Reply to Referee #1

September 20, 2019

We thank the referee for their careful reading and for taking time to communicate many detailed comments, suggestions and questions. This is much appreciated and in the revision, we have taken the referee's points into account.

In the following, we insert our comments (marked by •) into the referee's text *which is in italics.*

1. After Eq. (1.4) it is said that "To describe irreversible effects it is necessary to pass to a limit where the oscillator frequencies $\omega(k)$ take on continuous value". Is it needed that $\omega(k)$ take on continuous values or would it be enough discrete infinitely many?

• Continuous spectrum is needed for true irreversible effects, such as thermalization and decoherence. For discrete values of $\omega(k)$, the system dynamics will exhibit recurrences. These recurrences might happen at very large times and in numerical experiments, one might work with discrete $\omega(k)$ to see the onset of irreversibility. Since we deal with the dynamics for all times and for $t \to \infty$ in particular, we do assume continuous spectrum.

2. It would be very interesting to comment about which variables affect C in bounds like (1.15). I guess that C depends on the form factor g and β, for instance. Actually, I would naively expect that C decreases if the width of the correlation functions decreases. Is this correct?

• The referee is addressing a good point here. Yes, C and λ_0 will depend on g and β, and also the dimension N of the small system. We do not have an immediate answer to this question – it necessitates analyzing the remainders of the approximation of the dynamics, which contain all orders in λ. We mention this now in the text (end of Section 2) and we propose a way of how this can be analyzed in a numerical experiment.

3. On page 10, it is written In this setup, the approach to equilibrium is driven by a spectral gap of the (complex deformed) Liouville operator for the resonance located at the origin. I think this is a very important point and connected to my comment 1. One naively say that unitary evolutions does not produce any irreversible dynamics, so the system plus reservoir state should not have any asymptotic limit. However, the point here is that the reservoir is a continuous of modes, which makes the situation more intricate. I think it would be worth to comment a bit more the main idea behind this mechanism of the spectral gap in the Liouville operator which allows it to produce convergence towards equilibrium $\rho_{\mathrm{SR},\beta,\lambda}$ for the system plus environment dynamics. Surely many people from quantum information/optics/chemistry will be confused that the total dynamics, which is unitary, displays convergence to some steady state.

• Ok, we have added a paragraph explaining this now.

4. I think the proof of Eq. (2.10) is needed to be included, it could be as a footnote as the author does in other cases.

• We have added footnote 8, as suggested. It is a bit on the lengthy side, but it is a straight forward argument, and it is an important one, because it shows how to create purifications of perturbed equilibria.

5. It is a bit confusing the explanation on page 13. The terms $\mathbb{1} \otimes H_{\mathrm{S}}$ *and* $\mathbb{1} \otimes H_{\mathrm{R}}$ *seem to come from the commutator with* H_{S} *and* H_{R}*, respectively. However, the author says they are optional but otherwise the relations* $L_{\mathrm{S}}\Omega_{\mathrm{S},\beta} = 0$ *and* $L_\lambda \Omega_{\mathrm{SR},\beta,\lambda} = 0$ *are not satisfied. I do not understand how the dynamics can be the same without those terms if* $\Omega_{\mathrm{S},\beta}$ *and/or* $\Omega_{\mathrm{SR},\beta,\lambda}$ *are not steady states.*

• We have added a footnote to explain why the $\mathbb{1} \otimes H_{\mathrm{S}}$ is optional. Also, we give now a proof of $L_\lambda \Omega_{\mathrm{SR},\beta,\lambda} = 0$.

6. The procedure to formulate the Liouvillian operator for the purified state reminds me the vectorization approach (I guess they are essentially the same thing for finite dimension. See, for instance, Def. 4.2.9. and Sec. 4.3 of R.A. Horn and C.R. Johnson. Topics in Matrix Analysis. Cambridge University Press, 1994), but taking the basis of the transposition to be the eigenbasis of H_{S} *(compare also with (3.18) in the manuscript). Perhaps this may be useful to explain the action of* J*, which I do not understand very well and it is used at the end of Appendix A.*

• Yes, in finite dimensions this is also called vectorization. The action of J can be expressed in part using the $\mathcal{C}$ in (3.18). We have modified the footnote 6 mentioning vectorization now. We have also included the definition of J, see (2.18), (2.19).

7. It is not fully clear to me why (2.19) holds and why this implies that B' *commutes with* $\mathrm{e}^{\mathrm{i}tL_\lambda}(X \otimes \mathbb{1}_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}})\,\mathrm{e}^{-\mathrm{i}tL_\lambda}$.

• The operator B' can be *chosen* to commute with $\mathrm{e}^{\mathrm{i}tL_\lambda}(X \otimes \mathbb{1}_{\mathrm{S}} \otimes \mathbb{1}_{\mathrm{R}})\,\mathrm{e}^{-\mathrm{i}tL_\lambda}$. This can be done since $\Omega_{\mathrm{SR},\beta,\lambda}$ has the separability property. We recall this now in the text. We also explain now (2.19) in detail in the revised text (now equation (3.35)).

8. It do not understand clearly the equation $L_{0,\theta} = L_0 + \theta N$ *after Fig. 1. Here the author writes* N *is the number operator, but which number operator?, I mean,* L_0 *is a Liouvillian acting on purifications. I think it may be worth to comment a little bit how this equation is obtained.*

• We have now clarified this point by introducing the paragraph **The glued Fock space representation** at the end of Section 3.2 and the new text around (3.39).

9. It is so evident the Eq. (2.26)? perhaps a reference and/or a footnote would be good. It is used at several instances.

• This is second order perturbation theory. We now give three references.

10. After Eq. (2.27), it is written "The remainder decays at rate $-\frac{3}{4}\theta_0$ due to the factor $\mathrm{e}^{\mathrm{i}tz}$. I think it is worth to add few more details on this. It is crucial to understand how the contribution of remaining points of the spectrum of L_λ decays.

• We have added a sentence here. The explanation is simple, we hope the referee agrees.

11. After (2.33), it is written that "$L_\lambda \Omega_{\mathrm{SR},\beta,\lambda} = 0$ implies that $L_0 \Omega_{\mathrm{S},\beta} = 0$". Is this so evident? perhaps a reference and/or a footnote would be good.

• This follows from the so-called isospectrality property of the Feshbach map. It is rather standard perturbation theory. We give a reference for this now.

12. After (3.22), it is written that "the eigenvalues of $\widetilde{L}_0$ y $\widetilde{L}_{\mathrm{S}}$ are the same. Why are they? ($\widetilde{L}_0$ contains in addition the reservoir part L_{R}).

• By eigenvalues we mean points in the spectrum which have normalized associated eigenvectors (not continuous spectrum points). The spectrum of those two operators is *not* the same of course, right, but the eigenvalues are the same. We have now added a footnote to clarify this.

13. It is difficult to follow the last two results in the footnote 13. I think more details are needed.

• Ok. We have explained the details now in the text (and we have removed the footnote).

Minor points:

- References are not ordered according to the order they are cited.

• For now they are in alphabetical order. We will change this if the paper gets accepted in the AOP.

- There are several instances where the identity map is written with double struck and other ones it is written in boldface. It would be good the make the notation uniform.

• Yes, thank you, we have made this change now.

- It is not written what Ω is in Eq. (2.4). I have assumed Ω is the vacuum of $\mathcal{F}$, if so, it would be convenient to write it.

• Done.

- After (2.7), it is written: "The initial states we consider are exactly those which are represented by a vector (or a density matrix) on the space $\mathcal{H}_{\mathrm{ref}}$". Why density matrices? all states are represented as vectors on $\mathcal{H}_{\mathrm{ref}}$, right? $\mathcal{H}_{\mathrm{ref}}$ is the purification space.

• The referee is right. However, it might be useful in practice to take a collection of states represented by vectors $\Psi_1, \ldots$ in $\mathcal{H}_{\mathrm{ref}}$ – say if each Ψ_n has a specific physical meaning – and consider the initial state $\sum_n p_n |\Psi_n\rangle\langle\Psi_n|$ which is a mixture of them. It is then maybe not easy to find the $\Psi \in \mathcal{H}_{\rm ref}$ which represents this mixture explicitly. This is why it may be useful to allow for density matrices on $\mathcal{H}_{\rm ref}$.

- In (2.19) I assume that $\mathcal{B}(\mathcal{H})$ is the set of bounded operators, but it is not defined.

• Yes, corrected.

- After (2.19), it is written $\omega_{{\rm SR},\beta,\lambda} = \Omega_{{\rm S},\beta} \otimes \Omega_{\rm R} + O(\lambda)$. However, I understand the left hand side is a functional and the right hand side is a vector.

• Correct, $\omega_{{\rm SR},\beta,\lambda}$ should be $\Omega_{{\rm SR},\beta,\lambda}$. This is corrected now.

- This is quite optional, but maybe the notation θ in (A) on page 14 and subsequents may induce to think that θ is an angular variable which is not true.

• θ is for *t*ranslation. We mention this now after stating condition (A), which has moved to the front of the paper.

- I think it would be good to add a reference in the footnote 7.

• We have done this now, and also explained this point in more detail, see discussion point (2) after (A).

- After (2.29) it is written "Note that the remainder is proportional to lambda as the integral over the zeroth order term vanishes", do you mean the "zeroth order term" of which equation? (2.27) perhaps?

• Yes, correct. This is now explained in more detail.

- In (2.33), by using the $\subset$ symbol I guess you mean that the subspace where $Q_e^{(s)}$ projects onto is contained in the one where $P(L_{\rm S} = e)$ projects onto, is this right? (I'm used to see the symbol $<$ for this).

• Right. We changed the notation.

- Before Eq. (3.3) it is written the power series expansion of the exponential and (2.44), I think you mean (2.45).

• Correct. Changed.

- In (A3) it might be clearer to remove ℓ in the first sum writing it just in terms of the summation variable k.

• In the anti-commutator part we have $k = \ell$. But in the term $P_k G P_k \rho P_\ell G P_\ell$ we must have both, k and ℓ. So we prefer to leave the sum as it is.

- In Appendix A there are two J's: one stands for the reservoir spectral density and the other for the anti-unitary involution.

• Yes. This ambiguity is now removed.