_1, \ldots, t_n$, for some constant C (which depends on n). It thus follows, using the Lebesgue Dominated Convergence Theorem, that the integral on the r.h.s. of (2.14) is analytic in $\theta \in \kappa_{\theta_0}$. To show (2.15), we note that the derivative $\partial_\theta h$ is a sum of terms where ∂_θ is either applied to one of the W_θ or to an exponential. In the latter case, we have

$$\|\partial_\theta \mathrm{e}^{\mathrm{i}t\theta N}\| = \|tN\mathrm{e}^{-t\Im\theta N}\| = [\Im\theta]^{-1}\|t\Im\theta N\mathrm{e}^{-t\Im\theta N}\| \leq [\Im\theta]^{-1}\sup_{x\geq 0} x\mathrm{e}^{-x},$$

where $\partial_\theta \mathrm{e}^{\mathrm{i}t\theta N}$ is understood in the strong sense.

The norm of the integral on the r.h.s. of (2.14) is bounded above by

$$\frac{[\tau \sup_{0\leq\Im\theta<\theta_0} \|W_\theta\|]^n}{n!},$$

uniformly in $\theta \in \kappa_\theta$. It follows that the series (2.14) converges uniformly (Weierstrass M-test) and therefore the r.h.s. of (2.14) is an analytic function in $\theta \in \kappa_{\theta_0}$.

Using the Lebesgue Dominated Convergence Theorem and the Weierstrass M-test as above, one readily shows that the series in (2.14) is strongly continuous as $\Im\theta \downarrow 0$. Since equality (2.14) holds for real θ, this means that indeed the Dyson series is an analytic extension of $T(\theta)^{-1}\mathrm{e}^{\mathrm{i}\tau K}T(\theta)$ into $\theta \in \kappa_{\theta_0}$. Note that the series is indeed the Dyson series of $\mathrm{e}^{\mathrm{i}\tau K_\theta}$.

Finally, analyticity in λ_1 and λ_2 is clear from (2.14) and (2.1). ■

The following result follows immediately from Theorem 2.2 and definition (2.10).

Corollary 2.3 *Recall the definition (2.10) of $M(\lambda)$. If assumption H2 is satisfied, then $T(\theta)^{-1}M(\lambda)T(\theta)$ has an analytic continuation into $\theta \in \kappa_{\theta_0}$, denoted $M_\theta(\lambda)$, and this continuation is continuous at $\mathbb{R}$. We have $M_\theta(\lambda)\Psi_{\mathcal{SR}} = \Psi_{\mathcal{SR}}$.*

2.4 Convergence to asymptotic state

In order to make a link with the dynamics of observables, we insert $\mathbb{1} = T(\theta)T(\theta)^{-1}$ (with $\theta \in \mathbb{R}$) into equation (2.11) to obtain

$$\begin{aligned}&\langle\Psi_0|(B')^*B'\alpha^m(O_m)\Psi_0\rangle = \\ &\quad\langle\Psi_0|(B')^*B'(\widetilde{U}_N^+)^*\mathrm{e}^{\mathrm{i}\tau K_1}\cdots\mathrm{e}^{\mathrm{i}\tau K_N}P_{\mathcal{SR}}T(\theta)M_\theta^{m-l-N-1}T(\theta)^{-1}\mathrm{e}^{\mathrm{i}\tau K_{m-l}}\cdots\mathrm{e}^{\mathrm{i}\tau K_m}\mathcal{N}(O_m)\Psi_0\rangle.\end{aligned} \tag{2.16}$$

If O_m is analytic (see Definition 1.2) then so is $\mathcal{N}(O_m)$ (see (2.4)), and therefore, by Theorem 2.2, $\mathrm{e}^{\mathrm{i}\tau K_{m-l}}\cdots\mathrm{e}^{\mathrm{i}\tau K_m}\mathcal{N}(O_m)$ is analytic as well (and continuous on the real axis). The r.h.s. of (2.16) is thus an analytic function in $\theta \in \kappa_{\theta_0}$ (c.f. Theorem 2.2 and

Corollary 2.3). Moreover, this function is continuous on $\mathbb{R}$, and, by unitarity of $T(\theta)$ for real θ, it is constant for real θ. Hence (2.16) is valid for all $\theta \in \kappa_{\theta_0} \cup \mathbb{R}$.

One expects that the operator $M_\theta(\lambda)^{m-l-N-1}$ converges to the projection onto the manifold of its fixed-points, as $m \to \infty$. Under certain (physically reasonable) conditions, this projection has rank one and is given by $|\Psi_{\mathcal{SR}}\rangle\langle\psi^*_\theta(\lambda)|$, where $\psi^*_\theta(\lambda)$ is the unique invariant vector of the adjoint operator $[M_\theta(\lambda)]^*$, normalized as $\langle\Psi_{\mathcal{SR}}|\psi^*_\theta(\lambda)\rangle = 1$.

Lemma 2.4 *Assume that Condition FGR holds (see before (1.12)). Then for* $0 < |\lambda| < \lambda_0$*,*

$$\lim_{n\to\infty} M_{\theta_1}(\lambda)^n = P_{1,M_{\theta_1}(\lambda)} = |\Psi_{\mathcal{SR}}\rangle\langle\psi^*_{\theta_1}(\lambda)|,$$

where $P_{1,M}$ *denotes the spectral projector of the operator* M *corresponding to the eigenvalue 1. The convergence is in operator norm, and occurs exponentially quickly.*

Proof. We write $M \equiv M_{\theta_1}(\lambda)$ and $\psi^* \equiv \psi^*_{\theta_1}(\lambda)$ in this proof. Using FGR, we have

$$M = P_{1,M} + M_Q, \quad \text{where} \quad M_Q = QMQ, \quad Q = \mathbb{1} - P_{1,M},$$

with

$$P_{1,M} = |\Psi_{\mathcal{SR}}\rangle\langle\psi^*|, \quad \langle\psi^*|\Psi_{\mathcal{SR}}\rangle = 1.$$

Moreover, there exists $\gamma = \gamma_{\theta_1}(\lambda) > 0$ s.t.

$$\sigma(M_Q) \subset \{z \in \mathbb{C} : |z| < e^{-\gamma}\}. \tag{2.17}$$

Therefore, the spectral radius of M_Q satisfies

$$\mathrm{spr}\,(M_Q) = \lim_{n\to\infty} \|M_Q^n\|^{1/n} < e^{-\gamma} < 1.$$

This, together with the identity

$$M^n = P_{1,M} + M_Q^n$$

yields for n large enough,

$$\|M^n - P_{1,M}\| \leq e^{-n\gamma} \to 0 \quad \text{as} \quad n \to \infty.$$

(For some slightly smaller γ than in (2.17) above.) ∎

It is now apparent from (2.16) and Lemma 2.4 how to complete the proof of Theorem 1.3: the increasing power of M_θ drives the system to an asymptotic state. Some care has to be exercised in the implementation of the complex deformation in the remaining part of the proof. Here are the details.

Recall that the initial state is a vector state with Ψ_ρ of the form (2.6)-(2.7). Let $\sigma \geq 0$ and define the spectral cutoff operator $\chi_\sigma := \chi(|\mathrm{d}\Gamma(\partial_s)| \leq \sigma)$, acting (non-trivially only) on $\mathcal{H}_{\mathcal{R}}$. (Here, $\chi(|x| \leq \sigma)$ equals one if $|x| \leq \sigma$ and zero otherwise.) The role of χ_σ is to smoothen the deformation operators: indeed, $\chi_\sigma T(\theta)^{-1}$ is analytic entire

in θ, see also (1.10). As $\sigma \to \infty$, χ_σ approaches the identity in the strong operator topology. Consequently, we have (see (2.11))

$$\begin{aligned}
&\rho(\alpha^m(O_m)) \\
&\quad = \langle\Psi_0|(B')^*B'(\tilde{U}_N^+)^*\chi_\sigma \mathrm{e}^{\mathrm{i}\tau L_1}\cdots\mathrm{e}^{\mathrm{i}\tau L_m}\mathcal{N}(O_m)\mathrm{e}^{-\mathrm{i}\tau L_m}\cdots\mathrm{e}^{-\mathrm{i}\tau L_1}\Psi_0\rangle + R_2(O_m) \\
&\quad = \langle\Psi_0|(B')^*B'(\tilde{U}_N^+)^*\chi_\sigma \mathrm{e}^{\mathrm{i}\tau K_1}\cdots\mathrm{e}^{\mathrm{i}\tau K_N}M^{m-l-N-1}\mathrm{e}^{\mathrm{i}\tau K_{m-l}}\cdots\mathrm{e}^{\mathrm{i}\tau K_m}\mathcal{N}(O_m)\Psi_0\rangle \\
&\qquad + R_1(O_m), \qquad (2.18)
\end{aligned}$$

where R_1 is a bounded linear functional on $\mathfrak{M}$ satisfying $\lim_{\sigma\to\infty} R_1 = 0$, uniformly in m.

We now introduce the spectral deformation in the main term on the r.h.s. of (2.18). We have for all $\theta \in \mathbb{R}$

$$\begin{aligned}
&\langle\Psi_0|(B')^*B'(\tilde{U}_N^+)^*\chi_\sigma \mathrm{e}^{\mathrm{i}\tau K_1}\cdots\mathrm{e}^{\mathrm{i}\tau K_N}M^{m-l-N-1}\mathrm{e}^{\mathrm{i}\tau K_{m-l}}\cdots\mathrm{e}^{\mathrm{i}\tau K_m}\mathcal{N}(O_m)\Psi_0\rangle \\
&\qquad = \langle\Psi_0|(B')^*B'(\tilde{U}_N^+)^*\chi_\sigma T(\theta)\mathrm{e}^{\mathrm{i}\tau K_{1,\theta}}\cdots\mathrm{e}^{\mathrm{i}\tau K_{N,\theta}}M_\theta^{m-l-N-1}\times \\
&\qquad\quad \times T(\theta)^{-1}\mathrm{e}^{\mathrm{i}\tau K_{m-l}}\cdots\mathrm{e}^{\mathrm{i}\tau K_m}\mathcal{N}(O_m)\Psi_0\rangle, \qquad (2.19)
\end{aligned}$$

where $K_{j,\theta} = T(\theta)^{-1}K_jT(\theta)$. Since O_m is an analytic observable, and according to Theorem 2.2, the right side of (2.19) has an analytic continuation into $\theta \in \kappa_{\theta_0}$ and this continuation is continuous at $\mathbb{R}$. Moreover, on $\mathbb{R}$, this continuation is a *constant* function (equal to the left side of (2.19)). It follows that the analytic continuation is constant on the whole region of analyticity plus the real axis, and (2.19) holds for all $\theta \in \kappa_{\theta_0} \cup \mathbb{R}$. Due to the Condition FGR and Lemma 2.4, we have

$$\begin{aligned}
&\langle\Psi_0|(B')^*B'(\tilde{U}_N^+)^*\chi_\sigma \mathrm{e}^{\mathrm{i}\tau K_1}\cdots\mathrm{e}^{\mathrm{i}\tau K_N}M^{m-l-N-1}\mathrm{e}^{\mathrm{i}\tau K_{m-l}}\cdots\mathrm{e}^{\mathrm{i}\tau K_m}\mathcal{N}(O_m)\Psi_0\rangle \\
&\quad = \langle\Psi_0|(B')^*B'(\tilde{U}_N^+)^*\chi_\sigma T(\theta_1)\mathrm{e}^{\mathrm{i}\tau K_{1,\theta_1}}\cdots\mathrm{e}^{\mathrm{i}\tau K_{N,\theta_1}}\Psi_0\rangle \times \\
&\qquad \times\langle\psi_{\theta_1}^*|T(\theta_1)^{-1}\mathrm{e}^{\mathrm{i}\tau K_{m-l}}\cdots\mathrm{e}^{\mathrm{i}\tau K_m}\mathcal{N}(O_m)\Psi_0\rangle + \mathrm{e}^{-m\gamma}R_2(O_m) \\
&\quad = \langle\psi_{\theta_1}^*|T(\theta_1)^{-1}\mathrm{e}^{\mathrm{i}\tau K_{m-l}}\cdots\mathrm{e}^{\mathrm{i}\tau K_m}\mathcal{N}(O_m)\Psi_0\rangle + \mathrm{e}^{-m\gamma}R_2(O_m), \qquad (2.20)
\end{aligned}$$

where $\gamma > 0$ (see proof of Lemma 2.4) and where

$$\begin{aligned}
\|R_2(O_m)\| &\leq \|\tilde{U}_N^+(B')^*B'\Psi_0\| \times \|\chi_\sigma T(\theta_1)\| \times \prod_{j=1}^N \|\mathrm{e}^{i\tau K_{j,\theta_1}}\| \\
&\qquad \times\|T(\theta_1)^{-1}\mathrm{e}^{\mathrm{i}\tau K_{m-l}}\cdots\mathrm{e}^{\mathrm{i}\tau K_m}\mathcal{N}(O_m)\Psi_0\|\ \mathrm{e}^{N\gamma} \\
&\leq C(\theta_0,N)\|T(\theta_1)^{-1}\mathrm{e}^{\mathrm{i}\tau K_{m-l}}\cdots\mathrm{e}^{\mathrm{i}\tau K_m}\mathcal{N}(O_m)\Psi_0\|\mathrm{e}^{\theta_1\sigma}.
\end{aligned}$$

The latter quantity is bounded uniformly in m. To arrive at the second line in (2.20) we made use of the fact that Ψ_0 is invariant under the action of all of χ_σ, $T(\theta)$ and $\mathrm{e}^{\mathrm{i}\tau K_{j,\theta}}$ and that $\|B'\Psi_0\|^2 = 1$ ($B'\Psi_0$ is the initial vector state). We combine estimates (2.18) and (2.20) to arrive at

$$\left|\rho(\alpha^m(O_m)) - \langle\psi_{\theta_1}^*|T(\theta_1)^{-1}\mathrm{e}^{\mathrm{i}\tau K_{m-l}}\cdots\mathrm{e}^{\mathrm{i}\tau K_m}\mathcal{N}(O_m)\Psi_0\rangle\right| \leq \|R_1(O_m)\| + \|R_2(O_m)\|\mathrm{e}^{-m\gamma}.$$

Hence,

$$\limsup_{m\to\infty}\left|\rho(\alpha^m(O_m)) - \langle\psi_{\theta_1}^*|T(\theta_1)^{-1}\mathrm{e}^{\mathrm{i}\tau K_{m-l}}\cdots\mathrm{e}^{\mathrm{i}\tau K_m}\mathcal{N}(O_m)\Psi_0\rangle\right| \leq \limsup_{m\to\infty}|R_1(O_m)|. \qquad (2.21)$$

Finally, by taking $\sigma \to \infty$ and since $\lim_{\sigma\to\infty} R_1 = 0$ uniformly in m, this completes the proof of Theorem 1.3. ■

3 Analysis of $M(\lambda)$

An important issue in the analysis of concrete models is the verification of the *Fermi Golden Rule* assumption FGR (see before Theorem 1.3). We have introduced the description of the two types of systems, 'small' and 'reservoir' in Section 1.1. For a more detailed analysis, we need to complement that description.

We denote the eigenvalues and associated eigenvectors of $H_{\mathcal{S}}$ by

$$E_1, \cdots, E_d, \quad \text{and} \quad \varphi_1, \cdots, \varphi_d.$$

Before analyzing the spectrum of $M_\theta(\lambda)$ in general, we mention some easier special cases.

• In the unperturbed case ($\lambda = 0$), we have

$$\begin{aligned} M(0) &= \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{S}}} \otimes \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{R}}}, \text{ with} \\ \sigma(M(0)) &= \left\{\mathrm{e}^{\mathrm{i}\tau(E_j - E_k)}\right\}_{(j,k)\in\{1,2,\cdots,d\}^2} \cup \left\{\mathrm{e}^{\mathrm{i}l},\ l \in \mathbb{R}\right\}, \end{aligned}$$

where the eigenvalues $\mathrm{e}^{\mathrm{i}\tau(E_j-E_k)}$ are embedded and have corresponding eigenvectors $\varphi_j \otimes \varphi_k \otimes \Psi_{\mathcal{R}}$. The eigenvalue 1 is at least d-fold degenerate.

• In case the coupling λ_1 between the small system and the reservoir is zero, we have

$$M(0, \lambda_2) = \tilde{M}(\lambda_2) \otimes \mathrm{e}^{\mathrm{i}\tau L_R} \ \text{ on } \ \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}},$$

where

$$\tilde{M}(\lambda_2) \simeq P_{\mathcal{S}} \mathrm{e}^{\mathrm{i}\tau(L_{\mathcal{S}} + L_{\mathcal{E}} + \lambda_2 V_{\mathcal{S}\mathcal{E}})} P_{\mathcal{S}} \ \text{ and } \ P_{\mathcal{S}} = \mathbb{1}_{\mathcal{S}} \otimes |\Psi_{\mathcal{E}}\rangle\langle\Psi_{\mathcal{E}}|.$$

The results of [11] apply to $\tilde{M}(\lambda_2)$, which is nothing but the RDO corresponding to the repeated interaction quantum system formed by $\mathcal{S}$ and $\mathcal{C}$ only. In particular, we get that $M(0, \lambda_2)$ is power bounded, as $\tilde{M}(\lambda_2)$ is and $e^{i\tau L_R}$ is unitary. Moreover, assuming the interaction $V_{\mathcal{S}\mathcal{E}}$ "effectively" couples $\mathcal{S}$ and $\mathcal{C}$, hypothesis (E) in [11], we know that the spectrum of $\tilde{M}(\lambda_2)$ satisfies

$$\sigma(\tilde{M}(\lambda_2)) = \{\mu_j(\lambda_2)\}_{\ j=1,2,\cdots,d^2},$$

with $\mu_1(\lambda_2) = 1$ a simple eigenvalue with eigenvector $\Psi_{\mathcal{S}}$ and $\mu_j(\lambda_2) \in \{z \,|\, |z| < 1\}$. Hence,

$$\sigma(M(0, \lambda_2)) = \sigma(\tilde{M}(\lambda_2)) \cup \{|\mu_j(\lambda_2)|\mathrm{e}^{\mathrm{i}l},\ l \in \mathbb{R}\}_{j=1,\cdots,d^2},$$

where the eigenvalues are embedded in the absolutely continuous spectrum again.

• In case the chain is decoupled, i.e. if $\lambda_2 = 0$, we get

$$M(\lambda_1, 0) = \mathrm{e}^{\mathrm{i}\tau(L_{\mathcal{S}} + L_{\mathcal{R}} + \lambda_1 W_{\mathcal{S}\mathcal{R}})} \ \text{ on } \ \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{R}},$$

whose spectral analysis already requires the tools we will use for general λ.

We now turn to a perturbative analysis of $M_\theta(\lambda)$ (small λ). Take $\theta \in \kappa_{\theta_0}$ and let $\lambda_1 = \lambda_2 = 0$. Then

$$M_\theta(0) = \mathrm{e}^{\mathrm{i}\tau(L_{\mathcal{S}} + L_{\mathcal{R}} + \theta N)} = \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{S}}} \otimes \mathrm{e}^{\mathrm{i}\tau L_{\mathcal{R}}} \mathrm{e}^{\mathrm{i}\tau\theta N}$$

and

$$\sigma(M_\theta(0)) = \{\mathrm{e}^{\mathrm{i}\tau(E_j-E_k)}\}_{j,k\in\{1,\cdots,d\}} \cup \{\mathrm{e}^{\mathrm{i}l}\mathrm{e}^{-\tau j\Im\theta},\ l\in\mathbb{R}\}_{j\in\mathbb{N}^*}.$$

The effect of the analytic translation is to push the continuous spectrum of $M_\theta(0)$ onto circles with radii $\mathrm{e}^{-\tau j\Im\theta}$, $j = 1, 2, \ldots$, centered at the origin. Hence the discrete spectrum of $M_\theta(0)$, lying on the unit circle, is separated from the continuous spectrum by a distance $1 - e^{-\tau\Im\theta}$. Analytic perturbation theory in the parameters λ_1, λ_2 guarantees that the discrete and continuous spectra stay separated for small coupling. The following result quantifies this.

Proposition 3.1 *Let $C_0(\lambda) := \sup_{\theta\in\kappa_{\theta_0}\cup\mathbb{R}} \|W_\theta\|$. Take $\theta \in \kappa_{\theta_0}$ and suppose that*

$$\tau C_0(\lambda)\mathrm{e}^{\tau C_0(\lambda)} < \tfrac{1}{4}(1 - \mathrm{e}^{-\tau\Im\theta}). \tag{3.1}$$

Then the spectrum of $M_\theta(\lambda)$ splits into two disjoint parts,

$$\sigma(M_\theta(\lambda)) = \sigma_\theta^{(0)}(\lambda) \cup \sigma_\theta^{(1)}(\lambda) \quad \textit{with} \quad \sigma_\theta^{(0)}(\lambda) \cap \sigma_\theta^{(1)}(\lambda) = \emptyset.$$

These parts are localized as follows

$$\sigma_\theta^{(0)}(\lambda) \ \subset\ \big\{z\ :\ 1 - \tfrac{1}{4}(1 - \mathrm{e}^{-\tau\Im\theta}) < |z| \le 1\big\} \tag{3.2}$$

$$\sigma_\theta^{(1)}(\lambda) \ \subset\ \big\{z\ :\ 0 \le |z| < \mathrm{e}^{-\tau\Im\theta} + \frac{1}{4}(1 - \mathrm{e}^{-\tau\Im\theta})\big\}. \tag{3.3}$$

Moreover, the spectrum $\sigma_\theta^{(0)}(\lambda)$ is purely discrete, consisting of d^2 eigenvalues (counted including algebraic multiplicities).

Proof. According to the Dyson series expansion (2.14), we have

$$M_\theta(\lambda) = \mathrm{e}^{\mathrm{i}\tau(L_0+\theta N)} + S_2, \tag{3.4}$$

where

$$\|S_2\| \le \sum_{n=1}^{\infty} \frac{[\tau C_0(\lambda)]^n}{n!} \le \tau C_0(\lambda)\mathrm{e}^{\tau C_0(\lambda)}.$$

Since $\mathrm{e}^{\mathrm{i}\tau(L_0+\theta N)}$ is a normal operator, the spectrum of the perturbed operator $M_\theta(\lambda)$, (3.4), lies inside a set whose distance to $\sigma(\mathrm{e}^{\mathrm{i}\tau(L_0+\theta N)})$ does not exceed $\|S_2\|$. The spectrum of $\mathrm{e}^{\mathrm{i}\tau(L_0+\theta N)}$ consists of isolated eigenvalues lying on the unit circle and of continuous spectrum lying on concentric circles centered at the origin, with radii $\mathrm{e}^{-\tau n\Im\theta}$, $n = 1, 2, \ldots$

It follows that if (3.1) is satisfied, then the continuous spectrum of $M_\theta(\lambda)$ is located as in (3.3). Furthermore, the spectral radius of $M_\theta(\lambda)$ is determined by discrete eigenvalues only, and so by Lemma A.2 below, these eigenvalues cannot lie outside the unit circle, from which (3.2) follows. ■

Remark. We always have $1 \in \sigma_\theta^{(0)}(\lambda)$, with eigenvector $\Psi_{\mathcal{SR}}$.

As a consequence of Proposition 3.1, a verification of FGR for concrete models, like the one of Section 1.4, is done via (perturbative) analysis *only of the discrete eigenvalues* of $M_\theta(\lambda)$.

4 Energy fluxes, entropy production

We use the notation and definitions of Section 1.3 and assume throughout that Assumption H3 is satisfied. We have

$$\alpha^{m+1}(L_{\mathcal{R}}) - \alpha^m(L_{\mathcal{R}}) = \alpha^m(\alpha^\tau_{m+1}(L_{\mathcal{R}}) - L_{\mathcal{R}}),$$

with

$$\alpha^\tau_m(\cdot) = \mathrm{e}^{\mathrm{i}\tau\widetilde{L}_m} \cdot \mathrm{e}^{-\mathrm{i}\tau\widetilde{L}_m}, \qquad \widetilde{L}_m = L_0 + \vartheta_m(V).$$

Thus

$$\alpha^\tau_{m+1}(L_{\mathcal{R}}) - L_{\mathcal{R}} = \mathrm{i}\int_0^\tau \alpha^t_{m+1}([\widetilde{L}_{m+1}, L_{\mathcal{R}}])\,\mathrm{d}t = \mathrm{i}\int_0^\tau \alpha^t_{m+1}([\lambda_1 V_{\mathcal{SR}}, L_{\mathcal{R}}])\,\mathrm{d}t,$$

and we arrive at the expression for the variation of energy in the reservoir

$$\Delta E^{\mathcal{R}}(m) = \alpha^m\left(\mathrm{i}\int_0^\tau \alpha^t_{m+1}([\lambda_1 V_{\mathcal{SR}}, L_{\mathcal{R}}])\,\mathrm{d}t\right).$$

From definition (1.7) and assumption **H3** (see before (1.20)), together with the analyticity of the dynamics (Theorem 2.2), it is clear that $\alpha^t_{m+1}([\lambda_1 V_{\mathcal{SR}}, L_{\mathcal{R}}])$ is an analytic instantaneous observable. By approximating the integral $\int_0^\tau \mathrm{d}t$ by a Riemann sum (converging uniformly in m) we see that the integral in question is the limit of a sum of instantaneous observables all having uniformly bounded indices l, r (see (1.7)). Furthermore, α^m is bounded uniformly in m and therefore we can apply Theorem 1.3 to conclude that

$$\lim_{m\to\infty} \rho\left(\Delta E^{\mathcal{R}}(m)\right) = \rho_{+,\lambda}\left(\mathrm{i}\int_0^\tau \alpha^t(\lambda_1[V_{\mathcal{SR}}, L_{\mathcal{R}}])\mathrm{d}t\right). \tag{4.1}$$

Next, we examine the variation of energy in the chain $\mathcal{C}$. During the time interval $[m\tau, (m+1)\tau)$, the energy of the element $\mathcal{E}_k$, $k \neq m+1$, of the chain is invariant. Hence, the variation of energy in the whole chain between the times m and $m+1$ coincides with that of the element $\mathcal{E}_{m+1}$ only. Thus,

$$\Delta E^{\mathcal{C}}(m) = \alpha^{m+1}(L_{\mathcal{E}_{m+1}}) - L_{\mathcal{E}_{m+1}} \equiv \alpha^{m+1}(L_{\mathcal{E}_{m+1}}) - \alpha^m(L_{\mathcal{E}_{m+1}}).$$

Proceeding as above, we arrive at

$$\Delta E^{\mathcal{C}}(m) = \alpha^m\left(\mathrm{i}\int_0^\tau \alpha^t_{m+1}([\lambda_2\vartheta_{m+1}(V_{\mathcal{SE}}), L_{\mathcal{E}_{m+1}}])\,\mathrm{d}t\right),$$

where $\mathrm{i}\int_0^\tau \alpha^t_{m+1}([\vartheta_{m+1}(V_{\mathcal{SE}}), L_{\mathcal{E}_{m+1}}])\,\mathrm{d}t$ is an instantaneous observable (Assumption H3).

Finally, for the variation of energy in the small system $\mathcal{S}$ we get

$$\Delta E^{\mathcal{S}}(m) = \alpha^m\left(\mathrm{i}\int_0^\tau \alpha^t_{m+1}([\lambda_2\vartheta_{m+1}(V_{\mathcal{SE}}) + \lambda_1 V_{\mathcal{SR}}, L_{\mathcal{S}}])\,\mathrm{d}t\right).$$

It follows now from Theorem 1.3 (by the same reasoning leading to (4.1), see also [11]) that

$$\mathrm{d}E_+^{\mathrm{tot}} = \rho_{+,\lambda}(j^{\mathrm{tot}}) = \rho_{+,\lambda}(V - \alpha^\tau(V)),$$

where

$$j^{\mathrm{tot}} = V - \alpha^\tau(V) = -\mathrm{i}\int_0^\tau \alpha^t([L_{\mathcal{S}} + L_{\mathcal{R}} + L_{\mathcal{E}}, \lambda_2 V_{\mathcal{SE}} + \lambda_1 V_{\mathcal{SR}}])\,\mathrm{d}t. \tag{4.2}$$

On the other hand, by the same reasoning, $\mathrm{d}E_+^{\#}$, for $\# = \mathcal{S}, \mathcal{E}, \mathcal{R}$ (see (1.20)), is given by $\rho_{+,\lambda}(j^{\#})$, where

$$j^{\mathcal{S}} = \mathrm{i}\int_0^\tau \alpha^t([\lambda_2 V_{\mathcal{SE}} + \lambda_1 V_{\mathcal{SR}}, L_{\mathcal{S}}])\,\mathrm{d}t, \tag{4.3}$$

$$j^{\mathcal{E}} = \mathrm{i}\int_0^\tau \alpha^t([\lambda_2 V_{\mathcal{SE}}, L_{\mathcal{E}}])\,\mathrm{d}t, \tag{4.4}$$

$$j^{\mathcal{R}} = \mathrm{i}\int_0^\tau \alpha^t([\lambda_1 V_{\mathcal{SR}}, L_{\mathcal{R}}])\,\mathrm{d}t. \tag{4.5}$$

The following result relates the various flux observables $j^{\#}$.

Proposition 4.1 *We have $j^{\mathrm{tot}} = j^{\mathcal{S}} + j^{\mathcal{E}} + j^{\mathcal{R}}$. Furthermore, $\mathrm{d}E_+^{\mathcal{S}} := \rho_{+,\lambda}(j^{\mathcal{S}}) = 0$.*

Proof. The relation $j^{\mathrm{tot}} = j^{\mathcal{S}} + j^{\mathcal{E}} + j^{\mathcal{R}}$ follows directly from (4.2) and (4.3)-(4.5). To see that $\mathrm{d}E_+^{\mathcal{S}} = 0$ we note that since $L_{\mathcal{S}}$ is bounded we have

$$\frac{1}{N}(\alpha^N(L_{\mathcal{S}}) - L_{\mathcal{S}}) = \frac{1}{N}\sum_{m=1}^{N} \alpha^m(L_{\mathcal{S}}) - \alpha^{m-1}(L_{\mathcal{S}}) \to 0, \quad \text{as} \quad N \to \infty.$$

■

The main ingredient in the analysis of the entropy production is the following entropy production formula, established in [20],

$$\begin{aligned}&\Delta S(m)\\ &\quad = \rho\Big(U(m)^*\Big[\beta_{\mathcal{E}}\sum_k L_{\mathcal{E},k} + \beta_{\mathcal{S}} L_{\mathcal{S}} + \beta_{\mathcal{R}} L_{\mathcal{R}}\Big]U(m) - \beta_{\mathcal{E}}\sum_k L_{\mathcal{E},k} - \beta_{\mathcal{S}} L_{\mathcal{S}} - \beta_{\mathcal{R}} L_{\mathcal{R}}\Big).\end{aligned}$$

Following the proof of Proposition 2.6 of [11], we get

$$\begin{aligned}\Delta S(m) \;=\; &(\beta_{\mathcal{R}} - \beta_{\mathcal{E}})\sum_{k=1}^{m-1}\rho(\Delta E^{\mathcal{R}}(k)) + (\beta_{\mathcal{S}} - \beta_{\mathcal{E}})\sum_{k=1}^{m-1}\rho(\Delta E^{\mathcal{S}}(k)) + \beta_{\mathcal{E}}\sum_{k=1}^{m}\rho(\Delta E^{\mathrm{tot}}(k))\\ &+\beta_{\mathcal{E}}\left[\rho(\lambda_1 V_{\mathcal{SR}}) - \rho(\alpha^m(\lambda_1 V_{\mathcal{SR}})) + \rho(\lambda_2\vartheta_1(V_{\mathcal{SE}})) - \rho(\alpha^m(\lambda_2\vartheta_{m+1}(V_{\mathcal{SE}})))\right].\end{aligned}$$

Hence using Proposition 4.1, and since $\rho(\alpha^m(V_{\mathcal{SR}}))$ and $\rho(\alpha^m(\vartheta_{m+1}(V_{\mathcal{SE}})))$ are bounded in m, we get

$$\mathrm{d}S_+ := \lim_{m\to\infty}\frac{\Delta S(m)}{m} = (\beta_{\mathcal{R}} - \beta_{\mathcal{E}})\mathrm{d}E_+^{\mathcal{R}} + \beta_{\mathcal{E}}\mathrm{d}E_+^{\mathrm{tot}}. \tag{4.6}$$

5 More detail on the concrete example

We consider the model described in Section 1.4. In order to write down explicitly all the interaction operators appearing in the C-Liouvillean, we need the explicit form of modular data of $\mathcal{S}$, $\mathcal{R}$ and $\mathcal{E}$.

The modular data of $\mathcal{S}$ and the elements $\mathcal{E}$ of the chain associated to the reference states $\rho_{\mathcal{S}}$ and $\rho_{\beta_{\mathcal{E}},\mathcal{E}}$ are given by

$$J_{\mathcal{S}}(\phi\otimes\psi)=\bar{\psi}\otimes\bar{\phi},\quad \Delta_{\mathcal{S}}=\mathbb{1}_{\mathbb{C}^2}\otimes\mathbb{1}_{\mathbb{C}^2}.$$

$$J_{\mathcal{E}}(\phi\otimes\psi)=\bar{\psi}\otimes\bar{\phi},\quad \Delta_{\mathcal{E}}=\mathrm{e}^{-\beta_{\mathcal{E}}L_{\mathcal{E}}}.$$

The one of $\mathcal{R}$ is given as follows (see also Theorem 3.3 of [19])

$$\begin{aligned} J_{\mathcal{R}}\varphi(f_{\beta_{\mathcal{R}}})J_{\mathcal{R}} &= \mathrm{i}\Gamma(-\mathbb{1})\varphi(f^{\#}_{\beta_{\mathcal{R}}}) \\ \Delta_{\mathcal{R}} &= e^{-\beta_{\mathcal{R}}L_{\mathcal{R}}} \\ J_R &= (-1)^{N(N+1)/2}\mathcal{C}\circ\mathcal{F}. \end{aligned}$$

Here, we have introduced the notation $f^{\#}_{\beta}(s)=\mathrm{i}\mathrm{e}^{-\beta s/2}f_{\beta}(s)=\bar{f}_{\beta}(-s)$, where the bar indicates the complex conjugate. Furthermore, $N=\mathrm{d}\Gamma(\mathbb{1})$ is the number operator, $\mathcal{C}$ is the complex conjugation operator and $\mathcal{F}$ is the sign flip operator acting on $f\in\otimes_{j=1}^{n}L^2(\mathbb{R},\mathfrak{G})$ as

$$(\mathcal{F}f)(s_1,s_2,\ldots,s_n)=f(-s_1,-s_2,\ldots,-s_n).$$

An easy computation leads to the following expression for the "interaction" part W of the C-Liouvillean,

$$\begin{aligned} W(\lambda) &= \lambda_1\Big(\sigma_x\otimes\mathbb{1}_{\mathbb{C}^2}\otimes\varphi(f_{\beta_{\mathcal{R}}}(s)) \\ &\qquad -\mathbb{1}_{\mathbb{C}^2}\otimes\sigma_x\otimes\Gamma(-\mathbb{1})\left(a^*(f_{\beta_{\mathcal{R}}}(s))-a(\mathrm{e}^{-\beta_{\mathcal{R}}s}f_{\beta_{\mathcal{R}}}(s))\right)\Big) \\ &\quad +\lambda_2\Big(a_{\mathcal{S}}\otimes\mathbb{1}_{\mathbb{C}^2}\otimes a^*_{\mathcal{E}}\otimes\mathbb{1}_{\mathbb{C}^2}+a^*_{\mathcal{S}}\otimes\mathbb{1}_{\mathbb{C}^2}\otimes a_{\mathcal{E}}\otimes\mathbb{1}_{\mathbb{C}^2} \\ &\qquad -\mathrm{e}^{\beta_{\mathcal{E}}E_{\mathcal{E}}/2}\mathbb{1}_{\mathbb{C}^2}\otimes a^*_{\mathcal{S}}\otimes\mathbb{1}_{\mathbb{C}^2}\otimes a_{\mathcal{E}}-\mathrm{e}^{-\beta_{\mathcal{E}}E_{\mathcal{E}}/2}\mathbb{1}_{\mathbb{C}^2}\otimes a_{\mathcal{S}}\otimes\mathbb{1}_{\mathbb{C}^2}\otimes a^*_{\mathcal{E}}\Big). \end{aligned}$$

Assumption (H4) on the form factor f ensures that assumption (H3) of Section 2.3 is satisfied with $\theta_0=\delta$. We can thus apply the general results of Section 3. In particular, the map $\theta\mapsto T(\theta)^{-1}M(\lambda)T(\theta)$ has an analytic continuation in the strip κ_{θ_0} (see Corollary 2.3). We then fix some $\theta_1\in\kappa_{\theta_0}$ such that $1-\mathrm{e}^{-\tau\Im(\theta_1)}>0$. For λ small enough eq. (3.1) is therefore satisfied, so that we can verify the (FGR) hypotheses using perturbation theory for a finite set of eigenvalues, those four eigenvalues which are located in $\sigma^{(0)}_{\theta}(\lambda)$ (see (3.2)). When the coupling constants are turned off, we have

$$\sigma^{(0)}_{\theta}(0)=\sigma(\mathrm{e}^{\mathrm{i}\tau L_{\mathcal{S}}})=\{1,\mathrm{e}^{\mathrm{i}\tau E_{\mathcal{S}}},\mathrm{e}^{-\mathrm{i}\tau E_{\mathcal{S}}}\}$$

where the eigenvalue 1 has multiplicity 2. In order to make the computation in perturbation theory as simple as possible, we will assume that these eigenvalue do not coincide, i.e. $\tau E_{\mathcal{S}}\notin\pi\mathbb{N}$. However, this assumption is certainly not necessary.

Using a Dyson expansion for $M_\theta(\lambda)$ as in the proof of Theorem 2.2 and regular perturbation theory (see e.g. [17, 22]) we compute the four elements of $\sigma_\theta^{(0)}(\lambda)$. We know that 1 always belongs to $\sigma_\theta^{(0)}(\lambda)$. The other ones respectively write

$$\begin{aligned}
e_0(\lambda) &= 1-\lambda_1^2\tau\frac{\pi}{2}\sqrt{E_{\mathcal{S}}}\|f(\sqrt{E_{\mathcal{S}}})\|_{\mathfrak{G}}^2-\lambda_2^2\tau^2\mathrm{sinc}^2\left(\frac{\tau(E_{\mathcal{E}}-E_{\mathcal{S}})}{2}\right)+O(\lambda^3),\\
e_+(\lambda) &= \mathrm{e}^{i\tau E_{\mathcal{S}}}\left[1-\lambda_1^2\tau\frac{\pi}{4}\sqrt{E_{\mathcal{S}}}\|f(\sqrt{E_{\mathcal{S}}})\|_{\mathfrak{G}}^2-\lambda_2^2\frac{\tau^2}{2}\mathrm{sinc}^2\left(\frac{\tau(E_{\mathcal{E}}-E_{\mathcal{S}})}{2}\right)\right.\\
&\quad\left.-i\left(\lambda_1^2\frac{\tau}{4}\,\mathrm{PV}\int_{\mathbb{R}}\frac{\sqrt{|s|}\|f(\sqrt{|s|})\|_{\mathfrak{G}}^2}{s-E_{\mathcal{S}}}\mathrm{d}s+\lambda_2^2\tau^2\frac{1-\mathrm{sinc}(\tau(E_{\mathcal{E}}-E_{\mathcal{S}}))}{\tau(E_{\mathcal{E}}-E_{\mathcal{S}})}\right)\right]+O(\lambda^3),\\
e_-(\lambda) &= \mathrm{e}^{-i\tau E_{\mathcal{S}}}\left[1-\lambda_1^2\tau\frac{\pi}{4}\sqrt{E_{\mathcal{S}}}\|f(\sqrt{E_{\mathcal{S}}})\|_{\mathfrak{G}}^2-\lambda_2^2\frac{\tau^2}{2}\mathrm{sinc}^2\left(\frac{\tau(E_{\mathcal{E}}-E_{\mathcal{S}})}{2}\right)\right.\\
&\quad\left.+i\left(\lambda_1^2\frac{\tau}{4}\,\mathrm{PV}\int_{\mathbb{R}}\frac{\sqrt{|s|}\|f(\sqrt{|s|})\|_{\mathfrak{G}}^2}{s-E_{\mathcal{S}}}\mathrm{d}s+\lambda_2^2\tau^2\frac{1-\mathrm{sinc}(\tau(E_{\mathcal{E}}-E_{\mathcal{S}}))}{\tau(E_{\mathcal{E}}-E_{\mathcal{S}})}\right)\right]+O(\lambda^3),
\end{aligned}$$

where $\mathrm{sinc}(x)=\frac{\sin(x)}{x}$ and PV stands for Cauchy's principal value. We thus get the following

Lemma 5.1 *Assume that $\|f(\sqrt{E_{\mathcal{S}}})\|_{\mathfrak{G}}\neq 0$ and $\tau(E_{\mathcal{E}}-E_{\mathcal{S}})\notin 2\pi\mathbb{Z}^*$, then (FGR) is satisfied.*

In order to compute the asymptotic state $\rho_{+,\lambda}$, we compute the (unique) invariant vector $\psi_\theta^*(\lambda)$ of $M_\theta(\lambda)^*$ (see (1.12)). Once again, standard perturbation theory shows that $\psi_\theta^*(\lambda)=\psi_{\mathcal{S}}^*(\lambda)\otimes\Psi_{\mathcal{R}}+O_\theta(\lambda)$ with

$$\begin{aligned}
\psi_{\mathcal{S}}^*(\lambda) &= \sqrt{2}\frac{\lambda_1^2\gamma_{\mathrm{th}}^{(2)}Z_{\beta_{\mathcal{R}},\mathcal{S}}^{-1}+\lambda_2^2\gamma_{\mathrm{ri}}^{(2)}Z_{\beta'_{\mathcal{E}},\mathcal{S}}^{-1}}{\lambda_1^2\gamma_{\mathrm{th}}^{(2)}+\lambda_2^2\gamma_{\mathrm{ri}}^{(2)}}\,|00\rangle\\
&\quad+\sqrt{2}\frac{\lambda_1^2\gamma_{\mathrm{th}}^{(2)}\mathrm{e}^{-\beta_{\mathcal{R}}E_{\mathcal{S}}}Z_{\beta_{\mathcal{R}},\mathcal{S}}^{-1}+\lambda_2^2\gamma_{\mathrm{ri}}^{(2)}\mathrm{e}^{-\beta'_{\mathcal{E}}E_{\mathcal{S}}}Z_{\beta'_{\mathcal{E}},\mathcal{S}}^{-1}}{\lambda_1^2\gamma_{\mathrm{th}}^{(2)}+\lambda_2^2\gamma_{\mathrm{ri}}^{(2)}}\,|11\rangle,
\end{aligned}$$

where $\gamma_{\mathrm{th}}^{(2)}$ and $\gamma_{\mathrm{ri}}^{(2)}$ are defined in (1.25)-(1.26). Inserting the above expression in (1.13), this proves Proposition 1.6. ∎

A Some operator theory

Our analysis of the spectrum of $M_\theta(\lambda)$ makes use of a translated version of (2.12), which replaces the powerboundedness of $M_\theta(\lambda)$ in our setup.

Lemma A.1 *Assume $A_{\mathcal{SR}}$ and $B_{\mathcal{SR}}$ are translation analytic in κ_{θ_0}. Then*

$$\sup_{m\in\mathbb{N}}|\langle B_{\mathcal{SR}}\Psi_{\mathcal{SR}}|M_\theta^m(\lambda)A_{\mathcal{SR}}\Psi_0\rangle|\leq\|A_{\mathcal{SR}}(\theta)\|\|B_{\mathcal{SR}}(\overline{\theta})\|.$$

Proof: Consider

$$\begin{aligned}\langle B_{\mathcal{SR}}(\overline{\theta})\Psi_{\mathcal{SR}}|\alpha^m(A_{\mathcal{SR}}(\theta))\Psi_{\mathcal{SR}}\rangle &= \langle B_{\mathcal{SR}}\Psi_{\mathcal{SR}}|T(\theta)M(\lambda)^m T^{-1}(\theta)A_{\mathcal{SR}}\psi_{\mathcal{SR}}\rangle \\ &= \langle B_{\mathcal{SR}}\Psi_{\mathcal{SR}}|M_\theta(\lambda)^m A_{\mathcal{SR}}\psi_{\mathcal{SR}}\rangle\end{aligned}$$

■

We can use this property to bound the spectral radius of $M_\theta(\lambda)$ when it is determined by discrete eigenvalues only. This means that there are finitely many eigenvalues α_j, $j = 1, \ldots, N$, all of equal modulus α, such that $\sup\{|z| : z \in \sigma(M_\theta(\lambda))\} = \alpha$ and $\sigma_{\rm ess}(M_\theta(\lambda)) \cap \{|z| = \alpha\} = \emptyset$.

Lemma A.2 *Assume that for some $\theta \in \kappa_{\theta_0}$, $\mathrm{spr}\,(M_\theta(\lambda))$ is determined by discrete eigenvalues only. Then $\mathrm{spr}\,(M_\theta(\lambda)) = 1$.*

This is an application of the following result stated in a more abstract setting.

Proposition A.3 *Let M be a bounded operator on a Hilbert space $\mathcal{H}$ such that:*
i) there exists a dense set of vectors $\mathcal{C} \subset \mathcal{H}$ satisfying

$$\sup_{n\in\mathbb{N}} |\langle\varphi|M^n\psi\rangle| \leq C(\varphi,\psi), \quad \forall\varphi,\psi \in \mathcal{C},$$

ii) $\mathrm{spr}\,(M)$ is determined by discrete eigenvalues only, i.e.

$$\sigma(M) \cap \{z \in \mathbb{C} \mid |z| = \mathrm{spr}\,(M)\} \subset \sigma_{\rm d}(M).$$

Then,

$$\mathrm{spr}\,(M) \leq 1$$

and the eigenvalues of modulus one, if any, are semisimple.

Proof: Let $\{\alpha_j\}_{j=1,\cdots,N}$, be the discrete eigenvalues such that $|\alpha_j| = \alpha = \mathrm{spr}\,(M)$ and let P_j and D_j be the corresponding eigenprojectors and eigennilpotents. Recall that $[D_j, P_j] = 0$ and $P_jP_k = \delta_{jk}P_j$, for all $j,k \in \{1,\cdots,N\}$. Setting $Q = \mathbb{1} - \sum_{j=1}^N P_j$, we can write, by assumption ii)

$$M = \sum_{j=1}^N \alpha_j P_j + D_j + QMQ, \tag{A.1}$$

where

$$\|(QMQ)^n\| \leq e^{\beta n} \quad \text{with} \quad \beta < \ln\alpha.$$

Let $K \in \mathbb{N}^*$ be such that $D_j^{K+1} = 0$ for all $j \in \{1,\cdots,N\}$ and $D_{j_0}^K \neq 0$, for some $j_0 \in \{1,\cdots,N\}$. If all eigennilpotents are zero, we set $K = 0$. Using the properties of the spectral decomposition (A.1), we get for any $n \in \mathbb{N}$ large enough

$$M^n = \sum_{j=1}^N \left(\alpha_j^n P_j + \sum_{k=1}^K \binom{n}{k} \alpha_j^{n-k} D_j^k\right) + (QMQ)^n.$$

Consider first the case $K = 0$, where all $D_j = 0$. Assume that $\alpha > 1$ and consider $\varphi \in \mathcal{H}$ such that $P_{j_0}\varphi \neq 0$, for some $j_0 \in \{1, \cdots, N\}$. We define $\varphi_0 = P_{j_0}\varphi / \|P_{j_0}\varphi\|$ such that $M^n\varphi_0 = \alpha_{j_0}^n \varphi_0$. Now, $\mathcal{C}$ being dense, for any $\epsilon > 0$, there exists $\tilde{\varphi}_0 \in \mathcal{C}$ with $\|\tilde{\varphi}_0 - \varphi_0\| \leq \epsilon$ so that

$$M^n\tilde{\varphi}_0 = M^n\varphi_0 + \sum_{j=1}^{N} \alpha_j^n P_j(\tilde{\varphi}_0 - \varphi_0) + (QMQ)^n(\tilde{\varphi}_0 - \varphi_0),$$

where the norm of the last two terms is bounded by $\alpha^n\epsilon\left(\sum_{j=1}^N \|P_j\| + e^{(\beta - \ln\alpha)n}\right)$. Hence,

$$\langle\tilde{\varphi}_0|M^n\tilde{\varphi}_0\rangle = \alpha_{j_0}^n(\langle\tilde{\varphi}_0|\varphi_0\rangle + O(\epsilon)), \quad \text{with} \quad O(\epsilon) \ \text{ uniform in } n,$$

and $\langle\tilde{\varphi}_0|\varphi_0\rangle = 1 + O(\epsilon)$. Thus the modulus of the RHS goes to infinity exponentially fast with n (since $|\alpha_{j_0}| = \alpha > 1$), whereas the LHS should be uniformly bounded in n by assumption i).

Consider now $K > 0$ and let $\varphi \in \mathcal{H}$ be such that $D_{j_0}^K\varphi \neq 0$. Assume $\alpha \geq 1$ and set, as above, $\varphi_0 = P_{j_0}\varphi / \|P_{j_0}\varphi\|$. We have for n large enough

$$M^n\varphi_0 = \alpha_{j_0}^n\left(\varphi_0 + \sum_{k=1}^{K}\binom{n}{k}\alpha_{j_0}^{-k}D_j^k\varphi_0\right),$$

where, for $1 \leq k \leq K$ and n large,

$$\binom{n}{k} < \binom{n}{K} \simeq n^K/K!\ .$$

Let $\psi_0 = D_{j_0}^K\varphi_0 / \|D_{j_0}^K\varphi_0\|^2$, and, for any $\epsilon > 0$, $\tilde{\varphi}_0$, $\tilde{\psi}_0$ in $\mathcal{C}$ such that $\|\tilde{\varphi}_0 - \varphi_0\| < \epsilon$ and $\|\tilde{\psi}_0 - \psi_0\| < \epsilon$. Then, as $n \to \infty$,

$$\begin{aligned}\langle\psi_0|M^n\varphi_0\rangle &= \alpha_{j_0}^n\left(\binom{n}{K}\alpha_{j_0}^{-K} + \sum_{k=1}^{K-1}\binom{n}{k}\alpha_{j_0}^{-k}\langle\psi_0|D_j^k\varphi_0\rangle + \langle\psi_0|\varphi_0\rangle\right)\\ &= \alpha_{j_0}^n\alpha_{j_0}^{-K}\binom{n}{K}(1 + O(1/n)).\end{aligned}$$

Thus

$$\langle\tilde{\psi}_0|M^n\tilde{\varphi}_0\rangle = \langle\psi_0|M^n\varphi_0\rangle + \langle\tilde{\psi}_0 - \psi_0|M^n\tilde{\varphi}_0\rangle + \langle\psi_0|M^n(\tilde{\varphi}_0 - \varphi_0)\rangle,$$

where the vector

$$M^n\tilde{\varphi}_0 = \sum_{j=1}^{N}\left(\alpha_j^n P_j + \sum_{k=1}^{K}\binom{n}{k}\alpha_j^{n-k}D_j^k\right)\tilde{\varphi}_0 + (QMQ)^n\tilde{\varphi}_0$$

satisfies for n large enough and some constant C uniform in n,

$$\|M^n\tilde{\varphi}_0\| \leq C\alpha^n\binom{n}{K}\|\tilde{\varphi}_0\| \leq C\alpha^n\binom{n}{K}(1 + \epsilon),$$

and a similar estimate holds for $\|(M^n)^*\psi_0\|$. We finally get, for some constant $\tilde{C}$, uniform in n and ϵ,

$$\begin{aligned} |\langle\tilde{\psi}_0|M^n\tilde{\varphi}_0\rangle| &\geq |\langle\psi_0|M^n\varphi_0\rangle| - C\epsilon\alpha^n \binom{n}{K} (\|\tilde{\varphi}_0\| + \|\psi_0\|) \\ &= \alpha^{n-K} \binom{n}{K} (1 - \tilde{C}(1/n + \epsilon)). \end{aligned}$$

Again, if $\alpha \geq 1$, the RHS diverges as $n \to \infty$ whereas the LHS should be bounded by ii), and the result follows. ∎

Remark: To get Lemma A.2, from this Proposition, note that $\Psi_{\mathcal{SR}}$ is cyclic for $\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}}$ and the set of analytic observables $A_{\mathcal{SR}}$ is (strongly) dense in $\mathfrak{M}_{\mathcal{S}} \otimes \mathfrak{M}_{\mathcal{R}}$. Moreover, Lemma A.1 shows that the dense set of analytic vectors of the form $\{A_{\mathcal{SR}}\Psi_{\mathcal{SR}}\}$ satisfies assumption i). Finally, as $\Psi_{\mathcal{SR}}$ is invariant by $M_\theta(\lambda)$, the spectral radius is equal to 1.

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