M 3001

ASSIGNMENT 9: SOLUTION

Due in class: Wednesday, March 23, 2011

Name	M.U.N. Number

1. Find the interval of convergence for each of the following power series.

(i) $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+1}(x+2)^k$;

(ii) $1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \cdots$;

(iii) $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1}x^{2k+1}$;

(iv) $x + 1^2x^2 + \sqrt{1}x^3 + 2^2x^4 + \sqrt{2}x^5 + 3^2x^6 + \sqrt{3}x^7 + \cdots$.

(v) $\sum_{k=1}^{\infty}(\ln k)^k x^k$.

Solution. (i) First,

$$R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\frac{k+2}{k+1} = 1.$$

Next, the power series is centered at $x = -2$. When $x = -1$, the series becomes $\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k+1}$ which is the convergent alternating harmonic series. When $x = -3$, the series becomes $-\sum_{k=0}^{\infty}(k+1)^{-1}$ which is divergent. Hence $I = (-3, -1]$.

(ii) First,

$$R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\frac{2k+2}{2k+1} = 1.$$

Next, when $x = -1$ we get that the series is divergent by the Raabe's Test since

$$k\Big(\frac{|a_k|}{|a_{k+1}|} - 1\Big) = \frac{k}{2k+1} \to 1/2.$$

When $x = 1$, we get an alternating series whose general term decreases in absolute value (since $\left|\frac{a_{k+1}}{a_k}\right| < 1$), and approaches zero (the reasoning is given below). Hence $I = (-1, 1]$.

To prove

$$\lim_{k\to\infty}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k} = 0,$$

we verify

$$\sum_{k=1}^{\infty} b_k = \Big(\frac{1}{2}\Big)^3 + \Big(\frac{1\cdot 3}{2\cdot 4}\Big)^3 + \Big(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\Big)^3 + \cdots + \Big(\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\Big)^3 + \cdots < \infty.$$

By the Raabe's Test, we find

$$k\Big(\frac{b_k}{b_{k+1}} - 1\Big) = k\left(\Big(\frac{2k+2}{2k+1}\Big)^3 - 1\right) = \frac{2k}{2k+1}\cdot\frac{(1+t)^3 - 1}{2t},$$

where $t = (2k+1)^{-1}$. The limit of this expression, as $k \to \infty$, is (by l'Hospital's Rule):

$$\lim_{t\to 0}\frac{(1+t)^3-1}{2t}=\lim_{t\to 0}\frac{3(1+t)^2}{2}=\frac{3}{2}>1.$$

Therefore the last series is convergent. So the necessary condition of the convergence for a series implies $\lim_{k\to\infty} b_k = 0$, as desired.

(iii) First apply the Ratio Test:

$$\lim_{k\to\infty}\frac{\frac{x^{2k+3}}{2k+3}}{\frac{x^{2k+1}}{2k+1}}=\lim_{k\to\infty}\frac{2k+1}{2k+3}\cdot x^2=x^2.$$

So this series converges for $x^2<1$. When $x=-1$ we get $\sum(-1)^k/(2k+1)$, which converges by the Alternating Series Theorem. When $x=1$, we get $-\sum(-1)^k/(2k+1)$, whence $I=[-1,1]$.

(iv) Consider the odd powers and even powers separately: $\sum\sqrt{k}x^{2k+1}$ and $\sum k^2x^{2k}$. Both of these series have $R=1$, and at $x=\pm 1$, the general terms do not approach zero. Hence $I=(-1,1)$.

(v) Let R be an arbitrary positive number. Since $\lim_{k\to\infty}\ln k=\infty$, we have $\ln k>R$ for k sufficiently large. Therefore $\sum_{k=1}^{\infty}(\ln k)^k|x|^k$ diverges whenever $\sum R^k|x|^k$ diverges, and the latter series has $(-R^{-1},R^{-1})$ as its interval of convergence. Thus $\sum_{k=1}^{\infty}(\ln k)^k|x|^k$ diverges if $|x|\ge R^{-1}$. Since R can be any positive number, it follows that the series converges only when $x=0$. Hence $I=\{0\}$.

Another method: since $\ln k<\ln(k+1)$ and $\ln(k+1)\to\infty$, we conclude that

$$\lim_{k\to\infty}\Big|\frac{a_k}{a_{k+1}}\Big|=\lim_{k\to\infty}\Big(\frac{\ln k}{\ln(k+1)}\Big)^k\cdot\frac{1}{\ln(k+1)}=0.$$

This gives that the convergence radius is 0. Of course, the series converges at 0 only.

2. Evaluate the following sums:

(i) $\sum_{k=0}^{\infty}(k+1)x^k$, $|x|<1$;

(ii) $\sum_{k=0}^{\infty}k^2 2^{-k}$

(iii) $\sum_{k=0}^{\infty}(k^2+3k+1)2^{-k}$.

Solution. (i) Let $f(x)=\sum_{k=0}^{\infty}x^k=\frac{x}{1-x}$ for $x\in(-1,1)$. This yields

$$f'(x)=\sum_{k=0}^{\infty}(k+1)x^k=\frac{d}{dx}\Big(\frac{x}{1-x}\Big)=\frac{1}{(1-x)^2},\qquad x\in(-1,1).$$

(ii) Let $f(x)=\sum_{k=0}^{\infty}x^k=\frac{x}{1-x}$ for $x\in(-1,1)$. Then

$$f'(x)=\sum_{k=0}^{\infty}(k+1)x^k=\frac{d}{dx}\Big(\frac{x}{1-x}\Big)=\frac{1}{(1-x)^2},\qquad x\in(-1,1).$$

Multiplying through by x yields

$$\sum_{k=1}^{\infty}kx^k=\frac{x}{(1-x)^2},\qquad x\in(-1,1).$$

Now differentiate both sides to get

$$\sum_{k=1}^{\infty} k^2 x^{k-1} = \frac{1+x}{(1-x)^3}, \quad x \in (-1,1),$$

and then multiply by x; hence

$$\sum_{k=1}^{\infty} k^2 x^{k} = \frac{x(1+x)}{(1-x)^3}, \quad x \in (-1,1),$$

Let $x = 1/2$ to get $\sum k^2 2^{-k} = 6$.

(iii)

$$\sum_{k=0}^{\infty} (k^2+3k+1)2^{-k} = \sum_{k=0}^{\infty} k^2 2^{-k} + 3\sum_{k=0}^{\infty} k2^{-k} + \sum_{k=0}^{\infty} 2^{-k} = 6 + 3\cdot 2 + 2 = 14.$$