The Nine-Point Circle

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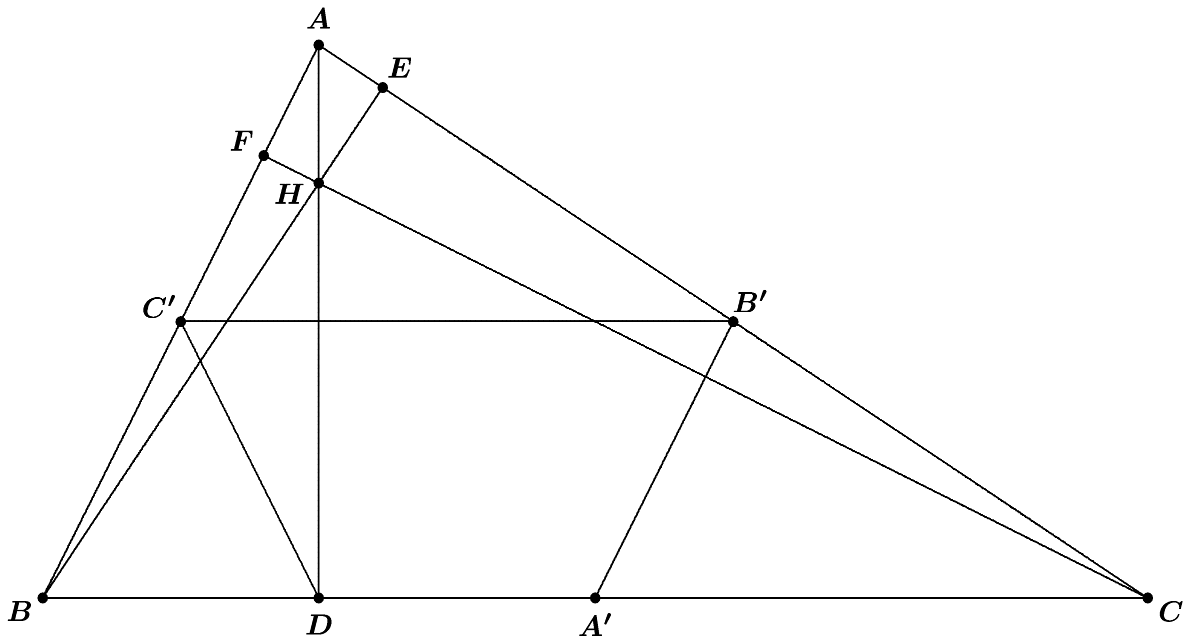
Given $\triangle ABC$ and mid-points of the sides A', B', C', and altitudes AD, BE, CE (intersecting at orthocentre H), prove that quadrilateral $DC'B'A'$ is cyclic.



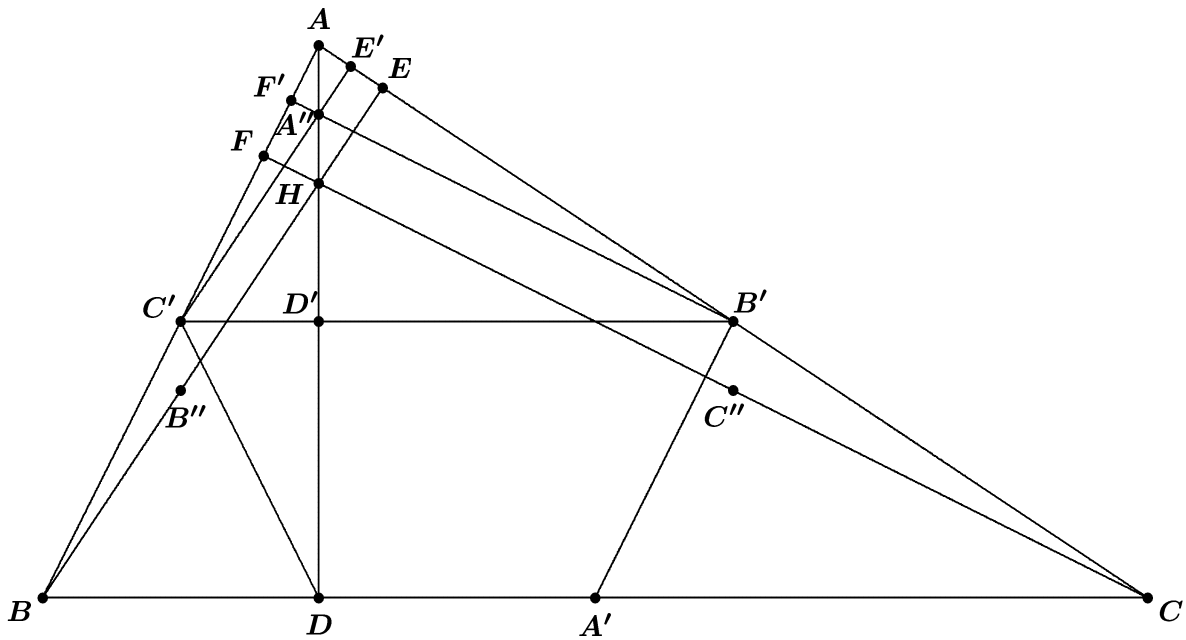
Note that C' is the centre of circle ABD. Hence, $C'B = C'D$, giving $\angle B = \angle C'DB$. Since $C'B' \parallel BC$, we get $\angle C'BD = \angle DC'B'$.

Also, $\triangle A'B'C' \sim \triangle ABC$ (sides in proportion), giving $\angle B = \angle C'B'A'$.

Hence, $\angle DC'B' = \angle A'B'C'$, and since $C'B' \parallel DA'$, we have that quadrilateral $DC'B'A'$ is a symmetric trapezoid, and so, is cyclic.

DEDUCTION. We could just as easily have chosen E or F instead of D: hence, the six points A', B', C', D, E, F, lie on a circle (the SIX POINT circle).
Let A'' be the mid-point of AH (B'' and C'' defined similarly).

Draw $C'A''$ to meet AC at $E'E$, draw $B'A''$ to meet AB at F', and let AD meet $B'C'$ at D'



Note that $AC' : C'B = AA'' : A''H = 1 : 1$, giving that $C'E' \parallel BH$. Similarly, $B'F \parallel CH$.

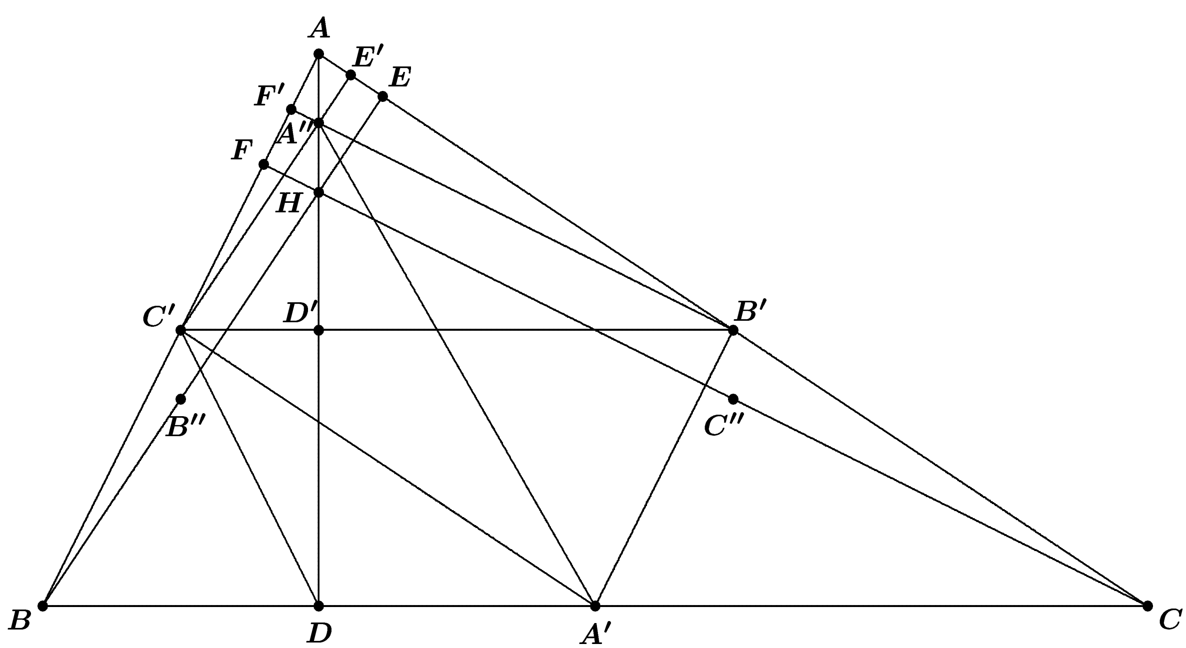
Thus, AD', $C'E'$, $B'F'$ are the altitudes of $\triangle AC'B'$.

Hence, quadrilateral $AF'A''E'$ is cyclic.

Thus, $\angle A = 180^\circ - \angle F'A''E' = 180^\circ - \angle C'A''B'$. Also, $\angle B'A'C' = \angle A$ (recall $\triangle ABC \sim \triangle A'B'C'$).

Thus, $\angle C'A''B' + \angle C'A'B' = 180^\circ$, giving that quadrilateral $A''B'A'C'$ is cyclic.

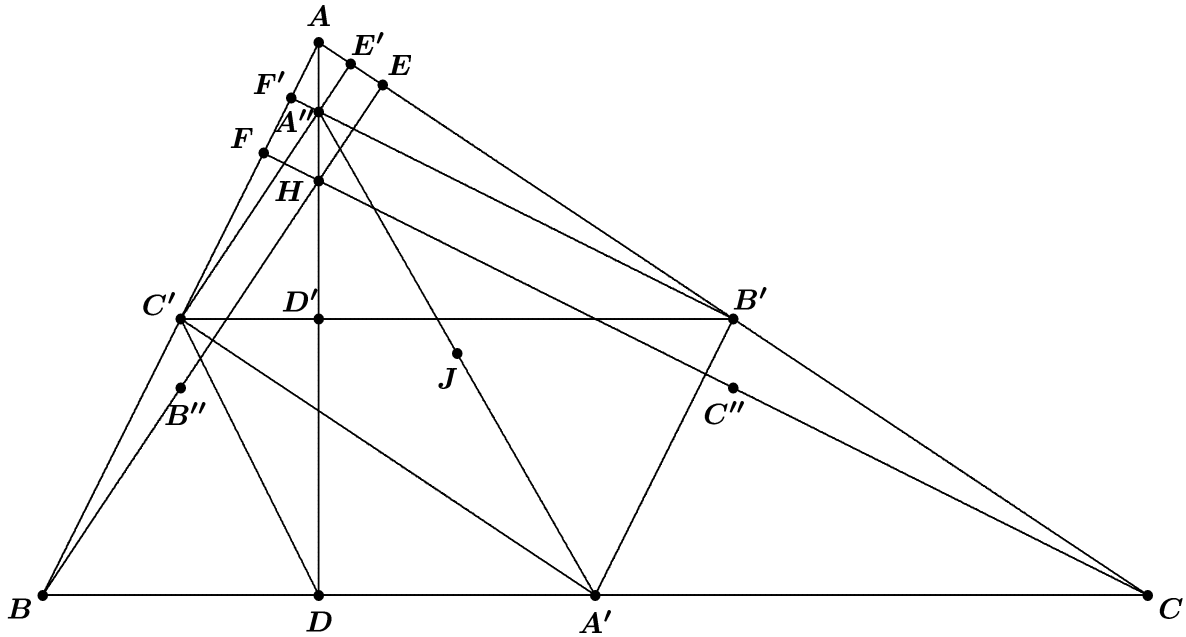
We could just as easily have chosen B'' or C''. Hence, the NINE POINTs A', B', C', D, E, F, A'', B'', C'' lie on a circle.



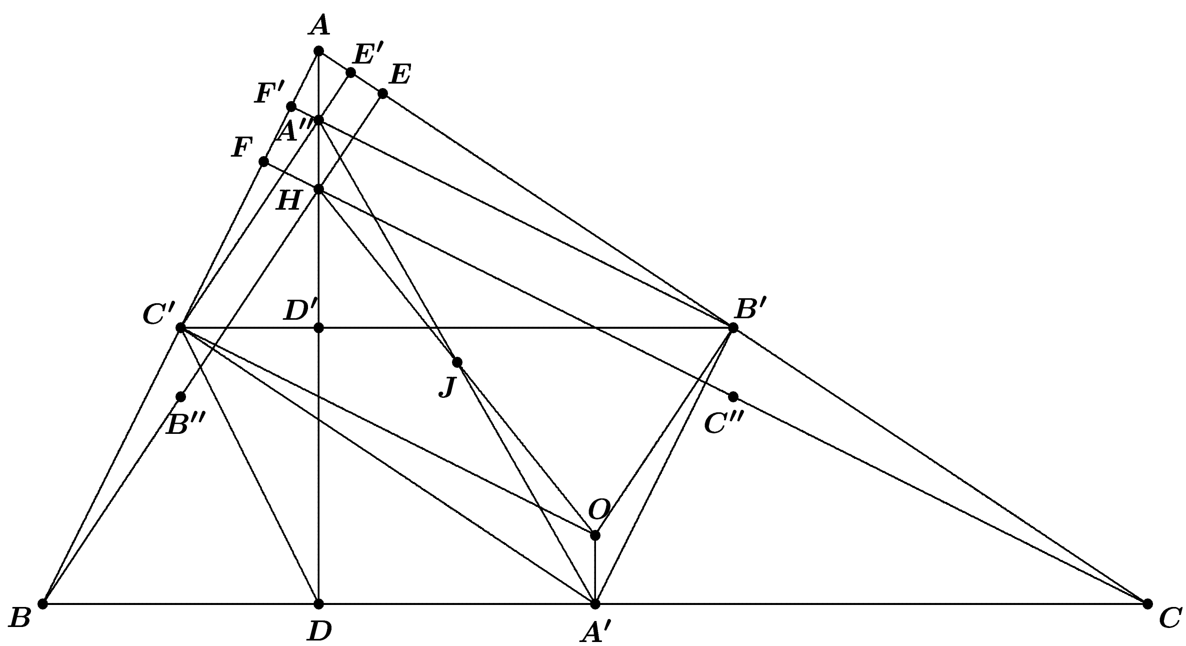
Note that $C'A'' \perp AC$ and that $C'A' \parallel AC$, giving that $\angle A''C'A' = 90°$.

Hence, $A''A'$ is a diameter of the nine-point circle.

Then J, the mid-point of $A''A'$, is the centre of the nine-point circle.



We will find the radius of the nine-point circle. First, we recall the circumcentre, O.



Note that $C'A'' = \frac{1}{2}BH$ and $C'A' = \frac{1}{2}AC$.

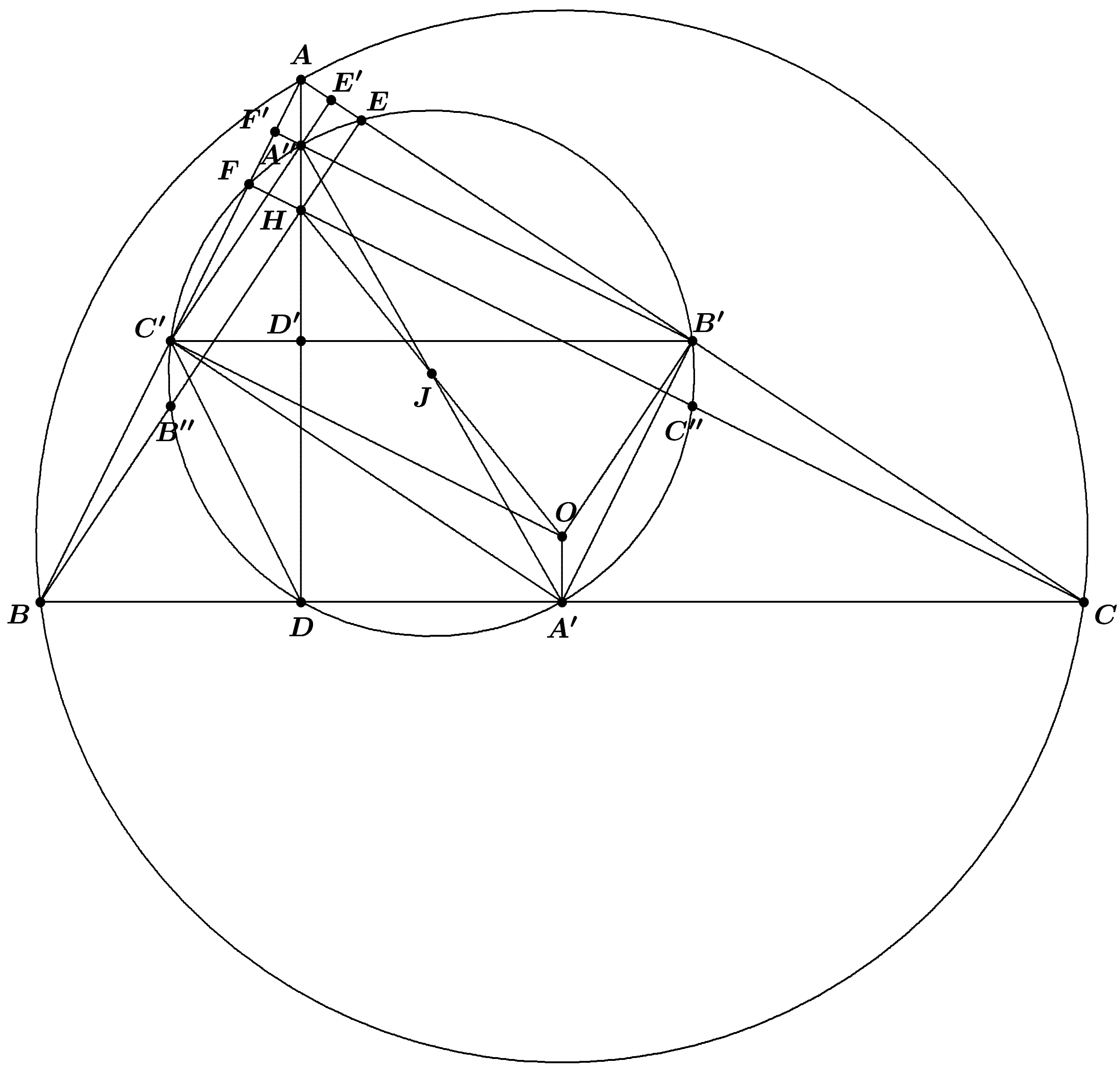
Thus, from $\triangle C'A''A'$, $4\rho^2 = \frac{1}{4}BH^2 + \frac{1}{4}AC^2$.

Also, $R^2 = OB'^2 + \frac{1}{4}AC^2$.

But, $OB' = C'A''$ (since $OB' \parallel C'A'' \perp AC$ and $B'A'' \parallel OC' \perp AB$).

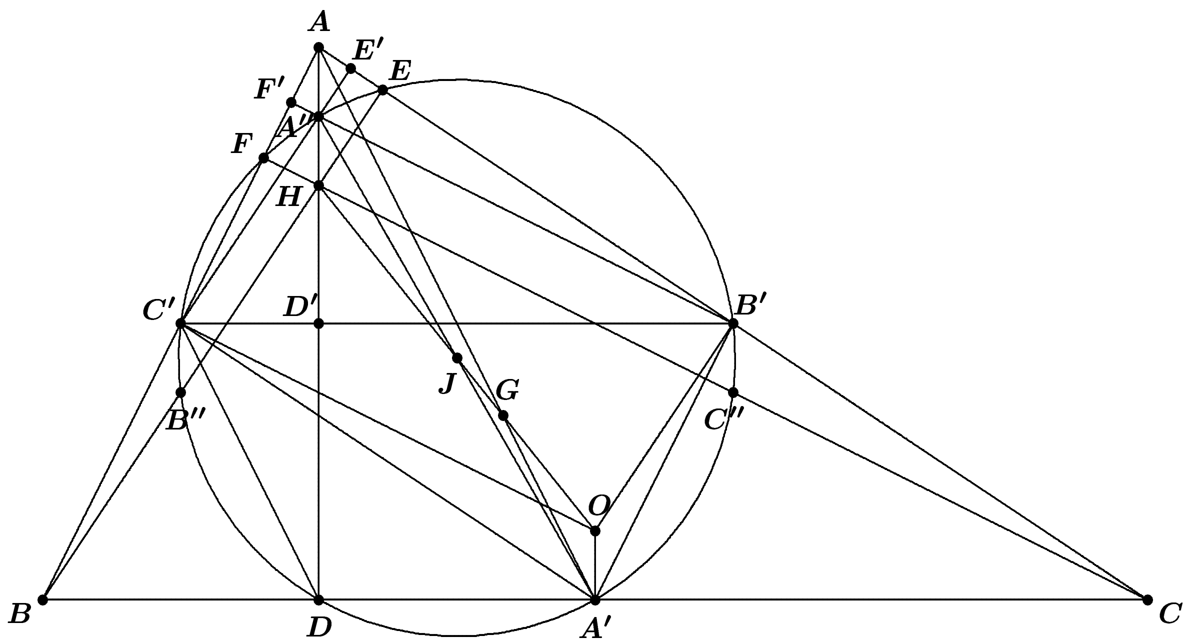
Therefore, $4\rho^2 = R^2$, and hence, $\rho = \frac{1}{2}R$.

The radius of the nine-point circle is one half of the radius of the circumcircle.



It looks as if the centre of the nine-point circle J lies on OH.

Now, recall the Euler line HGO, where H is the orthocentre, G is the centroid and O is the circumcentre.



Since $OG : GH = 1 : 2$ (property of Euler line) and $A'G : GA = 1 : 2$ (property of centroid on a median), and $AH \parallel OA'$, it follows that $AH = 2OA'$.

Thus, $A''H = OA'$. Since $A''J = JA'$ (centre is mid-point of a diameter), it follows that the mid-point of OH is J.

Thus, J lies of the Euler line, and we now have the distribution: